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On the Smallness of the Cosmological Constant

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ABSTRACT

We investigate the possible values of the cosmological constant allowed by quantum cosmology.

If we formulate quantum cosmology respecting the causal nature of fundamental equations in the semiclassical regime of the universe, then any classical universe should have at least one symmetric surface, i.e., a spatial surface on which every component of the extrinsic curvature vanishes. Combined with the Hamiltonian constraint, this implies that the allowed values of the cosmological constant are bounded from above.

Applying this argument to the Robertson-Walker universe, we obtain the theoretical upperbound for the cosmological constant, being of order $(H_0/c)^2$. This upperbound can also be interpreted as being determined by the adiabatic Schwarzshild radius of the whole universe. In this way, the question as to why the cosmological constant is so small is reduced to the question, why there is so much matter in our universe.

1. Introduction

The smallness of the cosmological constant has been a mysterious problem for physicists, although the inconsistency it implies has not yet affected the present activities in physics directly. The main problem consists in an extraordinary discrepancy between the two expected values of the cosmological constant, one from quantum field theory and particle physics and the other one from cosmological observations and observational cosmology.[1],[2]

Any kind of vacuum energy should be treated as a part of the cosmological constant, Λ, when general relativity is taken into account. The zero point energy, for instance, contributes to Λ as

$$\Lambda_{vac} = \frac{8\pi G}{c^2} \int \frac{1}{2} \hbar w / c^2 \frac{d^3p}{\hbar^3} \sim \left(\frac{l_{pl}}{l_0}\right)^2 \frac{1}{l_0^2} \sim (\frac{M_0}{M_{pl}})^2 M_0^2 \frac{c^2}{\hbar^2}$$

to each component of each quantum field. Here, l_0 and M_0 denote a cut-off scale, while l_{pl} and M_{pl} denote the Planck scale. As an illustration if we put $M_0 = M_{pl}$, the scale at which general relativity itself breaks down, we get $\Lambda_{vac} \sim l_{pl}^{-2} \sim (10^{-33}\mathrm{cm})^{-2}$.

On the other hand, we know from cosmological observations that the total matter density and spatial curvature do not greatly exceed 1 at present, in units of $\frac{3H_0^2}{8\pi G}$ and $(H_0/c)^2$, respectively. (Here, $H_0 = 50 - 100\ \mathrm{km \cdot s^{-1} \cdot Mpc^{-1}} =: h \cdot 100\ \mathrm{km \cdot s^{-1} \cdot Mpc^{-1}}$ is the Hubble parameter at present.) With the help of the Einstein equations, this implies that Λ_{obs} should also be of the order of $(H_0/c)^2 \sim (10^{28}\mathrm{cm})^{-2}$.

Thus, $\Lambda_{obs}/\Lambda_{vac} \sim 10^{-122}$. This discrepancy between theory and observation is the fundamental problem of the cosmological constant.[1],[2]

Here, let us note the special feature of this problem. Although there is a definite discrepancy between Λ_{vac} and Λ_{obs}, there remains a freedom to choose a (negative) bare parameter Λ_0 suitably to adjust $\Lambda_0 + \Lambda_{vac} \sim \Lambda_{obs}$. The queer

feature which bothers us, then, is that it looks as if this parameter Λ_0 had been so chosen in advance at the very early stage of the universe as to exactly cancel the present value of Λ_{vac}. Moreover, the exact value of Λ_{vac} at present is sensitive to the precise details of the dynamics of fields from the big-bang to the present - field species, symmetries, form of interactions and coupling constants. It is hard to think that the early universe have chosen Λ_0 correctly with the accuracy of $O(10^{-122})$, "knowing exactly" and "worrying about" the future value of Λ_{vac}. (The problem of "prearrangement".[3]) In this sense, this problem has more to do with the problem of "unnaturalness" rather than any theoretical contradictions. Thus, we are not interested in any *ad hoc* explanation for it, because such a theory tends to be a graceful rephrase of "unnaturalness".

