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Universe before Planck time: A quantum gravity model

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A model for quantum gravity can be constructed by treating the conformal degree of freedom of spacetime as a quantum variable. An isotropic, homogeneous cosmological solution in this quantum gravity model is presented. The spacetime is nonsingular for all the three possible values of three-space curvature, and agrees with the classical solution for time scales larger than the Planck time scale. A possibility of quantum fluctuations creating the matter in the universe is suggested.

I. INTRODUCTION

A formalism for quantum gravity can be developed by treating the conformal part of spacetime as a quantum variable (see the preceding paper[1] for details; see Ref. 2 for previous work on the subject). Such a formalism allows the light-cone structure of spacetime to be determined by a metric tensor g_{ik}, which satisfies the equation [Eq. (2.8) of Ref. 1]

$$\langle \Omega^2 \rangle (R_{ik} - \tfrac{1}{2} g_{ik} R) + 6 t_{ik} = -8\pi G T_{ik} + (g_{ik} \Box - \nabla_i \nabla_k) \langle \Omega^2 \rangle , \quad (1.1)$$

$$t_{ik} = -\langle \Omega_i \Omega_k \rangle + \tfrac{1}{2} g_{ik} \langle \Omega^a \Omega_a \rangle . \quad (1.2)$$

The angular brackets denote the expectation value of the quantum variables taken in the particular quantum state of the conformal factor. The spacetime geometry in that quantum state is described by the line interval

$$ds^2 = \langle \Omega^2 \rangle g_{ik} dx^i dx^k . \quad (1.3)$$

It was shown in Ref. 1 that this formalism leads to a consistent picture for quantum gravity. The classical solution is shown to be of no significance near the singular epoch. The simple, static solutions considered in Ref. 1 showed that, near the singularity, the quantum gravitational effects modify the classical solutions drastically, leading to nonsingular spacetimes.

In this paper we shall examine the isotropic, homogeneous cosmological solution to the above quantum gravitational equations. We take the point of view (which, in fact, motivated the whole formalism) that any theory of quantum gravity that does not remove the classical singularity is not acceptable as a physical theory. The removal of singularities is the major theoretical consistency criterion that can be used to distinguish the various formalisms of quantum gravity. Accepting a formalism which does not solve the singularity problem will again lead to lack of predictive power in a physical theory. Thus, it is vital to analyze the simplest cosmological model of our theory from this point of view.

The metric g_{ik} which determines the light-cone structure will depend on the quantum state of the conformal factor. Unfortunately, we have no theoretical principle to determine this quantum state. It turns out that this indeterminacy manifests itself in the solution, in the form of two arbitrary parameters, which have to be fixed from observation. Subject to this limitation, our solution is complete.

II. MAXIMALLY SYMMETRIC SPACETIME

We shall assume the spacetime to be isotropic and homogeneous at the quantum level. This implies that the line element has the form

$$ds^2 = \langle \Omega^2(t) \rangle \left[c^2 dt^2 - S^2(t) \left[\frac{dr^2}{1 - r^2/a^2} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right] \right] . \quad (2.1)$$

We have assumed here that the imposition of maximal symmetry rules out spatial dependence on Ω. (This assumption is discussed in Ref. 1.) As for g_{ik}, we have taken the most general form, with an expansion factor $S(t)$ (dimensionless). The variables r and a have the dimensions of length. We have written the form of the metric for the "closed" geometry. When necessary we shall give the relevant formulas for the other cases. The solution differs from those considered in Ref. 1 by the existence of a dynamical expansion factor $S(t)$. We shall assume that the source consists of isotropic radiation with a conformally invariant stress tensor,

$$T^i{}_k=\epsilon(t)(1,-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3}) . \quad (2.2)$$

Thus the theory contains three variables, say, $\epsilon(t)$, $S(t)$, and the wave function for the conformal factor $\psi(\Omega,t)$.

The quantum dynamics of Ω must be determined through the action [see Eq. (4.4) of Ref. 1]

$$J=\frac{Vc^4}{16\pi G}\int_{t_1}^{t_2} dt S^3(t)[R\Omega^2-6\dot{\Omega}^2] . \quad (2.3)$$

Here V is the volume under consideration,

$$V=\int_0^L \frac{4\pi r^2 dr}{(1-r^2/a^2)^{1/2}}=2\pi^2 a^3\beta , \quad (2.4)$$

and is written in terms of a dimensionless parameter β. (For example, $\beta=1$ would give the volume of the *closed* universe. Our main results are independent of the choice of β.)

