QUANTUM GRAVITY AND THE "FLATNESS PROBLEM" OF THE STANDARD BIG BANG UNIVERSE

T. PADMANABHAN
TATA Institute of Fundamental Research, Bombay 400 005, India

Received 13 April 1983

It is shown that, if the universe originated through quantum conformal fluctuations from the empty Minkowski space, then it is most likely to be spatially flat.

Our universe is very well described by a homogeneous, isotropic space–time. Direct observation shows that the energy density of the universe is very close to the critical density (obtained from Hubble's constant) implying that the spatial hypersurfaces of homogeneity have almost zero curvature. This requires a fine tuning of the initial conditions of the universe (the "flatness problem") [1]. One possible solution to this problem was proposed by Guth [2] recently. We present here an alternative discussion based on a model for quantum cosmology. It is shown here that, if the universe originated due to a quantum fluctuation, then, the spatially flat Friedmann model is predominantly preferred over the models with nonzero curvature for the spatial hypersurfaces. Thus this model for quantum gravity can resolve the "flatness problem" in a natural fashion.

Classical general relativity can be obtained from the variational principle based on the action,

$$J_E = \frac{1}{16\pi G}\int R\sqrt{-g}\,dx\,. \tag{1}$$

The equation $\delta J_E = 0$ leads to a definite evolution from one three-geometry ${}^3\mathcal{G}_1$ at $t = t_1$ to another ${}^3\mathcal{G}_2$ at $t = t_2$. quantum gravity can be obtained from this classical theory via the path-integral approach. Quantum theory replaces the unique classical trajectory by the probability amplitude,

$$K[{}^3\mathcal{G}_2 t_2; {}^3\mathcal{G}_1 t_1] = \int \mathcal{D}\mathcal{G}\exp[(i/\hbar)J_E(\mathcal{G})]. \tag{2}$$

This kernel gives the probability amplitude for the space–time to evolve from ${}^3\mathcal{G}_1$ at $t = t_1$ to ${}^3\mathcal{G}_2$at $t = t_2$. If the initial state is described by a wavefunction $\psi({}^3\mathcal{G}_1)$ then the final wavefunction can be written – symbolically – as,

$$\psi({}^3\mathcal{G}_2, t_2) = \int \mathcal{D}\,{}^3\mathcal{G}_1 K({}^3\mathcal{G}_2 t_2; {}^3\mathcal{G}_1 t_1)\psi({}^3\mathcal{G}_1)\,. \tag{3}$$

In practice, it is impossible to perform the path integral in eq. (2) taking all the degrees of freedom into account. In recent years, a model for quantum gravity was developed, treating the conformal part of the metric tensor to be a quantum variable (see e.g. refs. [3, 4]). In this approach, we confine ourselves to the metrics of the form

$$g_{ik} = \Omega^2(x)\bar{g}_{ik}\,, \tag{4}$$

with some fixed $\bar{g}_{ik}$. The kernel in eq. (2) becomes

$$K[\Omega_2(x)t_2; \Omega_1(x)t_1] = \int \mathcal{D}\Omega(x)\exp[(i/\hbar)J_E(\Omega)]\,, \tag{5}$$

which gives the probability amplitude for the space–time to evolve from the conformal factor $\Omega_1(x)$ at t_1 to $\Omega_2(x)$ at t_2. In terms of the "wave functions", one can write

$$\psi[\Omega_2(x), t_2] = \int \mathcal{D}\,\Omega_1(x)K(\Omega_2(x), t_2; \Omega_1(x)t_1)\psi(\Omega_1(x), t_1) \tag{6}$$

0 031-9163/83/0000–0000/$ 03.00

(for the details of the formalism, see ref. [3]). We shall now apply this formalism to the Friedmann universe.

A detailed analysis of this quantum gravity model shows that a flat vacuum (i.e. matterless space–time with $g_{ik} = \eta_{ik}$) is unstable to quantum fluctuations and will evolve into a homogenous isotropic space–time. In the semiclassical limit, conformal fluctuations act as a negative-energy scalar field and provide a mechanism for matter creation [5]. We arrive at a nonsingular, horizon-free universe via quantum fluctuations (for similar ideas that treat the origin of the universe as a quantum fluctuation see refs. [6–8]). By computing the probability for arriving at the universes with different spatial curvature, one can find out whether the flat model is preferred.

The flat space is generally assumed to be stable for finite perturbations that grow like a bubble. (This is because of the positive-mass theorem and related results; see for example, ref. [9].) The case we consider is different and is concerned with homogeneous perturbations in a space–time that is not asymptotically flat.

We will indicate briefly as to how this probability can be computed. Consider a quantum fluctuation that produces a maximally symmetric space–time with the metric ($k = 0, \pm 1$)

$$ds^2 = c^2\,dt^2 - S^2(t)\left[(1 - kr^2/a^2)^{-1}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right] \tag{7}$$

from the flat vacuum,

$$ds^2 = c^2\,dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\,. \tag{8}$$

We take advantage of the fact that the maximally symmetric universe in eq. (7) is conformally flat and hence can be written in the form

$$ds^2 = \Omega^2(x)\,ds^2_{\text{flat}}\,, \tag{9}$$

while the flat space has $\Omega = 1$. We can now use our formalism with $\bar{g}_{ik} = \eta_{ik}$ in eq. (4). The probability amplitude for transition from $\Omega(x) = \Omega_1(x)$ to $\Omega(x) = \Omega_2(x)$ is given by

$$K[\Omega_2(x)\,,t_2\,;1\,,t_1] = \int \mathcal{D}\,\Omega(x)\exp\left[(i/\hbar)(-3c^2/8\pi G)\int \Omega_i\Omega^i\,d^4x\right]\,. \tag{10}$$

