Probing the origin of large inhomogeneities in inflation using a toy quantum-mechanical model

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We study the quantum evolution of the homogeneous mode of a scalar field with simple potentials in Robertson-Walker spacetime. The simplicity of the model allows one to work out the *exact* quantum evolution of a wave packet. The model mimics the inflationary scenario quite well and is used to study the nature and origin of various constraints. We conclude that any model in which (i) the potential has only a single energy scale ($10^{14}-10^{15}$ GeV) and dimensionless parameters of order unity and (ii) the inflationary phase is endowed with a minimum temperature of $(H/2\pi)$ *will* produce density inhomogeneity which is too large.

I. MOTIVATION AND SCOPE

A rapid, exponential expansion of the Universe at very early stages of evolution may help to resolve some problems of the classical universe.[1-3] These problems, as well as their possible solutions through inflation, have been highlighted in recent years.[4] Unfortunately, successful implementation of inflation is yet to be realized. The simplest and the most attractive model for inflation, due to Guth, leads to an extremely inhomogeneous universe.[5] To avoid this difficulty, it is necessary to fine-tune the parameters in the scalar field potential in an *ad hoc* manner.[6] Since the only need for inflation is based on a desire to avoid fine-tuning, it is not clear whether we are any better off with a severely fine-tuned inflation.

Recently, yet another (possible) difficulty has been brought to light.[7] Do the initial conditions in the early Universe allow the Universe to go into an inflationary epoch? While workers in the field have not agreed on a final answer,[8-10] the analysis clearly brings out the hidden difficulties in the inflate-the-universe program.

There seems to be one more peril lurking in the background: *Every unworkable scenario for inflation constrains the parameters in the Higgs sector of the theory.* (Somehow this problem has not received sufficient attention in the literature before.) Consider, for example, a scalar potential defined by a set of parameters $(a_1,a_2,\ldots,a_n)$. Suppose that inflation works out satisfactorily (i.e., initial conditions allow inflation to begin, parameters ensure sufficient inflation and a correct value for density perturbations) *only* for a small range of parameters $\{a_i\}$. In addition to the unaesthetic fine-tuning which may be involved, the result directly excludes a large region in the parameter space. In other words, the coexistence of two features, (i) the initiation of an inflationary phase under a wide choice of conditions and (ii) the consequent production of a large fluctuation via inflation, can spell disaster to conventional cosmology. Inflation, therefore, is a double-edged sword. Stated in another fashion, every inflationary model that leads to unacceptable consequences contains valuable constraints on physical parameters.

The above difficulty, added to the realization that workable inflationary models do not exist for a wide range of physically meaningful parameters, forces one to take a second look at the nature and origin of constraints on the parameters of the potential. We analyze in this paper the origin of various constraints on inflation. In order to do this, we work with a drastically simplified toy model for the scalar field. While such a scalar field is far from a realistic candidate for inflationary model building, it does help a physical understanding of the various competing effects which are involved. In particular, the model allows one to work out the exact quantum solution of a wave packet so that both the "mean value" and the "fluctuations" can be traced out. Since the model permits exact analysis, this may be better suited to study the effects of initial conditions, etc.

We emphasize that this paper is not an attempt at producing a completely workable model for inflation. Our scope is limited to probing some features of inflation using a simple model theory.

II. COHERENT STATES IN A ROBERTSON-WALKER UNIVERSE

Consider a classical scalar field $\phi(\mathbf{x},t)$ described by the action

$$A=\int\sqrt{-g}\,d^4x\left[\tfrac{1}{2}\nabla^i\phi\nabla_i\phi-V(\phi)\right] \tag{1}$$

in a Robertson-Walker $(k=+1)$ spacetime with the line element

$$ds^2=dt^2-S^2(t)\left[\frac{dr^2}{1-r^2}+r^2(d\theta^2+\sin^2\theta\,d\phi^2)\right]. \tag{2}$$

The classical dynamics of such a system is described by the Einstein equation

$$R^i{}_k-\tfrac{1}{2}\delta^i{}_kR=-8\pi GT^i{}_k\ , \tag{3}$$

which is equivalent to the following two equations:

$$\frac{\dot S^2+1}{S^2}=\frac{8\pi G}{3}\left[\tfrac{1}{2}\dot\phi^2+V(\phi)\right], \tag{4}$$

$$2\frac{\ddot S}{S}+\frac{\dot S^2+1}{S^2}=8\pi G(-\tfrac{1}{2}\dot\phi^2+V)\ . \tag{5}$$

The requirement that $T^i{}_k$ due to $\phi(\mathbf{x},t)$ must lead to a homogeneous and isotropic universe has already forced us to make $\phi(\mathbf{x},t)=\phi(t)$. Simple algebra will now show that (4) and (5) are equivalent to (4) and the field equation

$$\frac{1}{S^3}\frac{d}{dt}\left[S^3\frac{d\phi}{dt}\right]=-\frac{dV}{d\phi} . \tag{6}$$

The solutions of (4) and (6) will determine the evolution of $S(t)$ and $\phi(t)$.