Hence, we need to find a natural explanation for the smallness of $\Lambda = \Lambda_0 + \Lambda_{vac}$, rather than Λ_{vac}. Quantum cosmology is supposed to be effective for this purpose, because it can handle a wide variety of universes simultaneously, including a wide variety of values for Λ_0. Works by Hawking[4] and by Coleman[3] are skillful attacks on the problem along this line. They have shown a promising possibility that the answer to the problem lies in the realm of quantum cosmology. We will discuss these theories in the final section. Here, it suffices to recall that their arguments heavily rely on the Euclidean formulation of quantum gravity, postulating some technical assumptions which are open to question at present.[5]

In this paper, we consider the cosmological constant from the viewpoint of quantum cosmology, but with a different perspective. Keeping the specific nature of this problem in mind, we examine the logical consequences of the pursuit of "naturalness", in the sense that we rely on a theory based on the assumptions which seem reasonable and minimum, avoiding uncertain impositions as far as possible. In other words, we investigate how far we can proceed along a path of "prearrangement".[3]

Here, we respect the pseudo-Riemannian characteristic of space-times. We assume that the path-integral formulation of quantum theory can also be applied

to general relativity at suitably low energy scales. We postulate that the stationary phase configurations have relevance, while we need not assign any probabilistic meaning to the absolute value of the "wave function", as far as our discussions below are concerned.

2. Quantum cosmology and selection of spacetimes

By combining general relativity with quantum theory in a suitable way we can restrict global properties of the universe to a great extent: quantum theory handles all possible histories at one time, compares them with each other and chooses preferable ones from among them.

Recently, one natural formulation of quantum cosmology, which respects the causal nature of the semiclassical universe, has been presented.[6] Since the closed-time path formalism[8] (the in-in formalism), which is one useful version of the path-integral formalism, plays the central role in this theory, let us recall this briefly. To make an explanation simple, we consider a mode $q(\tau)$ with action $S[q(\cdot)]$ and source $J(\tau)$.

In the closed-time path formalism, we perform the path-integration on a specific contour along the time axis: first go forward in time from $\tau = 0$ to $\tau = T$, turning at $\tau = T$, finally go backward in time from $\tau = T$ to $\tau = 0$ (we parametrize this backward-route as $\tau : T-2T$). Here, the boundary condition is $q(0) = q(2T) = q_0$. Let us denote symbolically the class of all paths of $q(\tau)$ subject to this special boundary condition as $c(q_0)$. Equivalently, we can restate this procedure as follows: We double the degrees of freedom from $q(\cdot)$ to $q_+(\cdot)$ and $q_-(\cdot)$. We assign actions $S[q_+]$ and $-S[q_-]$ for $q_+(\cdot)$ and $q_-(\cdot)$, respectively. We then perform the path-integral during $\tau : 0 - T$ with the boundary condition $q_+(0) = q_-(0) = q_0$, $q_+(T) = q_-(T) = q_1$. Finally, we integrate over q_1.

More specifically, we calculate the following quantity:

$$\exp\frac{i}{\hbar}W[J_+,J_-] := \int_{c(q_0)}(dq)\exp\frac{i}{\hbar}(S[q(\cdot)]+\int Jqd\tau)$$
$$= \int dq_1 \int (dq_+)_{|\substack{q_+(0)=q_0\\ q_+(T)=q_1}} (dq_-)_{|\substack{q_-(0)=q_0\\ q_-(T)=q_1}}$$
$$\exp\frac{i}{\hbar}\left\{(S[q_+(\cdot)]+\int_0^T J_+q_+d\tau)-(S[q_-(\cdot)]+\int_0^T J_-q_-d\tau)\right\} .$$

From $W[J_+,J_-]$, we can neatly obtain various expectation values, e.g.,

$$\frac{\hbar}{i}\frac{\delta W}{\delta J_+(\tau)} = \frac{{}_{J_-}< q_0|q(\tau)|q_0>_{J_+}}{{}_{J_-}< q_0|q_0>_{J_+}}_{|J_+\equiv J_-=J} =< q_0|q(\tau)|q_0>_J \quad .$$