Since the path integral is quadratic, it can be evaluated exactly. We take the initial state of the universe to be described by a wave function that is peaked at the flat-space value. The action in eq. (10) corresponds to a set of independent harmonic oscillators in Fourier space. The minimum uncertainty state – the best allowed by quantum theory – corresponds to a coherent state for each oscillator [5]. Using eq. (6) one can arrive at the wave function for the universe. A straightforward computation shows that the *probability* for the space–time to evolve into a geometry with the line element,

$$ds^2 = \phi^2(x, t_2)\,ds^2_{\text{flat}} \tag{11}$$

(starting from flat space at $t = t_1$) is given by

$$\mathcal{P}[\phi(x)] = N\exp\left(-\frac{3c^3}{8\pi G\hbar}\int\frac{d^3x\,d^3y}{2\pi^2}\,\frac{\nabla\phi(x)\cdot\nabla\phi(y)}{|x-y|^2}\right). \tag{12}$$

It can be easily shown that the integral is positive definite and thus N is a well-defined normalization constant. (Note the similarity with the vacuum functional for the electromagnetic field [10].) Quite clearly, the probability is maximum for geometries with $\nabla\phi = 0$ – i.e. for spatially flat universes of the form

$$ds^2 = \phi^2(t)\left[c^2dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right] = c^2\,dT^2 - \phi^2(T)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right]\,. \tag{13}$$

Thus if we accept the picture that the universe originated via a quantum fluctuation, it is most likely to be flat. To compute the probabilities, one has to assume specific forms for $\phi(x)$ and evaluate $\mathcal{P}[\phi]$. A rough estimate for maximally symmetric universes can be made in the following way. The three-dimensional hypersurfaces of a maximally symmetric universe are characterised by a single parameter a [see eq. (7)], which scales the "size" of the universe. For such a universe (with $k \neq 0$), the probability has the following form:

$$\mathcal{P}[a] = N\exp\left[-(3/8\pi)(a^2/L_p^2)I\right] \quad (k \neq 0)\,, \tag{14}$$

where $L_p^2 = (G\hbar/c^3) \sim 10^{-66}\ \text{cm}^2$, and I is a dimensionless definite integral of the order of unity. This result, which is obtainable from dimensional considerations, shows that a universe with scale sizes $\sim 10^{28}$

cm has negligible probability, unless it is spatially flat.

The above result may be understood physically by the following observation: Quantum fluctuations have very little probability to change the curvature of the three-dimensional hypersurfaces. One can see from the following argument that "direct transition" between hypersurfaces with different three-curvatures is forbidden. Consider the kernel in eq. (2), with the three-geometries characterized by the three-curvature (^3R). For calculating the transition amplitude (treating only ^3R as the dynamical variable)

$$K[^3R_2 t_2; {}^3R_1 t_1] = \int \mathcal{D}\, \mathcal{G} \exp\{(\mathrm{i}/\hbar) J_{\mathrm{E}} [\mathcal{G}(^3R)]\} \qquad (15)$$

one has to express J_{E} in terms of ^3R. But J_{E} involves ^3R only non-dynamically (i.e. no derivatives appear) and hence transitions between ^3Rs of different value cannot take place *directly*. This result, of course, should not be taken seriously, in general, because one is not justified in treating ^3R as single dynamical degree of freedom. (Only for a maximally symmetric universe is such an analysis possible.) The formalism presented above shows that transitions are in fact possible, but with negligible probability. A universe arising from the quantum fluctuation in flat space is *most likely* to be spatially flat.

In order to avoid any possible confusion, we repeat that the present paper attempts a purely quantum gravitational solution to the flatness problem. This is to be contrasted with the "inflationary scenario" attempts which take the origin of the universe for granted and attempt a classical solution.

The early universe is bedevilled by three difficulties – the initial singularity, the horizon problem and the flatness problem. Since these problems raise questions about the initial conditions and the quantum era, it is incumbent on a quantum gravity model to answer these questions. It has been shown [11] that quantum gravity cures the first two diseases as well. Notice that the solution presented here answers these questions at a fundamental level and does not use any feature of nongravitational physics. Other models proposed before [2] critically depend on the phase transition scenario in the early universe.

I thank Professor J.V. Narlikar for valuable guidance

References

[1] R.H. Dicke and P.J.E. Peebles, in: General relativity: an Einstein centenary survey, eds. S.W. Hawking and W. Israel (Cambridge Univ. Press, London, 1979).
[2] A.H. Guth, Phys. Rev. D23 (1981) 347.
[3] J.V. Narlikar, Found. Phys. 11 (1981) 473.
[4] T. Padmanabhan and J.V. Narlikar, Nature 295 (1982) 677.
[5] T. Padmanabhan, TIFR preprint (1982), to be published in Int. J. Theor. Phys.
[6] D. Lidley, Nature 291 (1981) 392.
[7] D. Atkatz and H. Pagels, Phys. Rev. D25 (1982) 2065.
[8] R. Brout et al., Nucl. Phys. B170 (1980) 228.
[9] E. Witten, Comm. Math. Phys. 80 (1981) 381.
[10] C.W. Misner, K.S. Thorne and J.A. Wheeler, Gravitation (Freeman, San Francisco, 1973) ch. 43.
[11] J.V. Narlikar, Phys. Lett. 96A (1983) 107.