

Generation of Seed Perturbations from Quantum Cosmology

by

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Abstract

The origin of seed perturbations in the Universe is studied within the framework of a specific minisuperspace model. It is shown that the ‘creation’ of the Universe as a result of a quantum transition from a flat empty spacetime would lead to a flat FLRW (Friedmann Lemaître Robertson-Walker) Universe with weak inhomogeneous perturbations at large wavelengths. The power spectrum of these perturbations is found to be scale invariant at horizon crossing (i.e., the Harrison-Zeldovich spectrum). It is also recognised that the seed perturbations generated in our model would be generically of the isocurvature kind.

subject headings: quantum cosmology; structure formation; seed perturbations.

1. Introduction

Classical models of cosmology till the turn of the last decade were unable to resolve three fundamental problems, commonly known as i) the singularity, ii) the horizon problem (Misner 1969), and iii) the flatness problem (Dicke and Peebles 1979).

An exciting development in the early 80's was the emergence of the concept of inflation as a promising remedy for the above problems (Guth 1981). Inflationary scenarios chiefly resolve the horizon and the flatness problems though a few do address the singularity problem (Starobinsky 1980; Linde 1982). The problem of the origin of structure in the universe, which ought to have been included in the above list, was relegated to an initial value problem until inflationary scenarios showed promises of successfully addressing it. It was then realised very soon that all 'natural models' of inflation produce density perturbations with the right spectrum but with an unacceptably large amplitude (Hawking 1982; Starobinsky 1982; Guth and Pi 1982; Bardeen 1987). Since then, cosmology has seen the rise and fall of numerous inflationary scenarios with various forms of fine-tuning to constrain the amplitude of the perturbations (Brandenberger 1985; Olive 1990; Narlikar and Padmanabhan 1991). Even granted the fine-tuned parameters needed to resolve the problem of density perturbations, the claim of inflation of generically producing the flat ($\rho_o/\rho_c \approx 1$) FLRW (Freidmann Lemaître Robertson Walker) universe has also been questioned by Ellis (1988; also see Madsen and Ellis 1988). These authors have pointed out that $\rho_o/\rho_c \approx 1$ is not a generic outcome of inflation and, in fact, has to be fine-tuned to its present value. Similarly, the horizon problem is also not permanently eliminated but remedied only for the present epoch pushing it to an epoch far in the future (Padmanabhan and Seshadri 1988; Ellis and Stoeger 1988). Thus the strong points in favour of the inflationary model which made it so attractive in the beginning now do not seem so strong. In particular, alternative scenarios for the very early universe can certainly be tried.

An attempt to resolve the above important problems by taking recourse to a simplistic but working model of quantum cosmology was made by Narlikar and Padmanabhan (1983). This approach to quantum gravity (Narlikar 1979, 1981; Padmanabhan and Narlikar 1982; Padmanabhan 1982) is based on a path integral formulation of the quantum version of classical geometrodynamics, where

only the conformal degree of freedom is quantised leaving the other degrees (extrinsic curvature) frozen. Although this restriction may oversimplify the problem of quantizing gravity, it has certain advantages. First, an exact nonperturbative solution of the quantum geometrodynamics evolution is possible. Secondly, restriction to conformal degrees of freedom only implies a quantum theory of gravity where the quantum fluctuations preserve the causal structure of spacetime. The conformal quantisation leads to many interesting results, e.g., cosmological spacetimes with singularity[†] (and the consequent horizon problem) appear to be a set of zero measure in the solution space and that the flat FLRW universe happens to be the most favoured conformally flat spacetime arising out of a quantum transition from the Minkowskian spacetime.

These successes prompt us to take up a problem at the next level of sophistication, viz. the origin of fluctuations against a homogeneous background. In this paper we extend the above approach to study the possible generation of primordial density fluctuations in a universe created by quantum transition from a flat empty spacetime. To this end we briefly review the Narlikar-Padmanabhan approach to quantum cosmology in §2 highlighting the resolution of the flatness problem that quantum conformal transitions of the universe from a quantum state peaked around the flat Minkowski spacetime ($\langle \Omega \rangle_i \equiv 1$) would, with maximum probability, end up in a state peaked around a flat FLRW universe ($\langle \Omega \rangle_f \equiv \Omega_o(t)$). Then in §3 we show that fluctuations around a background $\Omega_o(t)$ in the form of inhomogeneous modes, $\phi(\mathbf{x}, t)$, can be perturbatively introduced into the mean conformal factor $\langle \Omega \rangle$ with marginally diminished probability and obtain the power spectrum of the fluctuations $\phi(\mathbf{x}, t)$. It is also demonstrated that conformal fluctuations always imply isocurvature perturbations. The translation of the conformal fluctuation $\phi(\mathbf{x}, t)$ to density perturbations $\delta M/M$ once the hot radiation dominated matter loses its conformal invariance, is studied in §4.