Unfortunately, the real world is governed by the quantum laws of motion and not by classical equations. A complete quantum description of (4) and (6) would involve a quantum theory for gravity which is not available. Therefore, one has to be content with a hybrid description of our system, with $S(t)$ classical and $\phi(\mathbf{x},t)$ quantum mechanical. Once again the homogeneity and isotropy of the metric will force one to consider any spatial dependence of $\phi(\mathbf{x},t)$ as a small perturbation. The usual approach to inflation does exactly that. The scalar field $\phi(\mathbf{x},t)$ is separated as

$$\phi(\mathbf{x},t)=\bar{\phi}(t)+\phi_1(\mathbf{x},t) \tag{7}$$

in which $\bar{\phi}(t)$ is a solution to (6) with V replaced by the effective potential V_{eff}—the classical background field—and $\phi_1(\mathbf{x},t)$ is the fluctuation around the classical part. Expanding the action in (1) up to quadratic order in $\phi_1(\mathbf{x},t)$, one can easily develop a quantum theory for the fluctuations $\phi_1(\mathbf{x},t)$. To study perturbations due to inflation one has to essentially work out the root-mean-square value for ϕ_1 (Refs. 11–13).

It should be clear from the above description that one is *not* actually performing a quantum-mechanical study of $\phi(\mathbf{x},t)$ but *only a semiclassical* analysis. The "average evolution" is treated through classical equations while the quantum fluctuations about the classical part are retained up to quadratic order.

Is it possible to perform a completely quantum-mechanical analysis of the problem? The discussion above suggests that such an analysis should be possible whenever (i) only terms up to quadratic order exist and/or are retained in the potential and (ii) the homogeneous mode $\bar{\phi}(t)$ contributes the maximum to the energy density. There is, however, one feature that has to be ensured in the quantum-mechanical treatment: We want the quantum-mechanical evolution to follow the classical behavior as closely as possible. Mathematically, one would like to have

$$\langle\psi\,|\,\phi(\mathbf{x},t)\,|\,\psi\rangle=\bar{\phi}(t) , \tag{8}$$

where $|\,\psi\rangle$ is the quantum state of the field. There exists, of course, an infinity of states satisfying (8). How do we make a choice among them?

The simplest choice for a state that mimics classical behavior is a Gaussian coherent state. In this paper, we shall explore the consequences of making such a choice. This does not mean, in any way, that nature is expected to be kind enough to our individual preferences. The choice of a coherent state is made purely in an exploratory manner keeping the following features in mind: (a) coherent states are the best approximations for semiclassical behavior, (b) these states allow, as we shall see, an exact analysis of the problem, and (c) if the quantum fluctuations are not expected to be large, then any other choice for the state cannot lead to results which are very different.

Once a particular state is chosen, Eq. (4) is replaced by the hybrid version involving the expectation values in the right-hand side:

$$\frac{\dot{S}^2+1}{S^2}=\frac{8\pi G}{3}\left(\tfrac{1}{2}\langle\dot{\phi}^2\rangle+\langle V\rangle\right) . \tag{9}$$

So far we have been using the Heisenberg picture for the description of the quantum physics of ϕ. It happens to be more convenient to work with the Schrödinger picture especially if we are only interested in the homogeneous mode. In this case ϕ is the quantum-mechanical degree of freedom (like the generalized coordinate q for a particle) and the quantum state is described by the wave function $\Psi[\phi,t]$. For the homogeneous mode

$$A=2\pi^2\int dt\,S^3(t)\left[\tfrac{1}{2}\dot{\phi}^2-V(\phi)\right] . \tag{10}$$

so that the Hamiltonian is

$$H=\frac{p^2}{4\pi^2S^3(t)}+2\pi^2S^3(t)V(\phi) . \tag{11}$$

The wave function satisfies the "Schrödinger equation"

$$i\frac{\partial\psi}{\partial t}=-\frac{1}{4\pi^2S^3}\frac{\partial^2\psi}{\partial\phi^2}+2\pi^2S^3V(\phi)\psi , \tag{12}$$

and our task is to find $\Psi[\phi,t]$ and $S(t)$ which satisfy (9) and (12). Note that

$$\langle\dot{\phi}^2\rangle=\frac{1}{4\pi^4}\frac{1}{S^6(t)}\langle p^2\rangle=-\frac{1}{4\pi^4S^6(t)}\int_{-\infty}^{+\infty}\psi^*\frac{\partial^2\psi}{\partial\phi^2}d\phi . \tag{13}$$

To avoid any possible misunderstanding, we stress the fact that we are actually solving *a quantum-mechanical problem related to the homogeneous, zero-frequency mode of the scalar "field."* The expectation value of the Hamiltonian which governs this quantum-mechanical evolution acts as the source of gravity [see (9)]. Whenever we use the terminology "scalar field" in this paper, it is necessary to remember that we are actually considering *only* the homogeneous mode of the "field" which has only a single degree of freedom. It is a quantum-mechanical system rather than a field-theoretic system, even though we may call it a scalar field.