The most important point is that, only after putting $J_+(\cdot)\equiv J_-(\cdot)$, we obtain the real, causal quantity $< q_0|q(\tau)|q_0>_J$.[9] Here, "causal" means the property that $< q_0|q(\tau)|q_0>_J$ depends only on $\{J(\tau')\}_{0\le\tau'\le\tau}$, i.e. only on the past information of J. In this way, the closed-time path formulation yields expectation values naturally.[8],[9]

Now, let us return to the main subject. The fundamental object $(h\ \phi)$, which is of a central importance in the theory of our concern, is defined as

$$(h\ \phi) = \int_{c(h,\phi)}(dg)(d\phi)\exp\frac{i}{\alpha}(S_g+S_m(\phi:g)) \quad . \tag{1}$$

Here, $S_g := \int R\sqrt{-g}$, $S_m := 16\pi G/c^3 \times$ (matter action) and $\alpha := l_{pl}^2 = 16\pi G\hbar/c^3$. In this context, h is always an abbreviation for h_{ab}. For the technical convenience, we will confine ourselves to the case of the spatially closed spacetimes throughout this paper. The symbol $c(h,\phi)$ implies the class of closed-time paths with the boundary value h_{ab} and ϕ. If we need, it is convenient to choose the gauge-fixing condition for eq.(1) as $\int_0^T N_+d\tau = \int_0^T N_-d\tau$, $N_{+i} = N_{-i} = 0$.

We include the cosmological term into the matter action S_m. First, the original bare cosmological constant is included in S_m in the form $(-2\Lambda_0)\sqrt{-g}$. One can

attach to this term a dynamical meaning, if, e.g., one introduces a special kind of field with 3 anti-symmetric indices.[10],[4],[1] Hence, we understand that $d\Lambda_0$ is included formally in $(d\phi)$ of eq.(1). Secondly, other contributions, Λ', to the cosmological constant of dynamical origin, e.g. the zero point energy of quantum fields, are also included in S_m implicitly. In this way, we can understand that the symbolical argument ϕ in $(h\ \phi)$ includes the cosmological constant Λ, in the sense that $\Lambda = \Lambda_0 + \Lambda'$.

Parallel to the previous example with $q(\cdot)$, we can define $W[g_+, g-]$ as

$$\exp \frac{i}{\alpha} W[g_+, g_-] := \int_{c(h,\phi)} (d\phi) \exp \frac{i}{\alpha} S_m(\phi : g) \quad . \tag{2}$$

This time, the metrics $g_\pm$ play the role of source $J_\pm$ automatically. Hence, eq.(1) can be expressed as

$$(h\ \phi) = \int_{c(h)} (dg) \exp \frac{i}{\alpha} (S_g + W[g]) \quad . \tag{3}$$

When we are interested in the semiclassical regime of the universe, the typical fluctuations in gravity induce a macroscopic change in the action, $\mid S_g \mid \gg \alpha$, so that the stationary phase approximation for $(h\ \phi)$ becomes appropriate.

From the first variation of the phase in eq.(3), we obtain

$$G_{ab}^{g_\pm} + \frac{\delta W}{\delta g_\pm^{ab}} = 0 \quad ,$$

or,

$$G_{ab}^{g_\pm} = \frac{8\pi G}{c^3} \frac{{}_{g_-} < \phi | \tilde{T}_{ab}^{g_\pm} | \phi >_{g_+}}{{}_{g_-} < \phi | \phi >_{g_+}} \quad . \tag{4}$$

Here, the suffix "$g_\pm$" attached to G_{ab} and $\tilde{T}_{ab}$ implies that these quantities should be calculated using g_+ or g_-, respectively. We use the symbol $\tilde{T}_{ab}$ to express the energy-momentum tensor including the cosmological constant, $\tilde{T}_{ab} :=$