In §5, we evolve the fluctuations $\phi(\mathbf{x}, t)$ in a universe with some form of coupling of matter to the conformal degree of gravity in the matter Lagrangian. $\Omega_o(t)$ is obtained through the

[†] To be precise, we are referring to solutions with curvature singularities, i.e., points where the curvature invariants become unbounded.

evolution equation for $\phi(\mathbf{x}, t)$. However, we outline in §6, an alternative prescription for evolving the perturbations wherein the form of the homogeneous background conformal factor $\Omega_o(t)$ is independently fixed which then determines the evolution of $\phi(\mathbf{x}, t)$.

We end the paper with a discussion (§7) comparing the results in the present ‘standard’ and ‘non-standard’ scenarios of generation of primordial density fluctuations.

2. Quantum conformal fluctuations in Cosmology

At a classical level, general relativity can be formulated as a dynamical theory by considering a 3+1 York decomposition (York 1972) of the spacetime manifold $\mathcal{M}$ into 3-hypersurfaces $\Sigma(t)$ evolving along a timelike curve, parameterised by a time t , between the boundaries Σ_i and Σ_f — the specified initial and final hypersurfaces at times t_i and t_f . The 3-geometry on Σ at a given time t is specified by the scale factor Ω and the extrinsic curvature K_{ab} . The classical solution to general relativity is the trajectory $\Gamma_{cl}(t)$ in the superspace $\mathcal{G}$ of 3-geometries which extremises the action

$$J = \frac{1}{16\pi} \int_{\mathcal{V}} R \sqrt{-g} d^4x + J_m \quad (2.1)$$

(where J_m is the action for matter fields) over the 4-volume $\mathcal{V}$ between Σ_i and Σ_f (Isenberg and Wheeler 1979).

At the quantum level, the above picture translates to calculating the probability amplitude $K[\mathcal{G}_i; \mathcal{G}_f]$ for transition from a 3-geometry $\mathcal{G}_i$ on $\Sigma_i(t_i)$ to $\mathcal{G}_f$ on $\Sigma_f(t_f)$. In exact analogy to the path integral formulation of quantum mechanics (Feynman and Hibbs 1965), the transition amplitude $K[\mathcal{G}_i; \mathcal{G}_f]$ can be formally expressed as a sum over all trajectories $\Gamma(t)$ in the superspace $\mathcal{G}$ joining $\mathcal{G}_i$ and $\mathcal{G}_f$ as

$$K[\mathcal{G}_i; \mathcal{G}_f] = \sum_{\Gamma} \exp \left[i \frac{J[\Gamma]}{\hbar} \right]. \quad (2.2)$$

The evaluation of $K[\mathcal{G}_i; \mathcal{G}_f]$, which contains the complete essence of quantum gravity, is however beset with conceptual and technical difficulties. The expression in (2.2) gets considerably simplified and well defined if one demands the preservation of the causal structure of spacetime for all paths

Γ . In this case, the paths Γ which are summed over are such that the 4-geometries $g_{\mu\nu}(\Gamma)$ along them are all conformally related to the classical 4-geometry along Γ_{cl} i.e., the metrics allowed are

$$g_{\mu\nu}(\Gamma) = \Omega^2(\mathbf{x}, t) \tilde{g}_{\mu\nu}(\Gamma_{cl}) \quad (2.3)$$

where $\Omega(\mathbf{x}, t)$ is a C^2 function of the spacetime coordinates. This conformal degree of freedom corresponds to the volume of the 3-hypersurface and has a special status because it contributes a negative term to the kinetic energy in the Wheeler-DeWitt (W-D) equation. In line with Wheeler's philosophy that "the 3-geometry is the carrier of information about time" (Kuchär 1971; York 1971; Misner et al 1973), the conformal factor $\Omega(\mathbf{x}, t)$ can play the role of 'time' to describe evolution in the superspace $\mathcal{G}$ through the 'time-less' W-D equation. In fact, the conformal factor does appear to play the role of time in the semi-classical regime of the W-D equation (Padmanabhan 1989). The conformal degree of freedom is sometimes dismissed as 'unphysical', a point of view with which we disagree. In §7, we shall deal with this particular issue.

The expression for $K[\mathcal{G}_i; \mathcal{G}_f]$ in (2.2) thus reduces to

$$K[\Omega_i; \Omega_f] = \int \mathcal{D}\Omega \exp \left[\frac{i}{12 l_p^2} \int_{\mathcal{V}} (\tilde{R}\Omega^2 - 6\Omega^\mu \Omega_\mu) \sqrt{-\tilde{g}} d^4x \right]. \quad (2.4)$$

Being a quadratic a functional integral over $\Omega(\mathbf{x}, t)$, it can be explicitly evaluated in a non-perturbative manner.