We shall now show that coherent-state solutions exist for these equations whenever the potential is dominated by a quadratic part, i.e., if

$$V(\phi)=a\phi^2+b\phi+c . \tag{14}$$

To do this, first introduce a new time coordinate T via the transformation

$$T=\int\frac{dt}{S^3(t)} , \tag{15}$$

so that Eq. (12) becomes

$$i\frac{\partial\psi}{\partial T}=-\frac{1}{4\pi^2}\frac{\partial^2\psi}{\partial\phi^2}+u(\phi,T)\psi\ , \tag{16}$$

where

$$u(\phi,T)=2\pi^2S^6(T)(a\phi^2+b\phi+c)$$
$$\equiv\alpha(T)\phi^2+\beta(T)\phi+\gamma(T)\ . \tag{17}$$

Consider an ansatz for $\Psi(\phi,T)$ in the form

$$\Psi=A(t)\exp\{-B(t)[\phi-f(t)]^2\}\ . \tag{18}$$

Substituting (18) in (16) and equating coefficients of ϕ^2, ϕ^1, ϕ^0 leads to the equations

$$i\frac{dB}{dT}=\frac{B^2}{\pi^2}-\alpha(T)\ , \tag{19}$$

$$i\frac{df}{dT}=\frac{\alpha}{B}f+\frac{\beta}{2B}\ , \tag{20}$$

$$\frac{i}{A}\frac{dA}{dT}=\frac{B}{2\pi^2}+\alpha f^2+\beta f+\gamma\ . \tag{21}$$

These equations can be set into a more transparent form. Note that

$$|\psi|^2=N(T)\exp\left[-\frac{1}{2|\sigma(T)|^2}[\phi-\bar\phi(T)]^2\right]\ , \tag{22}$$

where

$$\bar\phi(T)=\frac{Bf+B^*f^*}{B+B^*}\ , \tag{23}$$

$$|\sigma|^2(T)=\tfrac{1}{2}(B+B^*)^{-1}\ , \tag{24}$$

and

$$N(T)=|A(T)|^2\exp\left[-\left[(Bf^2+B^*f^{2*})-\frac{(Bf+B^*f^*)^2}{B+B^*}\right]\right]\ . \tag{25}$$

Detailed algebraic manipulations show that $\bar\phi(T)$ satisfies the classical equation of motion in the potential $u(\phi,T)$:

$$2\pi^2\frac{d^2\bar\phi}{dT^2}=-\frac{\partial u}{\partial\phi}\ , \tag{26}$$

which is the same as the "field equation"

$$\frac{1}{S^3}\frac{d}{dt}\left[S^3\frac{d\bar\phi}{dt}\right]=-\frac{\partial V}{\partial\phi}\ . \tag{27}$$

The quantum-mechanical spread $\sigma(T)$ satisfies the equation

$$\pi^2\frac{d^2\sigma}{dT^2}+\alpha(t)\sigma=0 \tag{28}$$

or, equivalently,

$$\frac{1}{S^3}\frac{d}{dt}\left[S^3\frac{d\sigma}{dt}\right]+2a\sigma=0\ . \tag{29}$$

Also note that

$$B=-i\pi^2\left[\frac{\dot\sigma}{\sigma}\right]\ . \tag{30}$$

The normalization constant $N(t)$ is given by

$$N(t)=[2\pi\sigma^2(t)]^{-1/2}\ . \tag{31}$$

To summarize, the Schrödinger equation for a quadratic potential with time-dependent coefficients (12) possesses solutions of the form (22). The mean value of the Gaussian follows the classical evolution [see (27)] while the quantum-mechanical spread is determined by (29). We shall now use these solutions to study the evolution of a scalar field.

III. "SINGLE PUSH" MODEL

To set the stage, we begin with the simplest possible form of potential, viz., a piecewise constant potential, shown in Fig. 1 by the solid black line. [We have also indicated (by the dashed line) in the same figure the approximate shape of the usual inflation potential.] A scalar field does not feel any driving force in such a potential for $\phi<\phi_f$. Thus in order to make it "roll along" the potential from $\phi=0$ to $\phi\sim\phi_f$, it is necessary to give it an initial "push" toward positive ϕ direction. (The physics of such a model is analyzed in detail by one of the authors in Ref. 14; we shall merely summarize the results in this paper.) It can be easily shown that the initial state

$$\psi[\phi,0]=\left[\frac{1}{2\pi\sigma_0{}^2}\right]^{1/4}\exp\left[-\frac{(\phi-\phi_i)^2}{4\sigma_0{}^2}+iv\phi\right] \tag{32}$$

will evolve into a state with the following probability distribution:

$$P(\phi,t)\equiv|\psi(\phi,t)|^2=\left[\frac{1}{2\pi\sigma^2(t)}\right]^{1/2}\exp\left[-\frac{1}{2\sigma^2(t)}\left[\phi-\phi_i-\frac{v}{\pi^2}\int\frac{dt}{S^3}\right]^2\right]\ , \tag{33}$$

where

$$\sigma^2(t)=\sigma_0{}^2\left[1+\frac{1}{4\pi^4\sigma_0{}^4}\left[\int\frac{dt}{S^3}\right]^2\right]\ . \tag{34}$$

