

Power spectrum nulls due to non-standard inflationary evolution

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(Dated: January 7, 2011)

The simplest models of inflation based on slow roll produce nearly scale invariant primordial power spectra (PPS). But there are also numerous models that predict radically broken scale invariant PPS. In particular, markedly cuspy dips in the PPS correspond to nulls where the perturbation amplitude, hence PPS, goes through a zero at a specific wavenumber. Near this wavenumber, the true quantum nature of the generation mechanism of the primordial fluctuations may be revealed. Naively these features may appear to arise from fine tuned initial conditions. However, we show that this behavior arises under fairly generic set of conditions involving super-Hubble scale evolution of perturbation modes during inflation. We illustrate this with the well-studied examples of punctuated inflation and the Starobinsky-break model.

PACS numbers:

The paradigm of cosmological inflation explains not only a set of peculiarities, such as the flatness problem, the horizon problem, etc., in the hot big bang model, but also the origin of the initial metric perturbations that led to formation of the large scale structure in the distribution of matter in the universe [1–3]. The simplest models of inflation achieve this by assuming that at high enough energy scales, the dynamics of the universe is as if it was dominated by a single scalar (inflaton) field. The inevitable quantum fluctuations seed the primordial metric perturbations.

The primordial power spectrum (PPS) is connected to observed angular power spectrum ($C_{\ell s}$) of the temperature fluctuations in the CMB sky through the radiative transport kernel. Alternatively, for a given cosmology (which determines the transport kernel), the primordial power spectrum can be deconvolved from the observed $C_{\ell s}$ [4]. These results seem to indicate that the PPS may have features, e.g., a sharp infrared cutoff on the horizon scale, a bump (i.e. a localized excess just above the cut off) and a ringing (i.e. a damped oscillatory feature after the infrared break). While the statistical significance of such a feature is still being assessed [5], this led to a lot of activity in building up models of inflation that can give large and peculiar features in primordial power spectrum (see [6] and references therein, along with [8–12, 14, 15, 17]). Many such models tend to assume very special initial conditions at the beginning of inflation. In contrast, others postulate quite fine tuned values of the parameters of the Lagrangian of the inflaton at tree level to produce features in the scalar PPS [6, 8, 11]). In many such scenarios, the scalar PPS has cuspy dips [6, 11, 12, 14] (also, see Fig. 1) that actually correspond to a null in the PPS i.e. precisely zero scalar power at some wave number. Also for a range of modes near such a feature, the tensor power overtakes

scalar power [7]. Scalar PPS with cuspy dips turn up in many models of inflation with different forms of potential (such as false vacuum inflation model with quartic potential [14], double-well potential [12], Coleman-Weinberg potential [12] etc) and also in other ways (such as in dissipative models of inflation, see [13]). It is seen that when one tries to produce an enhanced power on some scale (such as when considering models which try to enhance the production of primordial black holes), it is accompanied with a sharp drop in power leading to cusps in scalar PPS. Similarly, models that tend to produce low power in low multipoles of CMB anisotropies end up having sharp cusps. Thus, cusps in scalar PPS have been reported in the literature but their origin has not been satisfactorily understood. An exact null in scalar PPS can have interesting consequences such as on processed non-linear power spectrum. On the other hand, this null in the power spectrum is found by doing a classical computation, thus, for a small range of modes near the one having a zero, quantum effects can not be neglected [19] and hence the null may not be present when the quantum effects are taken into account exposing the truly quantum nature of the generation mechanism of primordial perturbations. This motivates us to study the origin and conditions required for cuspy dips in scalar PPS.

In this paper we take a fresh look at the evolution of mode functions of cosmological perturbations during inflation, a subject that is well studied (see [16] for a recent treatment). We point out that there are some key properties that the mode functions follow as they evolve in the complex plane. We also realize that it is possible to cause a particular kind of non standard evolution of modes. This kind of non standard evolution is connected closely to the existence of sharp cuspy dips in the scalar PPS.