The quantum state of the universe is a wavefunctional $\Psi[\Omega, t]$. The probability amplitude $K[\Omega_i; \Omega_f]$ is the propagator which gives the quantum state Ψ_f at some time t_f given Ψ_i at time t_i through the relation

$$\Psi[\Omega_f] = \int \mathcal{D}\Omega K[\Omega_i; \Omega_f] \Psi_i[\Omega_i]. \quad (2.5)$$

The transition amplitude between Ψ_i and Ψ_f is given by

$$\langle \Psi_f | \Psi_i \rangle = \int \int \mathcal{D}\Omega_1 \mathcal{D}\Omega_2 \Psi_f^*[\Omega_2] K[\Omega_2; \Omega_1] \Psi_i[\Omega_1]. \quad (2.6)$$

The ‘creation’ of the universe as a quantum event (like vacuum fluctuations, tunnelling etc.) has been explored by many authors from various points of view (see for example, Tryon 1973; Brout 1980; Zeldovich 1981; Atkatz 1982; Vilenkin 1982 and for a review, Kandrup and Mazur 1991). In this framework, the ‘creation’ of the universe has been studied by evaluating the transition probability from a state Ψ_i peaked around the flat Minkowski spacetime ($\Omega_i \equiv 1$ and $g_{\mu\nu}(\Gamma_{cl}) = \eta_{\mu\nu}$) to some state Ψ_f around a conformally flat spacetime $\langle\Omega\rangle \equiv \Omega(\mathbf{x}, t)$. It is found that the former is unstable to such fluctuations (Atkatz and Pagels 1982; Brout et al 1980; Padmanabhan 1983) and the transition probability between Ψ_i — a gaussian wave packet around $\langle\Omega\rangle_i$ and a state Ψ_f — a gaussian packet around $\langle\Omega\rangle_f = \Omega_f(\mathbf{x}, t)$ can be written as

$$|\langle\Psi_f|\Psi_i\rangle|^2 = \mathcal{N} \exp \left[-\frac{1}{2}\mathcal{W} \right] , \quad (2.7a)$$

where

$$\mathcal{W} = \frac{1}{l_p^2} \int \int \frac{\nabla\Omega_f(\mathbf{x}_1) \cdot \nabla\Omega_f(\mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|} d^3\mathbf{x}_1 d^3\mathbf{x}_2 , \quad (2.7b)$$

and l_p is the planck length. The probability of transition is maximum when $\mathcal{W} = 0$ (since $\mathcal{W} \geq 0$) which occurs for $\nabla\Omega_f = 0$. This implies that $\Omega_f \equiv \Omega_o(t)$ which corresponds to a flat FLRW metric. Hence, we see that a Ψ_f peaked around the flat FLRW universe, $\langle\Omega\rangle \equiv \Omega_o(t)$, is the most probable outcome of a causal structure preserving quantum transition of the universe from a ‘ground’ state (peaked around the flat Minkowski geometry). The resultant universe being a strictly flat FLRW model (i.e., $\Omega_o(t) \equiv 1$, rather than $\Omega_o(t) \approx 1$ as in inflation), this approach is free from the type of criticism of Ellis (1988) against inflation.

3. The generation of inhomogeneties

In this section, we build upon the result reviewed in §2 to generate cosmological perturbations. The exponent $\mathcal{W}$ of the transition probability given by equation (2.7b), can be rewritten in the momentum space as

$$\mathcal{W} = \frac{1}{l_p^2} \int |Q_k(t)|^2 |\mathbf{k}| \frac{d^3 \mathbf{k}}{(2\pi)^3} \quad (3.1a)$$

where

$$\Omega_f(\mathbf{x}, t) = l_p^{\frac{3}{2}} \int Q_{\mathbf{k}}(t) e^{i\mathbf{k} \cdot \mathbf{x}} \frac{d^3 \mathbf{k}}{(2\pi)^3} . \quad (3.1b)$$

Clearly, the homogeneous mode with $Q_k = 0$ for $|\mathbf{k}| \neq 0$ (i.e., $\Omega_f \equiv \Omega_f(t)$) leads to the maximisation of $|\langle \Psi_f | \Psi_i \rangle|^2$. However, it is equally obvious that in the ‘next best’ situation a transition to Ω_f , with weak inhomogeneties can occur with a slightly reduced probability still quite close to unity. Suppose that, in the final state of process of the quantum creation, the universe has Ψ_f peaked around

$$\langle \Omega(\mathbf{x}, t) \rangle_f = \Omega_o(t) + \epsilon \phi(\mathbf{x}, t) \quad (3.2)$$

where ϵ is a small number. The $\langle \Omega(\mathbf{x}, t) \rangle$ in the above expression, involves a transition from the regime of quantum fluctuations to classical perturbations. In our case this identification of a classical field with the expectation value of quantum fluctuations of a quantum field is very well defined owing to the fact that the expectation values are calculated between coherent quantum states.

The probability of transition $\mathcal{P}$ to a given $\langle \Omega \rangle_f$ can be evaluated using equation (2.7) and (3.2). $\mathcal{W}$ now reads

$$\mathcal{W} = \frac{\epsilon^2 l_p}{2} \int |q_{\mathbf{k}}(t)|^2 |\mathbf{k}| \frac{d^3 \mathbf{k}}{(2\pi)^3} \quad (3.3a)$$

with

$$\phi(\mathbf{x}, t) = l_p^{\frac{3}{2}} \int q_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \frac{d^3 \mathbf{k}}{(2\pi)^3} . \quad (3.3b)$$

The dependence of $\mathcal{P}$ on the length scale of the inhomogeniety can obtained from (3.3) by substituting a form of $q_{\mathbf{k}}$ with a characteristic scale built into it. We take $q_{\mathbf{k}}$ to be of the form

$$q_{\mathbf{k}} = (4\pi\lambda)^{\frac{3}{2}} \exp \left[-\frac{k^2\lambda^2}{2} \right], \quad (3.4)$$

a gaussian around $\mathbf{k} = 0$ with a characteristic spread λ^{-1} in k ($\equiv |\mathbf{k}|$).