This action corresponds to the Hamiltonian

$$\hat{H}=\frac{\hbar^2}{2MS^3(t)}\frac{\partial^2}{\partial q^2}-\tfrac{1}{2}MS^3(t)\omega^2(t)q^2 , \quad (2.5)$$

where

$$q=a\Omega,\ \ M=\tfrac{2}{3}\pi\beta\left[\frac{ac^2}{G}\right] , \quad (2.6)$$

$$\omega^2(t)=\frac{\ddot{S}}{S}+\frac{1}{S^2}(\dot{S}^2\pm\nu^2),\ \ \nu=\frac{c}{a} . \quad (2.7)$$

(The plus sign corresponds to the closed model and the minus sign to the open model. Flat background can be achieved by dropping the ν term.) Thus our equation for the wave function reads

$$i\hbar\frac{\partial\psi}{\partial t}=\frac{\hbar^2}{2MS^3(t)}\frac{\partial^2\psi}{\partial q^2}-\tfrac{1}{2}MS^3(t)\omega^2(t)q^2\psi . \quad (2.8)$$

The energy density $\epsilon(t)$ and the expansion factor $S(t)$ are to be determined by Eqs. (1.1). For maximally symmetric spacetimes, there are only two independent equations in the set (1.1) which may conveniently be taken to be the trace equation and the $\binom{0}{0}$ component equation. Let us define (for reasons which will soon be clear)

$$A^i{}_k=(\delta^i{}_k\Box-\nabla^i\nabla_k)Q^2(t) , \quad (2.9)$$

$$Q^2(t)=\int_{-\infty}^{+\infty} dq\, \psi^*(q,t)\hat{q}^2\psi(q,t) =\langle q^2\rangle , \quad (2.10)$$

$$V^2(t)=\int_{-\infty}^{+\infty} dq\, \psi^*(q,t)\hat{\dot{q}}^2\psi(q,t) =\langle \dot{q}^2\rangle . \quad (2.11)$$

From the Lagrangian in Eq. (2.3), we know that the momentum conjugate to q is

$$p=\frac{\partial L}{\partial \dot{q}}=MS^3(t)\dot{q} . \quad (2.12)$$

Thus the operator for $\dot{q}^2$ [in Eq. (2.11)] must be taken to be $(\hat{p}^2/M^2S^6)$. In these notations, the trace of Eq. (1.1) reads

$$V^2(t)=\omega^2(t)Q^2(t)-A^i{}_i , \quad (2.13)$$

while the $\binom{0}{0}$ component equation reads

$$\frac{\dot{S}^2\pm\nu^2}{S^2}=\frac{8\pi Ga^2}{3c^2}\left[\frac{\epsilon}{Q^2}\right]-\frac{V^2}{Q^2}+A^0_0 . \quad (2.14)$$

Thus we are left with a set of coupled integrodifferential equations [see (2.10) and (2.11)] in the form of Eqs. (2.8), (2.13), and (2.14), solving which we should be able to determine $S(t)$, $\psi(q,t)$, $\epsilon(t)$. The functions $Q(t)$ and $V(t)$ are at once determined from $\psi(q,t)$.

To motivate the solution, let us consider the structure of the equations more carefully. Equations (2.13) and (2.8) form a set of coupled equations for $\psi(q,t)$ and $S(t)$ in the form

$$i\hbar\frac{\partial\psi}{\partial t}=\frac{\hbar^2}{2MS^3(t)}\frac{\partial^2\psi}{\partial q^2}-\tfrac{1}{2}M\omega^2(t)S^3(t)q^2\psi , \quad (2.15)$$

$$V^2(t)=\omega^2(t)Q^2(t)-A^i{}_i . \quad (2.16)$$

Once these two are solved, $\epsilon(t)$ can be trivially determined through Eq. (2.14), written in the form

$$\frac{8\pi Ga^2}{3c^2}\epsilon(t)=Q^2(t)\left[\frac{\dot{S}^2\pm\nu^2}{S^2}+\frac{V^2}{Q^2}-A^0_0\right] . \quad (2.17)$$