We study the complex plane trajectory that the Fourier mode function of perturbation variables such as the curvature perturbation on hypersurfaces orthogonal to comoving worldlines, R_k and the Mukhanov-Sasaki variable $v_k = zR_k$ (where $z = a\dot{\phi}/H$). The evolution of v_k is given by

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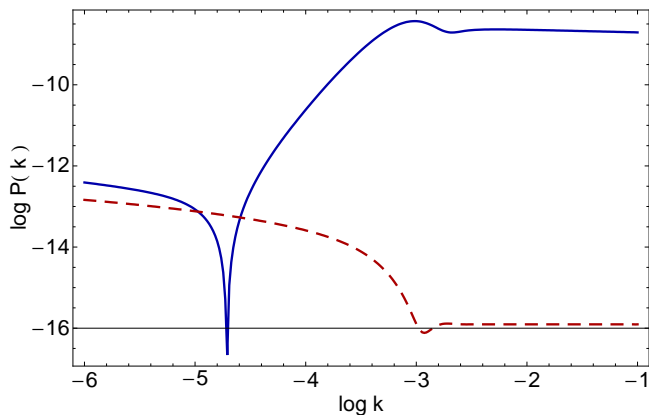


FIG. 1: The scalar (thick, blue) and tensor (dashed, red) PPS in Punctuated inflation model, an example of a sandwich model. Notice the sharp spike in scalar PPS and tensor power overtaking scalar power for for a range of modes near it.

$$v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0. \quad (1)$$

Above equation shows that the mode function of v goes along a circle of radius $1/\sqrt{2k}$ in clockwise sense (Bunch-Davies vacuum¹) in the complex plane when the mode is well inside the Hubble radius ($k \gg aH$) and goes radially outwards, (with $v_k \propto z$) when the mode is well outside the Hubble radius ($k \ll aH$), see e.g. [15]. Correspondingly, R_k just spirals-in (along the curve with polar equation $r \sim \theta^{\nu-1/2}$ for power law inflation) in extreme sub-Hubble regime, while it just freezes to some value when the mode is in super-Hubble regime. It is important to note that just prior to freezing, the tangent vector to the trajectory of R_k points radially inward since, z is negative, see the thin trajectory in Fig. 2. We will refer to this as standard evolution of the mode function of R_k . It is easy to confirm that in the simplest case of power law inflation, the trajectory of R_k in the complex plane would never cross the origin.

It is a well known fact that in a universe dominated by a single scalar field, R_k always freezes on super Hubble scales. This means that the amplitude as well as the phase of R_k freeze. Let us use the phrase “super Hubble evolution” to mean any evolution after R_k has completely frozen once. *In this super Hubble limit, it is impossible that the amplitude gets frozen while the phase does not, however, it is possible that the phase freezes but the amplitude does not.* Writing $v_k = re^{i\theta}$, Eq. (1) implies that

$$\theta'' + 2 \left(\frac{r'}{r}\right)\theta' = 0. \quad (2)$$

¹ It is important to note that except for the exact shape of the trajectory, the general arguments that we have given do not depend on the choice of vacuum.