The expression for $\mathcal{W}$ now becomes

$$\mathcal{W} = \frac{16\pi\epsilon^2 l_p}{\lambda} \quad (3.5)$$

and the transition probability $\mathcal{P}$ given by equation (2.7a) is

$$\mathcal{P} \equiv |\langle \Psi_f | \Psi_i \rangle|^2 = \mathcal{N} \exp \left[-\frac{\lambda_o}{\lambda} \right], \quad (\lambda_o = 16\pi\epsilon^2 l_p). \quad (3.6)$$

The expression (3.6) shows that the generation of inhomogeneity at small enough length scales ($\lambda \ll \lambda_o$) is exponentially suppressed.

The issue which is addressed next is to estimate the power on various scales of inhomogeneity. We calculate the energy content, $\mathcal{E}$, of the final wavepacket Ψ_f in terms of the Fourier modes $q_{\mathbf{k}}(t)$ of $\phi(\mathbf{x}, t)$. The expectation value of the energy density of the wavepacket Ψ_f is given by

$$\langle \Psi_f | T_{00}(\mathbf{x}) | \Psi_i \rangle \equiv \langle T_{00}(\mathbf{x}) \rangle = \int \mathcal{D}\Omega \Psi_f^*[\Omega] \hat{\mathcal{H}}_{\Omega} \Psi_f[\Omega], \quad (3.7)$$

where $\hat{\mathcal{H}}_{\Omega}$ is the Hamiltonian operator. The above functional integral can be evaluated explicitly (see appendix).

The energy $\mathcal{E}$ of the wavepacket, obtained by integrating $\langle T_{00}(\mathbf{x}) \rangle$ over all space, is of the form

$$\mathcal{E} = \int d^3\mathbf{x} \langle T_{00}(\mathbf{x}) \rangle = \frac{\epsilon}{6l_p^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} k^2 q_{\mathbf{k}} q_{-\mathbf{k}}. \quad (3.8)$$

We estimate the energy at a scale λ by evaluating the expectation value $\langle \mathcal{E}^2 \rangle$ (as outlined in the appendix) and arrive at the result that

$$\langle \mathcal{E}^2 \rangle \propto \lambda^{-4}. \quad (3.9)$$

A measure of power $P(\lambda)$, at a scale λ can then be taken to be the energy of a wavepacket with a characteristic scale of inhomogeneity λ , weighted by the relative probability of transition to such a state. The power spectrum so defined, reads

$$P(\lambda) \propto \lambda^{-4} \exp \left[-\frac{\lambda_o}{\lambda} \right]. \quad (3.10)$$

The power spectrum has a power law distribution for large wavelength modes

$$P(\lambda) \propto \lambda^{-4} \quad (\lambda \gg l_p) \quad (3.11)$$

with a exponential cut off at small values of λ . The peak occurs at a wavelength λ_{peak} ,

$$\lambda_{\text{peak}} = \frac{1}{4} \lambda_o = 4\pi \epsilon^2 l_p. \quad (3.12)$$

The above results show an interplay between the wavelength λ_{peak} at which inhomogeneity is generated and the amplitude ϵ .

We investigate the nature of these perturbations by looking at the gauge invariant potential Φ_H , introduced by Bardeen (1980), in a universe filled with two component (radiation and dust) hydrodynamic matter. The conformal fluctuations $\phi(\mathbf{x}, t)$ would not cause any deviation of the perturbed spacetime from conformal flatness, implying that the Weyl curvature tensor $C_{\mu\nu\lambda\delta}$ is identically zero. The Bardeen potential as a geometrical quantity is proportional to the square of the Weyl curvature,

$$\Phi_H \propto C_{\mu\nu\lambda\delta} C^{\mu\nu\lambda\delta}, \quad (3.13)$$

and relates to the matter perturbations (Starobinsky and Sahni 1984) as

$$\Phi_H \propto \delta\rho_{\text{tot.}} \quad (= \delta\rho_{\text{radn.}} + \delta\rho_{\text{dust}}). \quad (3.14)$$

The above equations (3.13) and (3.14), coupled with the fact that the Weyl tensor $C_{\mu\nu\lambda\delta} \equiv 0$ for conformal fluctuations leads to the conclusion that

$$\delta\rho_{\text{tot.}} = \sum_i \delta\rho_i = 0, \quad (3.15)$$

where we have stated our result as generalised to multi-component hydrodynamic matter. From the equation (3.15), the seed perturbations are recognised to isocurvature perturbations.

Isocurvature perturbations are a firm prediction of our scenario. The role of isocurvature perturbations in both the baryon dominated (Peebles 1987a, 1987b; Efstathiou and Rees 1988) and cold dark matter (CDM) models (Efstathiou and Bond 1987; Bardeen et al 1987; Starobinsky and Sahni 1984) of structure formation have been discussed in the literature. These perturbations are known to generate more power on large scales in CDM models which violates CMBR bounds, but can be exploited in Baryon dominated models because of characteristic features appearing in their final spectrum at astrophysically large scales.