Thus our task reduces to finding an $S(t)$ and $\psi(q,t)$ that will satisfy Eqs. (2.15) and (2.16). It is not clear *a priori* whether the solution would be unique. One has to use the fact that, except for quantum corrections, $S(t)$ should follow the classical evolution. Our previous experience with the formalism suggests that one should look for the stationary-state solutions with $\langle q^2\rangle=Q^2$ independent of time. In other words, any time dependence $\psi(q,t)$ must appear as a pure phase factor. This suggests the following ansatz:

$$S^6(t)\omega^2(t)=S^6(t)\left[\frac{\ddot{S}}{S}+\frac{1}{S^2}(\dot{S}^2\pm\nu^2)\right] =\alpha^2=\text{const} \quad (2.18)$$

and

$$\psi(q,t)=\exp\left[\frac{iE}{\hbar}\int\frac{dt}{S^3(t)}\right]\phi(q)\ , \qquad (2.19)$$

which leads to the following equation for $\phi(q)$:

$$-\frac{\hbar^2}{2M}\frac{d^2\phi}{dq^2}+\tfrac{1}{2}M\alpha^2q^2\phi=E\phi\ . \qquad (2.20)$$

Since this is just the harmonic-oscillator equation for $\phi(q)$, the solution is well known. Assuming the universe to be in the nth stationary state, we can compute Q^2 and V^2 to be [see Eqs. (2.10)–(2.12)]

$$Q^2=\left[\frac{\hbar}{M\alpha}\right](n+\tfrac{1}{2})\ , \qquad (2.21)$$

$$V^2=\frac{1}{M^2S^6}\langle\hat{p}^2\rangle=\frac{\hbar\alpha}{M}\frac{1}{S^6(t)}(n+\tfrac{1}{2})\ . \qquad (2.22)$$

Thus Q^2 is independent of time, making $A^i{}_k$ identically zero. Since the Hamiltonian has an explicit time dependence through $S^3(t)$, V^2 picks up a time dependence. But this is exactly the time dependence needed to satisfy the second equation (2.16). We have, from (2.21) and (2.22),

$$\frac{V^2}{Q^2}=\frac{\alpha^2}{S^6}=\omega^2(t)\ . \qquad (2.23)$$

The last equality follows from the definition of the constant α in Eq. (2.18). Thus our solution satisfies both Eqs. (2.15) and (2.16), provided the expansion factor satisfies the equation

$$S^6(t)\omega^2(t)=\alpha^2\ . \qquad (2.24)$$

The spacetime structure can be determined by solving this equation. Notice that α is a purely quantum-mechanical parameter. Its classical value is zero, since classical evolution for $S(t)$ comes from the equation $\omega=0$. Thus, in fixing the state of the universe to be a harmonic oscillator of frequency α, we have introduced an extra constant into the theory whose value can only be determined from observation. We shall now consider the solutions of Eq. (2.24) which determine the spacetime structure.

III. GEOMETRY OF THE QUANTUM UNIVERSE

If the spacetime is taken to be in a given quantum state $\psi(q,t)$, the line element has the form

$$ds^2=\frac{Q^2(t)}{a^2}\left[c^2dt^2-S^2(t)\times\left[\frac{dr^2}{1-r^2/a^2}+r^2(d\theta^2+\sin^2\theta\,d\phi^2)\right]\right]\ . \qquad (3.1)$$

In the stationary states we have assumed for the universe Q is independent of time and all the dynamics is contained in $S(t)$. The equation for $S(t)$,

$$S^6\left[\frac{\ddot{S}}{S}+\frac{\dot{S}^2\pm\nu^2}{S^2}\right]=\alpha^2\ , \qquad (3.2)$$

can be integrated once, to give

$$\dot{S}^2=-\frac{\alpha^2}{S^4}\mp\nu^2+\frac{\rho^2}{S^2}\ . \qquad (3.3)$$

Here ρ^2 is the integration constant that arises in the process. We stated earlier that once ψ and S are determined, the energy content $\epsilon(t)$ can be fixed using Eq. (2.17). Using Eq. (3.3) (and the fact that A^0_0 is zero), we get

$$\epsilon(t)=\frac{3c^2}{8\pi Ga^2}Q^2\left[\omega^2(t)+\frac{\dot{S}^2\pm\nu^2}{S^2}\right] \qquad (3.4)$$

$$=\frac{3c^2Q^2}{8\pi Ga^2}\left[\frac{\alpha^2}{S^6}-\frac{\alpha^2}{S^6}+\frac{\rho^2}{S^4}\right]$$

$$=\frac{3c^2Q^2}{8\pi Ga^2}\frac{\rho^2}{S^4}\ . \qquad (3.5)$$

In other words,

$$\epsilon(t)=\frac{9}{16\pi^2\beta}\frac{\hbar}{\alpha a^3}\frac{\rho^2}{S^4(t)}(n+\tfrac{1}{2})\ . \qquad (3.6)$$

Thus the integration constant ρ determines the energy content of the universe.