The quantity v signifies the amount of initial velocity ("push") given to the scalar field. Since there is no "force" acting on ϕ, the canonical momentum $p=2\pi^2S^3\dot\phi$ is constant and has the expectation value

$$\langle p\rangle=v\ .$$

Having solved the Schrödinger equation in a background metric, we have to tackle Einstein's equation (9). In the range $0<\phi<\phi_f$ (see Fig. 1), $\langle V\rangle$ is just V_0, while $\langle\dot\phi^2\rangle$ becomes, using (13),

$$\langle\dot\phi^2\rangle=-\frac{1}{4\pi^4S^6(t)}\int_{-\infty}^{+\infty}\psi^*\frac{\partial^2\psi}{\partial\phi^2}d\phi=\frac{\text{const}}{S^6(t)}\ . \tag{35}$$

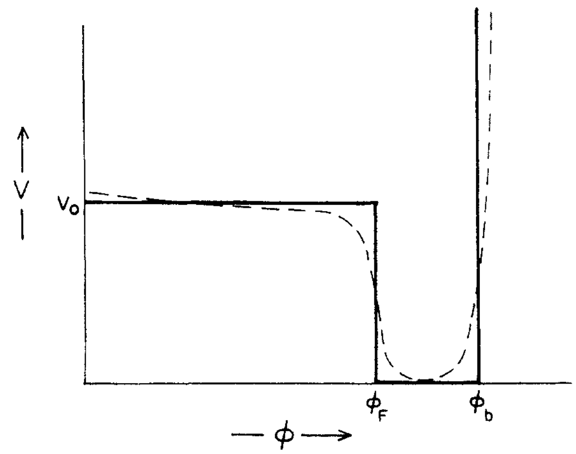

FIG. 1. Potential for the single push model. The solid line denotes the potential discussed in Sec. III. The dashed line denotes a realistic potential for which the solid line is an approximation.

Thus in Eq. (9), the constant term V_0 dominates over the "kinetic energy term" ($\langle\dot{\phi}^2\rangle \sim S^{-6}$) and the "curvature term" (kS^{-2}) allowing one to obtain inflationary solutions

$$S(t) \simeq \frac{1}{H}\exp(Ht) \tag{36}$$

with

$$H^2 = 8\pi G V_0/3 \,. \tag{37}$$

There are some peculiar features in this quantum-mechanical picture that deserve to be mentioned. Once the de Sitter expansion has started, the expectation value of the operator $\dot{\phi}$ starts to decrease rapidly:

$$\langle\dot{\phi}\rangle = \frac{1}{2\pi^3 S^3}\langle p\rangle \approx \frac{vH^3}{2\pi^2}e^{-3Ht} \,. \tag{38}$$

(Notice that the canonical momentum p is a constant of motion.) Similarly, the mean value of the field ϕ evolves as

$$\langle\phi\rangle = \phi_i + \frac{v}{\pi^2}\int \frac{dt}{S^3} \simeq \phi_i + \frac{vH^2}{3\pi^2}(1-e^{-3Ht}) \,. \tag{39}$$

In other words $\langle\phi\rangle$ approaches an asymptotic value of

$$\lim_{t\to\infty}\langle\phi\rangle \equiv \langle\phi\rangle_\infty = \phi_i + \frac{vH^2}{3\pi^2} \,. \tag{40}$$

Modeling our potential in Fig. 1 closely after the standard inflationary potential we may take

$$\phi_f \approx 1.2\times 10^{15}\ \text{GeV}, \quad V_0 \approx (10^{14}\ \text{GeV})^4 \tag{41}$$

giving

$$H = \left[\frac{8\pi G V_0}{3}\right]^{1/2} \approx \left[8\times\frac{(10^{14}\ \text{GeV})^4}{(10^{19}\ \text{GeV})^2}\right]^{1/2} \approx 2\times 10^9\ \text{GeV} \,. \tag{42}$$

The requirement that the initial state should be reasonably away from the "well" in the potential implies that

$$\phi_i \approx 0, \quad \sigma_0 \ll \phi_f \,. \tag{43}$$

On the other hand, we do expect the scalar field to have "plunged into" the well as $t\to\infty$. This implies, using (40) and taking $\langle\phi\rangle_\infty \gg \phi_i$,

$$\frac{vH^2}{3\pi^2} \gtrsim \phi_f \,, \tag{44}$$

or using (42) and (41),

$$vH \approx 1.6\times 10^7 \ \text{and}\ v \gtrsim 8\times 10^{-3}\ \text{GeV}^{-1} \,. \tag{45}$$

These are the constraints on the initial state through σ_0, v. Note that sufficient inflation can be easily achieved because $\phi_f \lesssim \langle\phi\rangle_\infty$. We do not get any new constraint by using this condition.

The dynamical constraints on the model arise from the constraints on the density perturbations produced by the model. While an exact computation would require incorporating the spatial degrees of freedom, an order-of-magnitude estimate can be made by the expression[6,11–13]

$$\frac{\delta\rho}{\rho} = \epsilon\left\langle\frac{H\Delta\phi}{\dot{\phi}}\right\rangle_{t=t_1}, \quad \dot{\phi} \equiv \frac{d\phi}{dt} \,. \tag{46}$$