4. Translation to mass perturbations

The early universe is expected to be hot i.e., matter is in thermal equilibrium at a very high temperature. At high temperatures, all particles would be relativistic hence one can consider the matter Lagrangian to be invariant under conformal transformations of the metric. However, as the universe cools down, some component of matter would break its conformal invariance. The simplest picture would be to have some particle become non-relativistic at some epoch t^* (corresponding to some mass scale) or, alternatively, a phase transition may cause some massless boson to acquire mass. This component of matter would then *see* the conformal fluctuations and chart out a corresponding density perturbation. A simple analysis shows the relation between the $\delta M/M$ and the conformal fluctuations $\phi(\mathbf{x}, t)$. In line with the conventional treatment of cosmological perturbations (Landau 1958; Mukhanov et al 1991), we compute the perturbations δM in some physical quantity M (say mass) as the difference between the value M calculated in the physical 3-hypersurface Σ with conformal factor $\Omega(\mathbf{x}, t)$, and the homogeneous background value $\overline{M}$ calculated in the background 3- hypersurface $\overline{\Sigma}$.

Consider a comoving 3-volume ΔV_c . The mass $M(\Delta V_c)$ contained in the corresponding physical

volume is given by

$$M(\Delta V_c) = \Omega^3(\mathbf{x}, t) \rho \Delta V_c = [\Omega_o(t) + \epsilon \phi(\mathbf{x}, t)]^3 \rho \Delta V_c, \quad (4.1)$$

and the mass $\overline{M}(\Delta V_c)$ contained in the background volume is

$$\overline{M}(\Delta V_c) = \Omega_o^3(t) \rho \Delta V_c. \quad (4.2)$$

As noted in §3, the perturbations are of the isocurvature kind, and hence the $\delta\rho$ term that one would expect in equation (4.1) is absent. The mass fluctuation at a point, $\delta M/M(x, t)$ can be expressed (upto linear order in ϵ) as

$$\frac{\delta M}{M}(\mathbf{x}, t) = \eta \left[\frac{M(\Delta V_c) - \overline{M}(\Delta V_c)}{\overline{M}(\Delta V_c)} \right] = 3\epsilon\eta \frac{\phi(\mathbf{x}, t)}{\Omega_o(t)}, \quad (4.3)$$

where the constant η (≤ 1) has been introduced to account for the fact that only a fraction of the matter would respond to the conformal fluctuations. The mass fluctuations $\delta M/M$ are directly proportional to the conformal fluctuations $\phi(\mathbf{x}, t)$. In the Fourier space, the mass fluctuation at a scale k (i.e., the power spectrum of $\delta M/M$) would be directly related to the power spectrum of the conformal perturbations obtained in §3. The power spectrum of the mass fluctuations, using (3.11) can be written as

$$\left| \frac{\delta M}{M} \right|^2(\mathbf{k}, t) = P(k^{-1}) f(t) \propto k^4 e^{-\lambda_o k} f(t), \quad (4.4)$$

where $f(t)$ incorporates the time dependence of $\phi(\mathbf{x}, t)$ which would be taken up in the next section. The framework within which we are working does not require us to invoke an inflationary stage during the evolution of the universe; hence, the astrophysically relevant scales are the modes with extremely large wavelengths ($k^{-1} \sim 10^{28} l_p$). Therefore, the power spectrum of fluctuations at the astrophysically relevant scales is given by

$$\left| \frac{\delta M}{M} \right|_k^2 \propto k^4 \quad (kl_p \ll 1). \quad (4.5)$$

This is the Harrison-Zeldovich spectrum (Harrison 1970; Zeldovich 1972). To state the above more clearly, we use

$$\left| \frac{\delta M}{M} \right|_k^2 \approx k^3 |\delta_k|^2 \quad (4.6a)$$

where

$$\delta_k = \int d^3 \mathbf{x} \ e^{i \mathbf{k} \cdot \mathbf{x}} \frac{\delta \rho}{\rho}(\mathbf{x}, t). \quad (4.6b)$$

Using (4.5) and (4.6) we get

$$|\delta_k|^2 \propto k \quad (k \ll 1), \quad (4.7)$$

which is the Harrison-Zeldovich spectrum in the more familiar form.

5. Evolution of fluctuations

We now address the question of evolution of the conformal fluctuations generated at the time of the ‘creation’ of the universe till the time they decay away into seed perturbations in the non-conformal component of the matter in the universe. After the quantum transition has occurred, the conformal factor can be treated as a classical field with an action J as given in equation (2.2) evolving in a background geometry. Our analysis would be only limited to the initial evolution of conformal fluctuations which later decay away transferring their energy to density perturbations (as outlined in §3) and we assume that these follow the well known evolution equations (Peebles 1980).