$T_{ab} - \frac{c^3}{8\pi G}\Lambda g_{ab}$. Since the left-hand side of eq.(4) is real, the right-hand side should also be real, implying that $g_+(\cdot) \equiv g_-(\cdot) = \exists g_0(\cdot)$. Thus, eq.(4) reduces to the semiclassical Einstein equation, which is believed to be appropriate for the description of (semi)classical gravity with quantum matter:

$$G_{ab}(x) = 8\pi G/c^3 < \phi|\tilde{T}_{ab}(x)|\phi > . \tag{5}$$

We should note that $(h\ \phi)$ correctly yields the causal equation of motion eq.(5) in the semiclassical epoch of the universe, which fits to our way of describing Nature. Here, by the term "causal", we mean that eq.(5) does not refer to the information outside the past light-cone of x. This causal character can be seen prominently in the right-hand side of eq.(5) : there appear expectation value $< \phi|\tilde{T}_{ab}(x)|\phi >$, and not $< out|\tilde{T}_{ab}(x)|in >$.

Even though we have constructed $(h\ \phi)$ in a general manner, we can extract from $(h\ \phi)$ a great deal of information about the global properties of the universe, some of which we will mention now.

Since the phase in eq.(3) should be stationary also at $\tau = T$, $g_+ = g_0$ and $g_- = g_0$ should be joined smoothly on the $\tau = T$ spatial surface Σ_T, because the Einstein equation produces a smooth manifold from a smooth initial condition. This means that $\pi^{ab} = 0$ on Σ_T, where π^{ab} is momentum conjugate to h_{ab}. Here, let us call a spatial surface Σ with $\pi^{ab} = 0$ a symmetric surface. The sense of this term comes from the fact that one can construct, if one wishes, a symmetric configuration of spacetime, with Σ as a reflection surface. Now, Σ_T: symmetric implies Σ_T: maximal, i.e., $\pi = 0$ on Σ_T. With the help of theorems on maximal surfaces, we find out that[6]

(a) If the strong energy condition for $< \tilde{T}_{ab} >$ is satisfied, any classical universe should be the one which starts from, and ends in a singularity.

(b) If the dominant energy condition for $< \tilde{T}_{ab} >$ is satisfied, possible 3-dimensional topologies of the classical universe are severely restricted, essentially elliptic.

By investigating the second variation of the phase in eq.(3), we can also obtain the validity conditions for semiclassical gravity.[7]

The present formulation of quantum cosmology, which respects fundamental concepts in physics (in the semiclassical regime), e.g. pseud-Riemannian structure of spacetime and causality, seems to be one of the reasonable ways to combine general relativity and quantum theory. Below, we investigate another implication of $(h\ \phi)$: the restriction on the value of the cosmological constant.

3. The symmetric surface and the cosmological constant

In this section, we investigate the constraint on the cosmological constant resulting from the existence of a symmetric surface.

We fix a spatial surface Σ. Let n^a be a unit normal vector to Σ. Using n^a, we can construct the spatial metric $h_{ab} = g_{ab} + n_a n_b$ and the extrinsic curvature $K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab} = h_a{}^c \nabla_c n_b = h_{(a}{}^c h_{b)}{}^d \nabla_c n_d$.

Then, the Hamiltonian constraint reads

$$\mathcal{A} = -(\mathbf{R} - \alpha/\hbar . T_{nn} - 2\Lambda), \qquad (6-a)$$

where

$$\mathcal{A} := \left(\frac{1}{n-2}\pi^2 - \pi_{ab}\pi^{ab}\right)\sqrt{\mathbf{h}}^{-2} = K^2 - K_{ab}K^{ab} \quad . \qquad (6-b)$$

Here, $\mathbf{R}$ is a spatial scalar curvature constructed from h_{ab}, $\sqrt{\mathbf{h}} := \sqrt{\det(h_{ab})}$, $\pi^{ab} := (K^{ab} - Kh^{ab})\sqrt{\mathbf{h}}$ and $T_{nn} := T_{ab}n^a n^b$.

On a symmetric surface, $\mathcal{A} = 0$ so that

$$\Lambda = \frac{1}{2}(\mathbf{R} - \alpha/\hbar . T_{nn})_s \quad ,$$

where the suffix "$_s$" denotes the quantity on the symmetric surface.