We have to solve Eq. (3.3) to determine the evolution of the universe, especially about the singularity. In the given form, we can give solutions to (3.3) only as parametric functions. It is therefore convenient to write the metric as (with $d\tau=dt/S$)

$$ds^2=Q^2S^2(\tau)\left[c^2d\tau^2-\frac{dr^2}{1-r^2/a^2}-r^2(d\theta^2+\sin^2\theta\,d\varphi^2)\right]\ . \qquad (3.7)$$

The equation in terms of the τ coordinate is

$$\left[\frac{dS}{d\tau}\right]^2=\rho^2-\frac{\alpha^2}{S^2}\mp\nu^2S^2 \qquad (3.8)$$

(the top sign corresponds to the closed model). Let us begin by considering a purely quantum gravitational solution with no matter (i.e., $\rho=0$). It is clear from Eq. (3.8) that only a positive sign is allowed for the third term. Integrating the equation gives

$$S^2(\tau)=\left[\frac{\alpha}{\nu}\right](1+2\sinh^2\nu\tau)\ . \qquad (3.9)$$

The metric reads ($L_p{}^2=G\hbar/c^3$)

$$ds^2=AL_p{}^2(1+2\sinh^2\eta)[d\eta^2-dx^2-\sinh^2x\,(d\theta^2+\sin^2\theta\,d\varphi^2)] \ . \tag{3.10}$$

Here we are using the geometric coordinates with $r=a\,\sinh x$. We have also set

$$A=\left[\frac{3}{2\pi\beta}\right](n+\tfrac{1}{2}) \ . \tag{3.11}$$

The spacetime is nonsingular, and (of course) not flat because $R\alpha\omega^2=\alpha^2/S^6$ is nonzero. [The solution to Eq. (3.8) with ρ and α set to zero, $S=\exp(\nu\tau)$ is a flat metric.] What is more, at large epochs this universe mimics the behavior

$$ds^2\cong 2AL_p{}^2\sinh^2\eta[d\eta^2-dx^2-\sinh^2x\,(d\theta^2+\sin^2\theta\,d\phi^2)] \ . \tag{3.12}$$

This would be taken to be a radiation-filled universe with the energy density (apparent)

$$\epsilon_{\text{app}}(t)=\frac{9}{8\pi^2\beta}(n+\tfrac{1}{2})\frac{\hbar c}{a^4S^4(t)} \ . \tag{3.13}$$

This leads to the concept of "matter without matter," wherein the quantum fluctuations lead to the same kind of evolution as produced by matter in expression (3.13).

Now consider the cases with $\rho^2\neq 0$. We have three models to consider depending whether the background metric is open, closed, or flat.

(a) Open Universe. The expansion factor is given by

$$S^2(\tau)=\left[\frac{\rho^4}{4\nu^4}+\frac{\alpha^2}{\nu^2}\right]^{1/2}\cosh(2\nu\tau)-\frac{\rho^2}{2\nu^2} \tag{3.14}$$

and the metric reads

$$ds^2=L_p{}^2A[(1+x^2)^{1/2}\cosh 2\eta-x]\times[d\eta^2-dx^2-\sinh^2x\,(d\theta^2+\sin^2\theta\,d\phi^2)] \ , \tag{3.15}$$

where

$$x=(\rho^2/2\alpha\nu) \ . \tag{3.16}$$

(b) "Flat" background. The corresponding expressions are

$$S^2(\tau)=\frac{\alpha^2}{\rho^2}+\rho^2\tau^2 \ , \tag{3.17}$$

$$ds^2=L_p{}^2A\left[\left[\frac{\alpha}{\rho}\right]^2+\rho^2\tau^2\right]\times[c^2d\tau^2-dr^2-r^2(d\theta^2+\sin^2\theta\,d\varphi^2)] \ . \tag{3.18}$$

(c) Closed model. This model exists only for $\rho^2>2\alpha\nu$. The expressions are

$$S^2(\tau)=\frac{\rho^2}{2\nu^2}-\left[\frac{\rho^4}{4\nu^4}-\frac{\alpha^2}{\nu^2}\right]^{1/2}\cos(2\nu\tau) \ , \tag{3.19}$$

$$ds^2=L_p{}^2A[x-(x^2-1)^{1/2}\cos 2\eta]\times[d\eta^2-dx^2-\sin^2x\,(d\theta^2+\sin^2\theta\,d\phi^2)] \ , \tag{3.20}$$

$$x=(\rho^2/2\alpha\nu) \ . \tag{3.21}$$

It is clear that all the spacetimes are nonsingular and "begin" with a minimum value for the expansion factor. The classical limits are achieved by setting $\alpha=0$, when these expressions go over to the corresponding classical solutions. The first two solutions do not contain any other new feature. However, notice that the third solution can exist only when there is "sufficient energy density," i.e.,