The action governing the conformal fluctuations $\Omega(\mathbf{x}, t)$ around a fiducial metric $\tilde{g}_{\mu\nu}$ has the form

$$J = \frac{1}{16\pi} \int [6\Omega^\mu \Omega_\mu - \tilde{R}\Omega^2] \sqrt{-\tilde{g}} d^4 x + J_m \quad (5.1)$$

The equation of motion of the conformal fluctuation $\Omega(\mathbf{x}, t)$ is

$$\tilde{\square} \Omega + \frac{1}{6} \tilde{R} \Omega = \frac{\delta \mathcal{L}_m}{\delta \Omega} \quad ; \quad (5.2)$$

akin to that of a conformally coupled scalar field with a potential $V(\Omega)$ (Narlikar and Padmanabhan 1983).

In particular, for $\tilde{g}_{\mu\nu} \equiv \eta_{\mu\nu}$, equation (5.2) reduces to

$$\partial_\mu \partial^\mu \Omega(\mathbf{x}, t) = \frac{\delta \mathcal{L}_m}{\delta \Omega} \equiv V'(\Omega) . \quad (5.3)$$

At this stage, we go ahead with the idea that the matter Lagragian is invariant under conformal transformations (i.e., set $V'(\Omega) \equiv 0$). Substituting the expression (3.2) into equation (5.3), we get at the zeroth order

$$\frac{d^2}{dt^2} \Omega_o(t) = 0, \quad \Rightarrow \quad \Omega_o(t) \propto t \quad (5.4a)$$

and to the first order in ϵ ,

$$\square \phi(\mathbf{x}, t) = 0 . \quad (5.4b)$$

The equation (5.4b) admits plane travelling wave solution for $\phi(\mathbf{x}, t)$.

However, in a more realistic treatment one would have a non-trivial form for $V(\Omega)$. We will bypass the question of determining the exact form of $V(\Omega)$ and assume a form

$$V(\Omega) = m^2 \Omega^2 \quad (5.5)$$

where m is the mass scale introduced to break the conformal invariance of $\mathcal{L}_m$. Using the form of $V(\Omega)$ given in (5.5) in the equation (5.3), we obtain at the zeroth order

$$\frac{d^2}{dt^2} \Omega_o(t) = m^2 \Omega_o(t) \quad (5.6a)$$

and at $\mathcal{O}(\epsilon)$

$$\square \phi(\mathbf{x}, t) = m^2 \phi(\mathbf{x}, t). \quad (5.6b)$$

The form of (5.6b) suggests that the Fourier components of $\phi(\mathbf{x}, t)$ would obey the evolution equation

$$\phi_{\mathbf{k}}(t) \propto \exp [\pm(m^2 - |\mathbf{k}|^2)^{\frac{1}{2}} t] . \quad (5.7)$$

The inhomogeneous modes $\phi_k(t)$, have a growing solution for $k \leq m$ (and also a decaying solution). This implies that in the case of $V(\Omega)$ of the form (5.5), the mass scale m introduces a lower cut off value for the wavelength above which one finds growing modes. This further strengthens our argument that inhomogeneties would exist only at the large wavelength modes which in our framework are the astrophysically relevant scales.

6. Alternative prescription for evolution

In the preceding section, the evolution of the conformal fluctuations was governed by equation (5.2) where the potential $V(\Omega)$ had to be fixed from some independent physical considerations regarding the coupling of matter to the conformal degree of freedom of gravity. The analysis of Narlikar and Padmanabhan (1983) outlined in §2, indicated that the conformal factor would be homogeneous at the leading order. However, the exact form of $\Omega_o(t)$ is not uniquely determined. This opens up the possibility of an alternative prescription where one could fix the form of $\Omega_o(t)$ first and then consider the evolution of $\phi(\mathbf{x}, t)$ through equation (5.3) using the corresponding form for $V(\Omega)$. In this section, we outline one possible physical consideration through which $\Omega_o(t)$ could be obtained. We rewrite the background metric in comoving coordinates as

$$ds^2 = \Omega^2(t) (dt^2 - d\mathbf{x}^2) = d\tau^2 - S^2(\tau) d\mathbf{x}^2 \quad (6.1)$$

and assume that the energy density of the universe in the early epoch would be given by some form of an uncertainty relation. The Heisenberg's uncertainty relation between energy and time, ($\Delta E. \Delta\tau = 1$), in the context of a quantum universe would involve the total energy and the time elapsed since its creation. The total energy, E in a physical volume $V_{\text{phy.}}$ can be expressed as

$$E = T_{00} V_{\text{phy.}} = T_{00} S^3(\tau) V_c \quad (6.2a)$$

where V_c is a constant. At early epoch (around the planck era)

$$\Delta\tau \approx \tau, \quad (6.2b)$$

which in conjunction with the uncertainty principle and equation (6.2) would lead to the following relation

$$T_{00} = \frac{K}{\tau S^3(\tau)}, \quad (K \equiv \text{constant}). \quad (6.3)$$