Here ϵ is of the order unity, $\Delta\phi$ is the quantum spread in the scalar field and $\dot{\phi}$ is the rollover velocity, and t_1 is the epoch at which the relevant scale "freezes out" of the horizon. Physically one may interpret

$$\Delta\tau \equiv \frac{\Delta\phi}{\dot{\phi}} \tag{47}$$

as the time lag between the instants at which the "leading edge" and the "trailing edge" will fall over the dip. Using this, Eq. (46) can be estimated for the model in a straightforward manner (the details can be found in Ref. 14) giving

$$\frac{\delta\rho}{\rho} \approx \frac{2\sqrt{2}}{3}\epsilon\left[1+\frac{1}{4v^2\sigma_0^2}\right]^{-1}\left[1+\frac{1}{8v^2\sigma_0^2}\right]^{1/2} \times \exp(3Ht_1) \,. \tag{48}$$

The disturbing exponential factor (which will lead to a high value of $\delta\rho/\rho$) has a simple physical origin: In our model, there is no "steady force" on ϕ. But $\langle\dot{\phi}\rangle$ dies down exponentially in the de Sitter space, making $\langle\dot{\phi}\rangle^{-1}$ in (46) and (47) very large. To the extent of "accuracy" we are maintaining, we may estimate t_1 to be about (5–10) H^{-1}. (Turner[15] has pointed out that galactic-size perturbations cross the horizon at about 50 H^{-1} before the end of inflation. Taking the inflation to last for about (55–60) H^{-1} leads to the above estimate.) Since we want $\delta\rho/\rho \lesssim 10^{-4}$, we should take $\sigma_0 v \ll 1$. Then (48) leads to the constraint

$$\frac{\delta\rho}{\rho} \approx (10^4)(v\sigma_0) \lesssim 10^{-4} \,. \tag{49}$$

Altogether, we have now arrived at the constraints (43), (45), and (49):

$$\sigma_0 \ll 10^{15}\ \text{GeV}, \quad v\sigma_0 \lesssim 10^{-8} \,, \quad vH \gtrsim 1.6\times 10^7 \,. \tag{50}$$

If these are the only constraints on the system, then, surprisingly enough, they can be satisfied. One possible set of parameters would be

$$v \gtrsim 8\times 10^{-3}\ \text{GeV}^{-1},\quad \sigma_0 \lesssim 10^{-6}\ \text{GeV}\ . \tag{51}$$

The real trouble actually comes from an entirely different quarter. It is believed that de Sitter spacetime has an intrinsic temperature[15] of $H/2\pi$. *If* such a result is applicable to inflationary models (which are only asymptotically de Sitter and involve only half the de Sitter manifold), *then* we must have

$$\sigma_0 \gtrsim \frac{H}{2\pi} \gtrsim 3\times 10^{8}\ \text{GeV}\ . \tag{52}$$

Since from (45) $v > 8\times 10^{-3}\ \text{GeV}^{-1}$, (49) gives

$$\frac{\delta\rho}{\rho} \gtrsim (10^4)(2.4\times 10^6) \simeq 2.4\times 10^{10}\ , \tag{53}$$

which is absurdly high. [Of course, the situation here is much worse than the usual inflation because of the 10^4 factor arising from $\exp(3Ht_1)$. This is an artificiality of our model which can be taken care of with a more realistic potential; see Sec. IV.]

Before proceeding further, let us take stock of the situation. We have been able to work out an exact quantum-mechanical evolution for the scalar field due to the following reasons: (i) We have considered only the homogeneous, zero-frequency mode of the scalar "field"—that is, we have not solved the field theory problem but only the quantum-mechanical problem; (ii) we have approximated the actual potential by a much simpler version. The evolution mimics the classical path on the average and the spread about this path is computable. We could also see the origin of each constraint. In particular, $\delta\rho/\rho$ is high essentially because of (52). The major artificiality in the model, of course, is the "no-force" potential because of which we had to resort to an "initial push." For this reason, we shall not attempt to work out spatial dependence, other initial conditions, etc., in this model. Instead, we shall now generalize the model by adding in a gentle slope.

IV. "STEADY SLOPE" MODEL

We shall now modify the potential by adding a steady slope towards the positive ϕ side (Fig. 2):

$$\begin{aligned} V(\phi) &= V_0 - \epsilon\phi,\quad 0<\phi<\phi_f \\ &= 0,\quad \phi_f<\phi<\phi_A \\ &= \infty,\quad \phi>\phi_A\ . \end{aligned} \tag{54}$$

Since we do want V_0 to dominate the dynamics, we will also assume that

$$V_0 \gg \epsilon\phi_f\ . \tag{55}$$

The quantum field ϕ evolves in this potential, starting with the initial Gaussian state

$$\Psi[\phi,0] = \left[\frac{1}{2\pi\sigma_0^2}\right]^{1/4} \exp\left[-\frac{1}{4\sigma_0^2}(\phi-\phi_i)^2\right]\ . \tag{56}$$

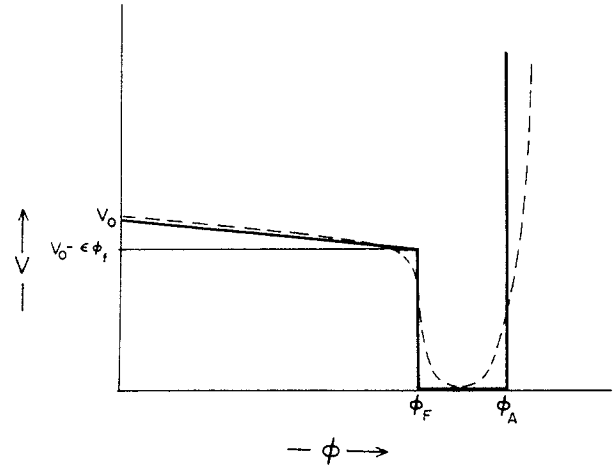

FIG. 2. Potential for the steady slope model. The solid line denotes the potential discussed in Sec. IV. This is an approximation to a more realistic potential which is denoted by a dashed line.