Using the equation of motion for matter, we can express T_{nn} in terms of h_{ab} and the initial condition for matter.

We parametrize the spatial compact foliations as Σ_τ ($\tau \in R$). If we choose one-parameter families of h_{ab} and ϕ, $\{h_{ab}(\cdot,\tau)\}$ and $\{\phi(\cdot,\tau)\}$, we obtain a function of τ,

$$f_{\{h\}\{\phi\}}(\tau) := \mathrm{Max}_{p\in\Sigma_\tau}\frac{1}{2}(\mathbf{R} - \alpha/\hbar.T_{nn})_{|p} \quad . \tag{7}$$

Then, the necessary condition for Λ to give a classical universe corresponding to fixed one-parameter families $\{h_{ab}(\cdot,\tau)\}$ and $\{\phi(\cdot,\tau)\}$ is,

$$\mathrm{Inf}_\tau f_{\{h\}\{\phi\}}(\tau) \leq \Lambda \leq \mathrm{Sup}_\tau f_{\{h\}\{\phi\}}(\tau) \quad , \tag{8}$$

since this condition is necessary for the existence of a symmetric surface.

Here, we should note that a symmetric surface favours a positive curvature as is obvious from eq.$(6-a)$. In fact, $\mathbf{R} = \alpha/\hbar.\tilde{T}_{nn}$ $(:= \alpha/\hbar.T_{nn} + 2\Lambda)$ on a symmetric surface, implying $\mathbf{R} \geq 0$ if we require the energy condition for $\tilde{T}_{nn}$. This induces a severe limitation of topology of Σ.[6]

Now, let us estimate a behavior of $f_{\{h\}\{\phi\}}(\tau)$ more specifically, introducing the "scale factor", $\gamma := \sqrt{\mathbf{h}^{\frac{1}{n-1}}}$. The typical behavior of $\mathbf{R}$ and T_{nn} are $\mathbf{R} \propto \gamma^{-2}$ and $T_{nn} \propto \gamma^{-(n-1+\delta)}$ $(\delta \geq 0)$. Thus, the typical behavior of $f_{\{h\}\{\phi\}}(\tau)$ is

$$f_{\{h\}\{\phi\}}(\tau) \sim \alpha\gamma^{-2} - \beta\gamma^{-(n-1+\delta)} \qquad (\alpha,\ \beta \geq 0) \quad ,$$

so that

$$\begin{aligned}\mathrm{Sup}_\tau f_{\{h\}\{\phi\}}(\tau) &\sim \left\{\left(\frac{2}{n-1+\delta}\right)^{\frac{2}{n-3+\delta}} - \left(\frac{2}{n-1+\delta}\right)^{\frac{n-1+\delta}{n-3+\delta}}\right\}\left(\frac{\alpha^{n-1+\delta}}{\beta^2}\right)^{\frac{1}{n-3+\delta}} \\ &=: f_0 \quad ,\end{aligned}$$

Thus, we obtain an upperbound for Λ, i.e., $\Lambda \lesssim f_0$. It is important to note that the existence of a symmetric surface imposes the upperbound on Λ, though it is

difficult to specify the numerical value for this bound for general $\{h_{ab}\}$ and $\{\phi\}$. In the next section, we will find an exact theoretical upperbound for Λ by restricting $\{h_{ab}\}$ to the Robertson-Walker metric and $\{\phi\}$ to a suitable matter.