Now we try to bridge the gap between the quantum universe created and the classical universe that follows later, by postulating that the classical homogeneous FLRW spacetime that emerges is dictated by the energy density given by equation (6.3). For a flat FLRW metric (6.1), equation (6.3) takes the form

$$\frac{1}{S^2(\tau)} \left(\frac{dS(\tau)}{d\tau} \right)^2 = \frac{K^3}{S^3(\tau)\tau}; \quad (6.4a)$$

which determines $S(\tau)$ (assuming $S(\tau) = 0$ at $\tau = 0$) to be

$$S(\tau) = K^{\frac{1}{3}} \tau^{\frac{1}{3}}. \quad (6.4b)$$

The above corresponds to the case of a universe filled with a hydrodynamic fluid with a stiff equation of state ($p = \rho$). The corresponding conformal factor $\Omega_o(t)$ reads

$$\Omega_o(t) = 2\sqrt{3}K^{\frac{1}{2}}t^{\frac{1}{2}}. \quad (6.4)$$

This would also change the evolution equation for $\phi(\mathbf{x}, t)$ since the form of T_{00} given by (6.1) would imply a particular form of $V(\Omega)$ in (5.3). The above consideration is more in the vein of illustration and one can generalise it to Ω_o of the form $\Omega_o(t) \propto t^\gamma$ to cover a broader range of possibilities (e.g. $\gamma = -1 \Rightarrow$ De Sitter spacetime).

7. Discussion

We have extended the concept of quantum conformal cosmology originally put forward by Narlikar and Padmanabhan (1983) to resolve the problems of the singularity, horizon and flatness problems in classical cosmology, to study the origin of density perturbations in our universe.

The scenario outlined harnesses the quantum fluctuations in the gravitational sector (e.g., see Halliwell 1985) in contrast to the inflationary ones which use quantum fluctuations in the matter sector. Furthermore, in our case the perturbations are not generated through microphysical processes, in fact the universe is ‘born’, with a high probability, as a flat FLRW spacetime with weak inhomogeneities on large scales. This approach therefore circumvents the need for inflation wherein causally produced small scale quantum fluctuations are stretched to exponentially large wavelengths which are then argued to be mimicing a classical, weakly inhomogeneous field (Vilenkin 1983; Brandenberger 1990). Furthermore, the identification of a classical field with the expectation values of the quantised (conformal) field is well defined in our scenario since we are dealing with coherent states.

The power spectrum of the inhomogeneties, $\phi(\mathbf{x}, t)$, in the conformal factor is directly related to the mass fluctuations $\delta M/M$ in the conformally non-invariant component of matter. This leads to a scale invariant spectrum (*Harrison-Zeldovich spectrum*) for $\delta M/M$ for the large wavelength modes which in our case are the astrophysically relevant scales. The conformal fluctuations are recognised to be of the isocurvature kind and here we have studied their evolution for conformal matter as well as for the case of quadratic coupling of the conformal sector of gravity to the matter Lagrangian.

At this stage, we would like to clarify that the often quoted unphysical nature of the conformal sector of gravity stems from considerations of a Euclideanised quantum gravity. The fact that, akin to theories with guage freedom, the kinetic term in the conformal sector (in euclideanised gravity) appears with a negative sign has prompted attempts to factor out the conformal factor and treat it like a non-dynamical (unphysical) degree of freedom (for e.g., see Mazur and Mottola 1990). In our case, we do not use any form of Euclideanisation and besides, it is easy to see that the conformal factor is as physical as the scale factor of the 3-hypersurfaces in FLRW spacetime (see equation

(6.1)) since the two are related by a gauge transformation (the choice of the lapse function in a 3+1 York decomposition).

Our discussion in this paper illustrates that quantum conformal cosmology, albeit arising out of a simplistic model of quantum gravity, can address all the long standing problems in big bang cosmology. We would conclude that though inflation is an attractive concept, it is not indispensable and one should keep an open mind for other alternative scenarios for solving the outstanding problems of classical big bang cosmology.

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Appendix : Energy in a wavepacket

Consider a state $\Psi_f[\Omega]$, a functional of the conformal factor, to be strongly peaked around $\langle \Omega \rangle_f = \Omega_o(t) + \epsilon \phi(\mathbf{x}, t)$. In the Fourier domain, this state is represented by a functional of the Fourier coefficients $Q_{\mathbf{k}}$ of $\Omega(\mathbf{x}, t)$ as

$$\tilde{\Psi}_f[Q_{\mathbf{k}}] = \mathcal{N} \exp \left[-\frac{3}{8\pi} \int |Q_{\mathbf{k}} - q_{\mathbf{k}}|^2 \frac{d^3 \mathbf{k}}{(2\pi)^3} \right] \quad (A.1)$$

which is peaked around $q_{\mathbf{k}}$, the Fourier coefficient of $\phi(\mathbf{x}, t)$.

The expression for $\langle T_{00}(\mathbf{x}) \rangle$ given in equation (3.7), rewritten in the Fourier domain reads

$$\langle T_{00}(\mathbf{x}) \rangle = \int \mathcal{D}\Omega \Psi_f^*[\Omega] \hat{\mathcal{H}}_{\Omega} \Psi_f[\Omega] = \int \mathcal{D}Q_{\mathbf{k}} \tilde{\Psi}_f^*[Q_{\mathbf{k}}] \hat{\mathcal{H}}_{Q_{\mathbf{k}}} \tilde{\Psi}_f[Q_{\mathbf{k}}] , \quad (A.2)$$

where the Hamiltonian operator,

$$\hat{\mathcal{H}}_{\Omega} = -\frac{l_p^2}{2} \frac{\delta^2}{\delta \Omega^2} + \frac{1}{6 l_p^2} \Omega_i \Omega^i + \frac{1}{12 l_p^2} R \Omega^2. \quad (A.3)$$

Since we are dealing with spacetimes which are close to $R = 0$, the third term will be ignored in the following calculation.