We have not introduced any "initial push" $v=0$ in contrast with (32). We shall also take ϕ_i to be an *infinitesimally* positive quantity ensuring the "rolling down" along the slope in Fig. 2.

Since (56) conforms with the general ansatz of (18) we can easily determine the evolution of (56). At any later time, we will have

$$|\Psi(\phi,t)|^2 = [2\pi\sigma^2(t)]^{-1/2} \times\exp\left[-\frac{1}{2\sigma^2(t)}[\phi-\bar\phi(t)]^2\right] \tag{57}$$

with $\sigma(t)$ and $\bar\phi(t)$ determined by (26) and (28). The spread $\sigma(t)$ is the same as in (34),

$$\sigma^2(t) = \sigma_0^2\left[1+\frac{1}{4\pi^4\sigma_0^4}\left[\int_0^t \frac{dt}{S^3}\right]^2\right]\ , \tag{58}$$

because Eq. (28) which governs the spread does not depend on the linear ($\alpha\phi$) term in $V(\phi)$. The mean value $\bar\phi(t)$ can be obtained by solving (27). We get

$$\bar\phi(t) = \phi_i + \epsilon\int_0^t \frac{dy}{S^3(y)}\int_0^y S^3(x)dx\ . \tag{59}$$

To determine $S(t)$ we need the expectation value of $\langle T^i{}_k\rangle$ which can be computed as before. We finally get

$$\begin{aligned} \langle T^0_0\rangle &= \tfrac{1}{2}\langle\dot\phi^2\rangle + \langle V\rangle \\ &= \frac{1}{16\pi^4\sigma_0^2}\frac{1}{S^6} + \left[\frac{\epsilon}{2\pi^2 S^3}\int_0^t S^3(x)dx\right]^2 \\ &\quad + [V_0 - \epsilon\bar\phi(t)]\ . \end{aligned} \tag{60}$$

It is easy to see that as long as $\epsilon\phi_f \ll V_0$ [see (55)], an exponential solution of the form

$$S \approx \frac{1}{H}\exp(Ht) \tag{61}$$

with

$$H^2 \approx \frac{8\pi G}{3} V_0 \tag{62}$$

exists to the Einstein equation. [The first term on the right-hand side of (60) dies down as S^{-6}; the last term $(-\epsilon\bar{\phi})$ is much smaller than V_0 because of (55); the second term is $\sim(\epsilon/H)^2$ which must be again smaller than V_0 if inflation lasts for about (at least) 20 H^{-1}.] We can write the explicit expressions for $Ht \gg 1$. Equations (58) and (59) give

$$\sigma^2(t) = \sigma_0{}^2 \left[1 + \frac{H^4}{36\pi^4\sigma_0{}^4}(1-e^{-3Ht})^2 \right], \tag{63}$$

$$\bar{\phi}(t) = \phi_i + \frac{\epsilon t}{3H} - \frac{\epsilon}{9H^2}(1-e^{-3Ht}) \approx \phi_i + \frac{\epsilon t}{3H} . \tag{64}$$

We see that the potential exerts a steady force on $\bar{\phi}$, keeping $\dot{\bar{\phi}}$ in (64) constant.

Let us now look at the constraints. We shall again take

$$\phi_f = 1.2\times 10^{15}\ \text{GeV}, \quad V_0 = (10^{14}\ \text{GeV})^4 , \quad H \simeq 2\times 10^9\ \text{GeV} , \tag{65}$$

but keep ϵ [i.e., $-(\partial V/\partial\phi)|_{\phi=0}$] free for the moment. The fact that the initial state should be well localized leads to the constraint

$$\sigma_0 \ll \phi_f \approx 10^{15}\ \text{GeV} . \tag{66}$$

Furthermore, we need sufficient inflation ($Ht_f \gtrsim 60$) by the time the field "plunges down" the well. From (64) (taking $\phi_i \ll \phi_f$),

$$t_f \approx \frac{3H\phi_f}{\epsilon} , \tag{67}$$

so that we must have

$$Ht_f = \frac{3H^2\phi_f}{\epsilon} \gtrsim 60 , \tag{68}$$

allowing us to constraint ϵ to be

$$\epsilon \lesssim 2\times 10^{32}\ \text{GeV}^3 \approx (6\times 10^{10}\ \text{GeV})^3 . \tag{69}$$

Our original constraint $\epsilon\phi_f \ll V_0$ is satisfied by 9 orders of magnitude

$$\frac{\epsilon\phi_f}{V_0} \lesssim \frac{(2\times 10^{32}\ \text{GeV}^3)(1.2\times 10^{15}\ \text{GeV})}{(10^{14}\ \text{GeV})^4} \approx 2.4\times 10^{-9} . \tag{70}$$