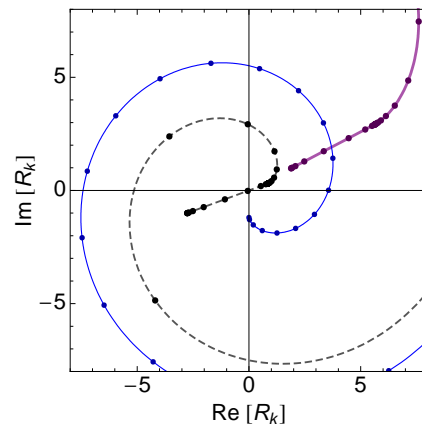


FIG. 2: The trajectories of mode function of R_k in complex plane. The dots on the trajectory are uniform in expansion e-folds, crowding of dots indicates that R_k freezes. The thin trajectory corresponds to power law inflation which never crosses the origin. Notice that when R_k freezes, the tangent vector to its trajectory points radially in. The other two trajectories show the behavior of R_k in the complex plane for two different modes in Punctuated inflation model. The mode denoted by continuous (thick) purple curve freezes once, then unfreezes, evolves radially in and freezes again before crossing the origin. Dashed gray curve corresponds to a mode with larger k^2 , it undergoes large radial super-Hubble evolution, crosses the origin and then freezes undergoing super-Hubble enhancement of power compared with the other mode that undergoes suppression. In between would lie a mode (not shown) that ends up right at the origin leading to a null in PPS.

Notice that, once the phase of v_k (and hence R_k) is frozen, it can not unfreeze. Also, the rate of change of θ' is directly proportional to θ' itself. Hence, the nearer we are to the epoch of phase freezing for a given mode, the less will the phase get affected by any background evolution. Thus, once the mode goes out of the Hubble radius, the phase freezes, in ordinary scenarios, the amplitude will also freeze. However, if it is arranged to unfreeze R_k , (by briefly decreasing z''/z) even then only the amplitude will unfreeze and not the phase. Thus, *if there is a possibility of any super-Hubble evolution of the mode, that should lead to only radial trajectory in the complex plane!* (see Fig. 2). Thus, provided that the phase of the mode function is already frozen, if such an evolution is sufficiently large, the mode must cross the origin in the complex plane. This is connected to a cuspy dip in the scalar PPS.

In the rest of the paper, we put forth the conditions that lead to the nulling in the PPS leading to cuspy features. Recall from Eq. (1) that it is the peculiarities in the dynamics of the quantity z''/z that can lead to super-Hubble evolution.

We seek an evolution of z''/z which is such that the mode function of R_k for at least some wavenumber, k , crosses the origin in the complex plane. Then simple continuity argument ensures that for at least one mode,

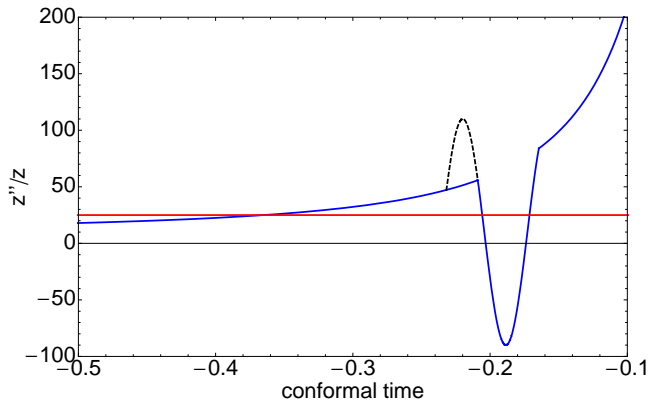


FIG. 3: Typical behaviour of z''/z against conformal time, η , in any sandwich model (blue curve). The horizontal red line is k^2 while the dashed curve is an alternative modified z''/z that may be needed to ensure that the mode of interest becomes super-Hubble (as explained in the text).

the mode function of R_k would freeze exactly at the origin.

For clarity let us recall that in the simplest case of power law inflation, z goes as $\eta^{1+\gamma}$ (where η is the conformal time and γ is a constant). In such a case, we shall have, $\frac{z''}{z} = \frac{a''}{a} = \frac{\gamma(\gamma+1)}{\eta^2}$.

Note that the first equality implies evolution of scalar modes and tensor modes are identical in these models, we will use this observation later. First, it is important to note the monotonic increasing form of $\frac{z''}{z}$. In this case, a mode which has once become super-Hubble ($k^2 < z''/z$) would continue to stay in that regime and the perturbation R_k will remain frozen. To unfreeze the mode, it is important for z''/z to decrease, and hence have a non-monotonic dip like feature (see Fig. 3).