In the Fourier domain

$$\hat{\mathcal{H}}_{Q_{\mathbf{k}}} = \int \int \frac{d^3 \mathbf{k}_1}{(2\pi)^3} \frac{d^3 \mathbf{k}_2}{(2\pi)^3} e^{i(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{x}} \left[-\frac{l_p^2}{2} \frac{\delta^2}{\delta Q_{\mathbf{k}_1} \delta Q_{\mathbf{k}_2}} + \frac{1}{6 l_p^2} \mathbf{k}_1 \cdot \mathbf{k}_2 Q_{\mathbf{k}_1} Q_{\mathbf{k}_2} \right]. \quad (A.4)$$

The expectation value of the energy density can be evaluated by substituting equations (A.1) and (A.4) into equation (A.2). The steps in the calculation involve standard functional differentiations and the following result of functional integration (Freidrich 1976),

$$\begin{aligned} \int \mathcal{D}\theta(s) \left(\prod_s \sqrt{\frac{\mu(ds)}{2\pi}} e^{-\frac{1}{2} \theta^2(s) \mu(ds)} \int \int \mu(ds_1) \mu(ds_2) B(s_1, s_2) \theta(s_1) \theta(s_2) \right. \\ \left. = \int \mu(ds) B(s, s) , \right. \end{aligned} \quad (A.5a)$$

where $\mu(ds)$ is a measure defined such that

$$\int_{-\infty}^{\infty} \sqrt{\frac{\mu(ds)}{2\pi}} e^{-\frac{1}{2}\theta^2(s)} \mu(ds) = 1 . \quad (A.5b)$$

In our case, $\tilde{\Psi}_f[Q_{\mathbf{k}}]$ being a normalised gaussian wavepacket, equation (A.5) leads to a useful result

$$\int \mathcal{D}f_{\mathbf{k}} \tilde{\Psi}_f[f_{\mathbf{k}}] \int \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \frac{d^3\mathbf{k}_2}{(2\pi)^3} B(k_1, k_2) f_{\mathbf{k}_1} f_{\mathbf{k}_2} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} B(\mathbf{k}, \mathbf{k}) . \quad (A.6)$$

where $f_{\mathbf{k}} = |Q_{\mathbf{k}} - q_{\mathbf{k}}|$. The final expression for $\langle T_{00}(\mathbf{x}) \rangle$ appears as a sum of integrals in the Fourier domain as

$$\langle T_{00}(\mathbf{x}) \rangle = \frac{\epsilon^2}{6 l_p^2} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \frac{d^3\mathbf{k}_2}{(2\pi)^3} e^{i(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{x}} \mathbf{k}_1 \cdot \mathbf{k}_2 q_{\mathbf{k}_1} q_{\mathbf{k}_2} + (\text{other terms}) , \quad (A.7)$$

where only the term that contributes to the energy $\mathcal{E}$ is written out explicitly. The energy

$$\mathcal{E} = \int d^3\mathbf{x} \langle T_{00}(\mathbf{x}) \rangle = \frac{\epsilon^2}{6 l_p^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} k^2 q_{\mathbf{k}} q_{-\mathbf{k}} , \quad (A.8)$$

is obtained using equation (3.8), by integrating $\langle T_{00}(\mathbf{x}) \rangle$ over all space.

We arrive at the result that, corresponding to every $\Omega_f(\mathbf{x}, t)$, there exists an energy functional $\mathcal{E}[\Omega_f]$ or equivalently $\mathcal{E}[q_{\mathbf{k}}]$. The equation (A.8) can be expressed formally as

$$\mathcal{E} = \sum_{\mathbf{k}} \frac{\epsilon^2}{8\pi} k^2 q_{\mathbf{k}}^2 \equiv \sum_{\mathbf{k}} \mathcal{E}_{\mathbf{k}} . \quad (A.9)$$

Clearly, $\mathcal{E}$ is a Gaussian random variable with a probability distribution proportional to that of $\Omega_f(\mathbf{x}, t)$

$$\mathcal{P}[\mathcal{E}] \propto \mathcal{P}[q_{\mathbf{k}}] = \prod_{\mathbf{k}} \exp \left[-16\pi^2 \frac{\mathcal{E}_{\mathbf{k}}}{|\mathbf{k}|} \right] , \quad (A.10)$$

where we have used equations (3.3) and (A.9).

The mean square fluctuation in energy in $\phi(\mathbf{x}, t)$ would be

$$\langle \mathcal{E}^2 \rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left(\frac{k}{16\pi^2} \right) \propto \lambda^{-4} , \quad (A.11)$$

λ being a cut off length scale for the above integral.

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