4. The theoretical upperbound for the cosmological constant

To make the discussion in the previous section a more specific one, let us confine ourselves to the case of the closed Robertson-Walker universe.[11] Then, the fundamental equations become

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{c^2}{a^2} = \frac{\alpha\Lambda_{(0)}c^2}{3} + \frac{8\pi G}{3}\rho \quad , \tag{9}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P/c^2) + \frac{\alpha\Lambda_{(0)}c^2}{3} \quad . \tag{10}$$

We consider matter which satisfies

$$P/c^2 = \mu\rho \quad (\mu : \text{constant}, \ -1 < \mu \leq 1) \ . \tag{11}$$

Especially, the case $\mu = 0$ (dust) is known as the Lemaitre model.[11]

Here, we explain the meaning of $\alpha\Lambda_{(0)}$ in eq.'s (9) and (10). For a given μ and a given ρ, the special value of the cosmological constant which allows a static solution ($\dot{a} = 0$, $\ddot{a} = 0$) is determined from eq.'s (9)-(11). Let us denote this value as $\Lambda_{(0)}$. Regarding $\Lambda_{(0)}$ as a unit, we can express the general cosmological constant as $\Lambda = \alpha\Lambda_{(0)}$, where α is a parameter.

Now, from eq.'s (9) and (10), $(\rho a^3)\dot{} + P/c^2.(a^3)\dot{} = 0$, so that, with the help of eq. (11),

$$(\rho a^{3(1+\mu)})\dot{} = 0 \quad , \tag{12}$$

or

$$\rho = \rho_0 a^{-3(1+\mu)} \quad , \tag{13}$$

where ρ_0 is a constant.

Putting $\alpha = 1$, $\dot{a} = 0$ and $\ddot{a} = 0$ in eq.'s (9) and (10), we can express ρ_0 in terms of μ and $\Lambda_{(0)}$, so that (13) becomes

$$\rho = \frac{1}{4\pi G}\chi\Lambda_{(0)}^{-\frac{1}{2}(1+3\mu)}c^2 a^{-3(1+\mu)}, \tag{14}$$

where $\chi := \frac{(1+3\mu)^{\frac{1}{2}(1+3\mu)}}{(1+\mu)^{\frac{3}{2}(1+\mu)}}$.

We summarize what we have done above. For a given μ, we get $\rho a^{3(1+\mu)} =$ const. By equating this constant to $\frac{1}{4\pi G}\chi\Lambda_{(0)}^{-\frac{1}{2}(1+3\mu)}$, we get a value $\Lambda_{(0)}$. This $\Lambda_{(0)}$ is the specific value of the cosmological constant which allows a static solution for given μ and $\rho a^{3(1+\mu)}$. We express the general cosmological constant in the unit of $\Lambda_{(0)}$: $\Lambda = \alpha\Lambda_{(0)}$.

Now, it is convenient to re-express eq.(9) as

$$\left(\frac{\dot{a}}{a}\right)^2 + U(a) = \frac{\alpha\Lambda_{(0)}c^2}{3}, \tag{15}$$

where

$$\begin{aligned} U(a) :&= \frac{c^2}{a^2} - \frac{8\pi G}{3}\rho \\ &= \frac{c^2}{a^2} - \frac{2}{3}\chi\Lambda_{(0)}^{-\frac{1}{2}(1+3\mu)}c^2 a^{-3(1+\mu)} \quad . \end{aligned} \tag{16}$$

The function $U(a)$ has the form of $A - B$ with $A \propto a^{-2}$ and $B \propto a^{-3(1+\mu)}$ $(-1 < \mu \leq 1)$, so that it has one and only one maximum U_{max}. It is obvious that $U_{max} = \frac{\Lambda_{(0)}}{3}c^2$. From eq.(10), it is also easy to show that $\ddot{a}$ $<, =, > 0$ $\Longleftrightarrow$ $\dot{a}$ $<, =, > 0$ (respectively). Thus, for any allowed Robertson-Walker universe, i.e. any one with a symmetric surface, α should be $\alpha < 1$ or,

$$\Lambda < \Lambda_{crit} := \Lambda_{(0)} \ . \tag{17}$$

Let us recast this Λ_{crit} in more convenient form. From eq.(14),

$$\begin{aligned} \Lambda_{crit} := \Lambda_{(0)} &= \left(\frac{c^2 \chi}{4\pi G \rho a^{3(1+\mu)}} \right)^{\frac{2}{1+3\mu}} \\ &= \left(\frac{2c^2 \chi}{3\Omega H^2 a^{3(1+\mu)}} \right)^{\frac{2}{1+3\mu}} \quad , \end{aligned} \tag{18}$$