Hence, we consider a class of approximate models that we refer to as ‘‘sandwich’’ models. These have three distinct stages of evolution during inflation. The quantity z''/z follows the monotonic power law inflation evolution in stage I. In stage II, there is a specific deviation from power law inflation leading to a desired dip feature in z''/z (Fig. 3). Stage III reverts to power law inflation. In what follows, we shall make a series of statements that will hold good for any such model. We will illustrate our arguments with the help of (i) Punctuated inflation (PI) model [6, 7] (see Fig. [1]), and (ii) Starobinsky-break model [17], which have been well-studied in literature.

The origin of cuspy dips in the power spectrum can now be understood in terms of the following salient features which are summarized below and elaborated afterwards:

- **Radial trajectory:** Super-Hubble evolution (any evolution after R_k for the mode freezes in stage I) always leads to only radial trajectory in the complex plane.
- **Inward motion:** The super-Hubble evolution in-

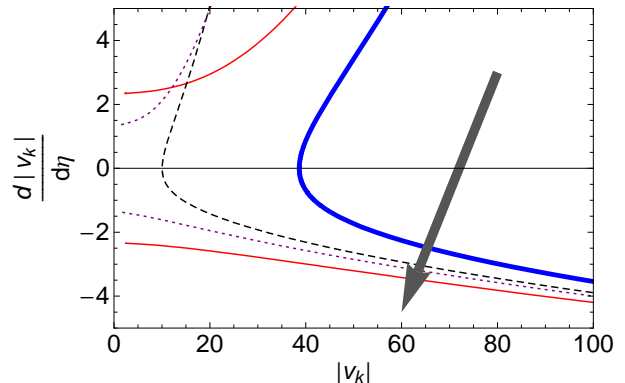


FIG. 4: The behavior of the mode in $(r \equiv |v_k|, r' \equiv |v_k'|)$ phase space for a given mode in Starobinsky model for different strengths of the delta function in z''/z . Stage I of the evolution shall make sure that θ' has already become zero. For low strengths of the delta function, blue(thick) and black(dashed), there is no origin crossing, while for high strengths, purple(dotted) and red(continuous thin), v_k passes through origin (leading to a discontinuity in r'). Notice that since this is not an autonomous system, the trajectories do cross each other. The thick arrow illustrates that the stage II is expected to take $(|v_k|, |v_k'|)$ to regions with sufficiently negative $|v_k'|$.

volves a radially inward motion (see Eq. (4)).

- **Amount of super-Hubble evolution:** The amount of super-Hubble evolution in stage II or III is determined by the depth of the dip in z''/z in stage II.
- **Continuity:** If a mode k_1 crosses the origin in the complex plane on a radial trajectory, there should exist a mode k_* (with $k_* < k_1$) that ends up right at the origin.

Thus, it is clear that one can easily construct sandwich models which will offer cuspy dips in scalar PPS under fairly general conditions.

For a mode whose phase is frozen in stage I, it is absolutely necessary that, in stage II, v_k should turn back and go radially in if it has to undergo origin crossing. This is a necessary but insufficient condition for origin crossing as is illustrated for Starobinsky-break model (in which stage II is just a Dirac delta function) in Fig. 4. For a given k , the dip in z''/z can be made sufficiently deep, to have origin crossing. Both the quantities $|v_k|$ and $|v_k'|$ are positive in stage I of a sandwich model, so we are in the upper half of Fig. 4. The desired model will have a stage II which brings us in the lower half of Fig. 4 which shows that if r' is sufficiently negative, origin crossing definitely take place. Thus, if after stage II, both $|v_k|$ and $\frac{d}{d\eta}|v_k|$ are sufficiently small, origin crossing must take place.