The density perturbations can be estimated as in the preceding section by evaluating (47) and using (46). The "leading edge" $[\bar{\phi}(t)+\sigma(t)]$ reaches ϕ_f at the time t_+ while the "trailing edge" $[\bar{\phi}(t)-\sigma(t)]$ reaches ϕ_f at t_-. Clearly t_+ and t_- are the roots of the equation

$$\bar{\phi}(t) \pm \sigma(t) = \phi_f . \tag{71}$$

For $Ht \gg 1$, we get from (63) and (64) (with $\phi_i \approx 0$),

$$\Delta\tau \equiv t_- - t_+ = a\frac{6H\sigma_0}{\epsilon}\left[1 + \frac{1}{36}\left[\frac{H}{\pi\sigma_0}\right]^4\right]^{1/2} , \tag{72}$$

so that

$$\frac{\delta\rho}{\rho} \simeq aH\Delta\tau = a^2\frac{6H^2\sigma_0}{\epsilon}\left[1 + \frac{1}{36}\left[\frac{H}{\pi\sigma_0}\right]^4\right]^{1/2} , \tag{73}$$

where a is a number of the order of unity. Because of the existence of de Sitter temperature $H/2\pi$, it is usual to take $\sigma_0 = H/2\pi$. [See comments following Eq. (51).] Then (73) gives

$$\frac{\delta\rho}{\rho} \simeq \frac{H^3}{\epsilon} . \tag{74}$$

Using (65) and (69) we obtain the coveted magic number

$$\frac{\delta\rho}{\rho} \simeq 4\times 10^{-5} . \tag{75}$$

How is it that our model produces the correct $\delta\rho/\rho$ while conventional inflationary scenarios do not? The answer has to do with the constraint in (69). Since we treated ϵ as a free parameter, we could use (69) and determine ϵ conveniently. In standard inflationary scenarios, the driving term $(-\partial V/\partial\phi)$ goes as $\lambda\phi^3$. Maintaining a constraint on ϵ would therefore imply a fine-tuning of λ.

We can reach the same conclusion in a different way: Our potential $V(\phi)$ has three parameters V_0, ϕ_f, ϵ. We may, however, write

$$V(\phi) = V_0 - \epsilon\phi \equiv \eta^4\phi_f{}^4 - \lambda^3\phi_f{}^3\phi = (\eta\phi_f)^4\left[1 - \frac{\lambda^3}{\eta^4}\frac{\phi}{\phi_f}\right] , \tag{76}$$

where η,λ are *dimensionless* constants. From (65) it follows that $\eta \sim 0.1$. But, the constraint (69) implies that

$$\lambda \lesssim 10^{-4} , \tag{77}$$

which corresponds to quite a bit of fine-tuning for a dimensionless constant. Thus we are really no better off than the conventional scenarios.

V. CAN SOPHISTICATION HELP?

The simple models discussed above illustrate the origin of various constraints in inflationary scenarios. These constraints clearly show that the basic idea will not work without fine-tuning the parameters in the potential.

From (73) it is clear that $\Delta\phi$ has to be kept smaller to produce smaller $\delta\rho/\rho$. It is easy to see that more sophisticated potentials cannot produce a $\sigma(t)$ which is smaller than that for the steady slope case. To see this, consider (28) written in the form

$$2\pi^2\frac{d^2\sigma}{dT^2} \approx -\left[\frac{\partial^2 u}{\partial\phi^2}\right]\sigma . \tag{78}$$

For potentials used for inflation $u'' \lesssim 0$ in the region of interest, so that the right-hand side of (78) will be positive or zero. In other words, $\sigma(T)$ for such a potential will have a higher value than the $\sigma(T)$ for the steady slope potential for which the right-hand side of (78) vanishes. [Even though coherent states of the form we are considering exist only for quadratic potentials, the comments above continue to be valid even for more general potentials; we merely have to interpret σ as $\langle(\phi-\bar{\phi})^2\rangle$.] Thus, as far as $\sigma(T)$ is concerned a more sophisticated potential will not help.

A different choice of potential will certainly change the nature of $\bar{\phi}(t)$ evolution. However, as long as (i) the potential contains only one scale ϕ_f, (ii) the rollover is slow allowing us to ignore $\ddot{\phi}$ terms, and (iii) there exists a minimum "noise" $\Delta\phi \sim H/2\pi$, our conclusions will still hold. To see this assume that

$$V(\phi)=(\eta\phi_f)^4\left[1-\lambda\left[\frac{\phi}{\phi_f}\right]^n\right]. \tag{79}$$

Neglecting the $\ddot{\bar{\phi}}$ term in

$$\ddot{\bar{\phi}}+3H\dot{\bar{\phi}}=-\frac{\partial V}{\partial\phi} \tag{80}$$

we can easily estimate the rollover time t_f to be

$$|t_f| \simeq \frac{3}{n(n-2)}\frac{1}{\eta^4\lambda}\frac{H}{\phi_f{}^2}, \tag{81}$$

so that the requirement of sufficient inflation implies

$$Ht_f \approx \frac{3}{n(n-2)}\frac{1}{\eta^4\lambda}\left[\frac{H}{\phi_f}\right]^2 \gtrsim 60 \tag{82}$$

or, equivalently,

$$\left[\frac{H}{\phi_f}\right] \gtrsim [20n(n-2)]^{1/2}\lambda^{1/2}\eta^2 . \tag{83}$$