where $H := \frac{\dot{a}}{a}$, $\Omega := \rho / \frac{3H^2}{8\pi G}$, as usual. From eq.'s (10) and (11),

$$\begin{aligned} \frac{\alpha \Lambda_{(0)} c^2}{3} &= \frac{\ddot{a}}{a} + \frac{4\pi G}{3}(1+3\mu)\rho \\ &= \left(\frac{\Omega}{2}(1+3\mu) - q \right) H^2 \quad , \end{aligned}$$

where, as usual, $q := -\frac{\ddot{a}}{a} H^{-2}$. Substituting this relation into eq.(9), we obtain

$$a^{-2} = \left(\frac{3\Omega}{2}(1+\mu) - q - 1 \right) (H/c)^2 \quad . \tag{19}$$

Substituting eq.(19) into eq.(18), we obtain finally,

$$\Lambda_{crit} = (1+3\mu) \left(\frac{2}{3\Omega} \right)^{\frac{2}{1+3\mu}} \left(\frac{3\Omega}{2} - \frac{q+1}{1+\mu} \right)^{\frac{3(1+\mu)}{1+3\mu}} (H/c)^2 \quad . \tag{20}$$

Since $\Lambda_{crit} = \Lambda_{(0)}$ remains constant during the evolution of the classical universe, we can calculate the right-hand side of eq.(18) or eq.(20) at any time in the classical epoch. Especially, we can substitute the present values $H = H_0$, $\Omega = \Omega_0$, $q = q_0$ into eq.(20). Regarding that $\Omega_0 = O(1)$ and $q_0 = O(1)$, we find out that

$$\Lambda_{crit} = O(1) (H_0/c)^2 \sim (10^{28}\mathrm{cm})^{-2} h^2 \sim (10^{61} l_{pl})^{-2} h^2 \quad .$$

This is the absolute upperbound for Λ which has been suggested by cosmological observations.

It is also helpful to recast Λ_{crit} in another form. When $\mu \neq 0$, the quantity of a mass-dimension $\rho V = 2\pi^2 \rho a^3$ does not remain constant during the evolution of the universe. It is more appropriate to define the adiabatic quantities. Let us define the adiabatic Schwarzshild radius with respect to the matter content of the universe:

$$L_s := \left(\frac{2G(2\pi^2 \rho a^3)}{c^2} a^{3\mu} \right)^{\frac{1}{1+3\mu}} \quad , \tag{21}$$

which remains constant during the evolution. Then, eq.(18) can be expressed as

$$\Lambda_{crit} = (\pi\chi)^{\frac{2}{1+3\mu}} L_s^{-2} = (1+3\mu)(1+\mu)^{-\frac{3(1+\mu)}{1+3\mu}} L_s^{-2} \quad . \tag{22}$$

Hence, the upperbound for the cosmological constant comes from the adiabatic Schwarzshild radius of the whole universe.

In this way, the question as to why the cosmological constant of our universe is so small has been reduced to the question why the adiabatic Schwarzshild radius of our universe is so large compared to the Planck length, which is equivalent to the question as to why the adiabatic mass of our universe is so large compared to the Planck mass.

The last question seems easier to answer compared to the original question. For instance, the inflationary scenario[12], which produces a large amount of matter after inflation, may be one possible path for it, although satisfactory version of this scenario has not yet been presented so far.[13]

We should remind ourselves that there are many origins of the cosmological constant. For example, as discussed in §1, the zero-point energy of quantum fields is expected to contribute enormously to Λ, typically in the order of M_{pl}^2. Whenever there is false-vacuum energy, it also contributes to Λ in the order of μ^4/M_{pl}^2, where μ denotes the scale of the spontaneous symmetry breaking. As discussed in §1, what is meant by the cosmological constant problem is a vast discrepancy between such an expected Λ, $O(M_{pl}^2)$, and the real Λ, $O(H_0/c)^2$.