$$\rho^2>2\alpha\nu \ . \tag{3.22}$$

(Classically $\alpha=0$ and hence a closed model can exist with any energy density.) Physically the conformal factor contributes a negative-energy density [as is evident from the matterless open model (3.10)]. This feature is purely quantum gravitational in origin. The closed model oscillates between the bounds

$$x-(x^2-1)^{1/2}\le S(\tau)\le x+(x^2-1)^{1/2}, \quad x>1 \ . \tag{3.23}$$

By the conventional procedure one can associate the temperature $T^1\propto E^{1/4}$ for the radiation. Equation (3.23) implies that the temperature will have a maximum and a minimum value. From the experimental value of helium abundance one can set a lower bound on $T_{\max}$ which will lead to an upper bound on α. This is similar to the "bouncing cosmologies" model suggested in a different context.[3] There is still a large degree of freedom left in the choice of α.

IV. CONCLUSION AND OUTLOOK

The present formalism of quantum gravity arose from an attempt to understand the question of singularites. Since the initial investigations revealed the importance of quantum conformal fluctuations, the formalism was developed with the conformal factor as the quantum variable. Two criteria must be satisfied by such a model. (i) The solutions for cosmological contexts must be nonsingular. (ii) The classical limit must be preserved. We have now demonstrated that the maximally symmetric solutions of our theory satisfy these criteria. Thus the formalism is theoretically consistent.

Can one do better than that? Are there any new features in the formalism? We believe that two such possibilities merit attention. These possibilities allow for the creation of the matter from the quantum gravitational processes.

Notice that what we have presented is only a special solution to the equations that assumes $S^6\omega^2$ to be a constant α^2. This allows, through Eq. (2.17), a $1/S^4$ dependence for $\epsilon(t)$. It is possible that there are other solutions in which ϵ will start at zero, rise to a large value, and start falling as $1/S^4$. Such models will involve a different quantum state for the universe, but will demonstrate the creation of the matter. While an analytic solution is difficult to obtain, we give below a qualitative argument to show what is involved.

Since such a model cannot accommodate a constant α (and since $\alpha \sim 0$ corresponds to the classical limit), let us assume

$$\begin{aligned} \alpha &= \alpha_1 \quad \text{for } \eta < \bar{\eta} \\ &= \alpha_2 \quad \text{for } \eta > \bar{\eta} \ . \end{aligned} \tag{4.1}$$

We assume that the universe made a transition from a quantum to a classical limit around $\eta = \bar{\eta}$, which is taken to be close to zero (very early epoch). Correspondingly, we expect α^2 to be almost zero with $\alpha_1 \gg \alpha_2$. In a realistic model, of course, $\alpha(t)$ will drop rapidly (but continuously) at $\eta \sim \bar{\eta}$. By assuming the universe to be an empty, quantum gravitational spacetime for $\eta < \bar{\eta}$ and an open model with a matter density ρ^2 for $\eta > \bar{\eta}$, one can accommodate the matter creation in the theory. Continuity of the metric across $\eta = \bar{\eta}$ leads to the condition

$$1 + 2\bar{\eta}^2 \approx 2\bar{\eta}^2 \left[\frac{\rho^2}{2\alpha_2 \nu} \right], \text{ i.e., } \bar{\eta}^2 \approx \left[\frac{\alpha_2 \nu}{\rho^2} \right] . \tag{4.2}$$

The energy density for $\eta > \bar{\eta}$ goes with time as

$$\epsilon(t) \cong \left[\frac{h\nu}{a^2} \right] \left[\frac{\rho^4}{\alpha_2{}^4} \right] \frac{1}{S^4(t)} . \tag{4.3}$$

Only an analytic solution (which would imply the continuity of all derivatives of metric at $\eta = \bar{\eta}$) can give more relations between ρ, $\bar{\eta}$, α_2, and α_1. Such a model is feasible because (i) nontrivial matterless solutions exist, (ii) a conformal factor provides a negative-energy density.

A less esoteric method of producing matter is the following. Let the universe start without any matter content and expand because of the negative-energy density of the conformal factor. This expansion will lead to pair creation in the standard fashion.[4] However, a universe at low-n states can have very high curvatures producing a large quantity of matter. This creation will lead the universe into one of the matter-filled solutions (for a similar idea, in an entirely different formalism, see Ref. 5).

Both the possibilities can lead to definite predictions based on quantum gravity and are under investigation.

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