The perturbations will be bounded by (since $\Delta\phi \geq H/2\pi$)

$$\frac{\delta\rho}{\rho} \simeq \left[\frac{H\Delta\phi}{\dot{\phi}}\right] \geq H\frac{H}{2\pi}\frac{3H}{\lambda\eta^4 n\phi_f{}^3}=\frac{3}{2\pi\lambda\eta^4 n}\left[\frac{H}{\phi_f}\right]^3 . \tag{84}$$

Combining (83) and (84), we obtain

$$\frac{\delta\rho}{\rho} \gtrsim 40n^{1/2}(n-2)^{3/2}\eta^2\lambda^{1/2} \tag{85}$$

requiring a fine-tuning of η and λ to obtain the correct $\delta\rho/\rho$. By putting in more effort one can make the above arguments more watertight, but these frills will not help one to achieve a 10^{-6} factor required in (85) if $\eta \sim \lambda \sim 1$.

Taking the spatial degrees of freedom into account will not, of course, change matters except in two inessential ways. (i) It will produce the logarithmic dependence of the wave vector $|\mathbf{k}|$ assuring the scale independence of $\delta\rho/\rho$. Our problem, however, is with the amplitude. (ii) By doing the quantum field theory "correctly" in the de Sitter spacetime. one would automatically incorporate the Hawking temperature noise $(H/2\pi)$. In our calculation, we had to incorporate it by hand. Once again, this does not affect the final result.

Any other form of initial conditions will only make matters worse. As long as the initial wave function is reasonably peaked about the origin and has a width $\sigma_0 \geq H/2\pi$ our conclusions will follow. If the spread is high [or, much worse, the wave function is peaked about the minima $(\pm\phi_f)$] then the scenario will be affected unfavorably.

We conclude that in order to produce a reasonable value for $\delta\rho/\rho$ in an inflation that lasts sufficiently long, it is necessary to invoke one or more of the following points.

(i) There exists (at least) two widely different scales in the potential $V(\phi)$.

For example, the simple linear model in Sec. III can work with the potential

$$V(\phi)=(\eta\phi_f)^4[1-\lambda(\phi/E)] ,$$

where $\eta \sim 0.1$, $\lambda \sim 0.1$, $\phi_f \sim 10^{15}$ GeV, $E \sim 10^{23}$ GeV. The disparity between the scales ϕ_f and E is obvious. If nonpolynomial terms in $V(\phi)$ are allowed to dominate, or if more free parameters are introduced into $V(\phi)$, then one can "distribute" the disparity among various terms, thereby effectively hiding it. Unless the choice of scales is justifiable in a noninflationary context, it will continue to remain as a fine-tuning.

(ii) Disregard the "thermal noise" in the inflationary epoch.

We showed in Sec. III that even the naivest model will produce correct $\delta\rho/\rho$ if one allows for the initial spread σ_0 to be much smaller than the Hawking temperature of the de Sitter space $(H/2\pi)$. Is it allowed?

The Hawking temperature depends on the choice of the vacuum state in the manifold. *The spacetime manifold describing the early Universe will never be truly de Sitter.* At best, it will asymptotically mimic the de Sitter expansion after the vacuum energy starts to dominate and before thermalization occurs. *Even then, the manifold considered in the early Universe is only a part of the de Sitter manifold.* The "correct" choice of "vacuum" in this situation is probably worthy of closer scrutiny.

(iii) Fine-tune the parameters of the potential.

This, clearly, would be the last resort if one wants to stick to the philosophy of inflation. One hopes that it would be avoided.

[1]A. H. Guth, Phys. Rev. D 23, 347 (1981).
[2]K. Sato, Phys. Lett. **99B**, 66 (1981).
[3]D. Kazanas, Astrophys. J. **241**, L59 (1980).
[4]A. D. Linde, Rep. Prog. Phys. **47**, 925 (1984).
[5]A. H. Guth and E. J. Weinberg, Nucl. Phys. **B212**, 321 (1983).
[6]M. S. Turner, in *Proceedings of the Fourth Workshop on Grand Unification, University of Pennsylvania, 1983*, edited by H. A. Weldon, P. Langacker, and P. J. Steinhardt (Birkhauser, Boston, 1983).
[7]G. F. Mazenko, W. G. Unruh, and R. M. Wald, Phys. Rev. D 31, 273 (1985); M. Evans and J. G. McCarthy, *ibid.* 31, 1799 (1985).
[8]R. H. Brandenberger, Nucl. Phys. **B245**, 328 (1984).
[9]A. H. Guth and S-Y. Pi, Phys. Rev. D 32, 1899 (1985).

[10]A. Albrecht, R. H. Brandenberger, and R. A. Matzner, Phys. Rev. D **32**, 1280 (1985).
[11]A. H. Guth and S-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982).
[12]A. A. Starobinsky, Phys. Lett. **117B**, 175 (1982).
[13]S. W. Hawking, Phys. Lett. **115B**, 295 (1982).
[14]T. Padmanabhan, J. Astron. Astrophys. **6**, 239 (1985).
[15]G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977).