Now, the symbol Λ in eq.$(6-a)$ means the summation of the original bare cosmological constant and all the other contributions of dynamical origins. Then, according to the present theory, only those universes are selected, in which the resultant total Λ obeys the condition (8) or (17) at sufficiently late stages in the classical epoch, irrespective of both the details of the origin of Λ and the detailed dynamics of fields before and after the Planck epoch. In this way, the suitable value for Λ_0 is chosen and fixed to adjust the value of Λ_{vac} at sufficiently late stages of the universe. Note that, in the early stages of the classical epoch (e.g. before and around $T = 100$Gev which is the scale of the electro-weak phase transition) in which the behavior of matter is quite different from the one in the later stages, Λ can take larger values than the present value because Λ_{vac} changes according to the dynamics of the matter fields, while Λ_0 is a fixed constant.

The above result, derived only from the combination of general relativity, quantum theory and causality, is natural enough to make one feel relieved to observe the smallness of the cosmological constant.

5. Discussion

There are two type of difficulties which we occasionally face in formulating various theories of physics (especially cosmology):

(a) those where the results/assumptions of the theory indicate a limitation or a deficiency and

(b) those where the results/assumptions are in contradiction with our sense of naturalness.

For example, difficulties of galaxy formation from nearly isotropic density perturbations may enter into class (a), while the so-called isotropy problem in cosmology, may enter into class (b), because the Robertson-Walker models themselves do not induce any fatal self-contradiction. For the purpose of removing this "unnaturalness" in (b), it is preferable to avoid *ad hoc* assumptions and arguments.

The cosmological constant problem, in this sense, may be appropriately classified into (b), because there is always the freedom to choose the value of the bare parameter Λ_0 to cancel Λ_{vac} sufficiently. In the resolution of this problem, we have tried to conform to a sense of "naturalness", though the precise definition of this term is difficult.

As mentioned briefly in §1, Hawking and Coleman gave impressive explanations for the cosmological constant problem. In the remaining part of this paper, we discuss a few specific points about their theories. They utilized skillfully the logic of the Euclidean quantum cosmology.[3],[4] However, it might be fair to say that their arguments do not completely satisfy our want of "natural" explanation of the problem. Both of their arguments heavily rely on the absolute value of the wave function of the universe, which is based on the Euclidean path-integral. If we analyze our feeling of uneasiness about these theories cautiously, we realize that it mainly comes from a lot of uncertainty in this Euclidean formulation, in which proof is replaced by belief.

Some essential points are left unquestioned in the shadow of fascinating features of the Euclidean formulation.[5] It is not clear whether saddle point configurations have dominant contributions to the divergent integral. It is uncertain whether such a divergence can be handled meaningfully without harming physical implications. It is also not clear whether the integral can be approximated by configurations of 4-spheres connected with each other by narrow wormholes.[14] It makes us feel uneasy to discuss about the infinite-height peak at $\Lambda = 0_+$ derived by a special approximation scheme in the integral expression which itself is subject to a serious divergence, or which requires a specific extra prescription for making sense. There appears also a possibility for the argument of wormholes to fail to answer the cosmological constant problem after in depth investigations of the integral.[15],[16]

We should also note another specific characteristic of these theories: their mechanisms work too well to allow non-vanishing Λ of order $(H_0/c)^2$. Recent developments of observational cosmology suggest that the non-vanishing Λ with a

cosmological scale is not unprovable.[2] Suppose that $\Lambda_{obs} \neq 0$ would be established by observations. Then, this would mean that there is at least one fatal defect in either the assumptions or arguments in those theories. This is especially true for the theory with wormholes because we need fine-tuning of the boundary condition for the wave function of the universe to allow $\Lambda \neq 0$.[3] If this might be the case, however, our gain of knowledge of a defect somewhere, either in the saddle point approximation, in the treatment of wormholes, in the Euclidean formulation or else, would compensate well for reluctantly abandoning those appealing theories. In this sense, the observational determination of whether the cosmological constant is exactly zero or not has an essential significance for the understanding of physics in the very early universe.

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