From Eq. (1), we can find the equation for the super-Hubble evolution of amplitude, r , of v_k . The condition for radial evolution means that θ' vanishes. The equation for r in this approximation is,

$$\frac{r''}{r} + \left[k^2 - \frac{z''}{z} \right] = 0. \quad (3)$$

One can estimate the super-Hubble evolution of r in stage II. We see from Eq. (3) above that the change in the derivative of the amplitude of v_k is:

$$\Delta r' = - \int \left(k^2 - \frac{z''}{z} \right) r \, d\eta. \quad (4)$$

The obvious conclusion that can be easily drawn from Eq. (4) is that for a fixed k , unless z''/z is such that $k^2 > z''/z$, the change in r' will be positive, so that at the end of stage II, instead of having v_k turn back in the complex plane, we shall have v_k going radially out at a quicker rate and origin crossing will most definitely not happen. This means that to cause origin crossing, for a given mode, we need a sufficiently deep dip such that $k^2 > z''/z$.

Let us fix z''/z and consider two modes (k_1 and k_2 with $k_1 > k_2$) with sufficiently small k values such that R_k corresponding to both have frozen in stage I (v_k radially outgoing). Stage I is power law inflation that gives tilted red spectrum so that $P_s(k_2) > P_s(k_1)$, which means that (since $k_1 > k_2$) $R_{k_2} > R_{k_1}$. Thus $v_{k_2} > v_{k_1}$. This means that the amplitude v_k in the complex plane at the end of stage one is smaller for a mode having larger k value. Also, since R_k 's have already frozen,

$$\frac{v'_2}{v'_1} = \frac{v_2}{v_1} > 1 \quad (5)$$

Thus, the speed with which $|v_k|$ increases in the complex plane at the end of stage I is also smaller for a mode having larger k value. From Eq. (4), we also know that $\Delta(r')|_1 > \Delta(r')|_2$. Thus, we conclude the following: for a fixed z''/z , even if origin crossing does not happen for a given k mode the chances that a larger k mode will undergo origin crossing is much larger (provided that in stage I, this larger k mode has become super-Hubble i.e. frozen once).

But increasing just the k value for a given model can potentially lead us to modes that have not become super-Hubble in stage I. In such a case, one could (i) delay the

location of dip (in z''/z) so that it occurs a little later and by that time v_k for that mode becomes radial, or, (ii) increase z''/z in stage I, (such as the dashed curve in Fig. 3). In either case, it is easy to see that stage I can be conveniently modified to cause origin crossing in stage III due to a given dip in stage II.

A deep dip in z''/z will cause z to quickly change, as a result, $|z|$ will actually decrease briefly. Since in an expanding universe, the scale factor can not decrease, the said evolution can never happen for tensor modes whose evolution is governed by a''/a (also see [14]). If the scalar power corresponding to a mode undergoes super Hubble suppression while the tensor power does not, it is not a surprise that the tensor to scalar ratio can become greater than one (see Fig. 1 and ref. [7]).

Thus, it is important to pay attention to the evolution of the mode functions (of various quantities related to perturbations of interest) in the complex plane. We learn that though the amplitude of R_k can be reawakened once it is frozen, frozen phase can not unfreeze i.e. super-Hubble evolution leads to radial trajectory in the complex plane. In Sandwich models, for modes that have taken up such a radial trajectory in the complex plane, the amount of super-Hubble evolution is determined by the depth of the dip in z''/z in stage II. If a given dip fails to bring the mode to the origin, we can easily modify stage I such that it does so. This ‘‘explains’’ the origin of (i) nulls in the scalar PPS (ii) tensor power overtaking scalar power, observed in the literature. There is no apriori reason that such zeros in PPS should survive even when higher order corrections are taken into account. Usually, quantum corrections to the PPS are too small compared to the classical result (see e.g. [20]), here, since the classical contribution is zero, the truly quantum nature of the primordial perturbations may get revealed.

Acknowledgment: The authors would like to thank L. Sriramkumar for comments and discussion at various stages of the work. We would also like to thank Alexei Starobinsky for illuminating comments and suggestions. GG thanks Council of Scientific and Industrial Research (CSIR), India, for the research grant award No. 10-2(5)/2006(ii)-EU II. TS acknowledges support from the Swarnajayanti Fellowship, DST, India.

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