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PATH AMPLITUDES FOR DIRAC PARTICLES*

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1. Introduction. The path integral approach of Feynman [1] provides an elegant link between the classical and the quantum physics. This approach takes as its starting point, the classical action S describing the physical system. In general the state of the system can be described by a point in a suitably defined phase space. As the state changes, the point moves along some path Γ. Although geometrically, any number of paths may be possible, classical physics requires the system to follow a unique path Γ_c, which satisfies the principle of stationary action:

$$\delta S = 0. \tag{1}$$

Here δS represents the change in S in going from Γ_c to another path in its neighbourhood. The uniqueness of Γ_c is responsible for the determinacy associated with classical physics.

In quantum physics (1) is replaced by a less definitive statement. Suppose the system is initially in a state described by point 1 and finally in a state described by point 2. Consider all the geometrically possible paths from 1 to 2. For each path Γ we can compute the action functional $S(\Gamma)$. Now a quantum system is permitted to follow the path Γ, even though $\Gamma \neq \Gamma_c$. There is, however, a probability amplitude that the system will follow the path Γ. Feynman gives a simple rule for computing the amplitude:

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$$P(\Gamma) = (\text{Constant}) \exp \{iS(\Gamma)/\hbar\}, \tag{2}$$

where $\hbar$ is Planck's constant divided by 2π. The constant in (2) can be determined by normalizing probabilities. If we are only interested in the total amplitude that the system starts at 1 and ends at 2, this is given by summing (2) over all permissible paths Γ. The final answer would depend on points 1 and 2 and may be written as $K(2, 1)$. Thus

$$K(2, 1) = \sum_{\Gamma} (\text{Constant}) \exp (iS/\hbar) = \int \exp\{iS(\Gamma)/\hbar\} \mathscr{D}\Gamma, \tag{3}$$

where the summation over Γ is replaced by an integral since usually there are uncountably many paths Γ. The constant factor has been absorbed in the measure of the path integral.†

The connection between the quantum and classical theories is now easy to establish. The latter follows from the former in the limit $\hbar \to 0$. When $\hbar \to 0$, the ratio $S/\hbar \to \infty$ in general, and the phase of the exponential $\exp (iS/\hbar)$ changes rapidly even if S changes slowly from path to path. The path integral (3) may be approximated by the method of stationary phase, the significant contributions to $K(2, 1)$ coming from those paths for which $\delta S \approx 0$. In the limit $\hbar \to 0$ we finally arrive at a unique path Γ_c which satisfies (1).

This approach not only brings out the connection between the classical and the quantum physics, but it also throws some light on why the principle of stationary action plays such an important part in the various branches of classical physics. The basic idea described above had been qualitatively stated first by Dirac [2]. By giving it a quantitative form as in (3), Feynman was able to connect it up with the more conventional Schrödinger approach to quantum mechanics.

In spite of its advantages, and the many applications [*cf.* (3) for details], the path integral picture is not widely used in quantum

†The definition of measure remains one of the difficult problems of path integral theory. Feynman was, however, able to arrive at important results without giving a precise general definition of measure.

theory. One reason is, that while it clarifies many of the conceptual difficulties of the non-relativistic theory, it is not so successful in describing the relativistic spin half particles. Since the concept of spin is lacking in the classical theory, and hence in the classical action, it is not obvious, how to define path amplitudes for spin half particles which satisfy the Dirac equation. In the present paper we shall discuss ideas which may lead to the solution of this formidable problem. To begin with, we shall consider a simplified problem, that of a free particle moving in one space and one time dimensions. In the rest of the paper we shall take $\hbar = 1$, and the velocity of light $c = 1$.

2. Motion in one space + one-time dimensions. In their book *Quantum Mechanics and Path Integrals*, Feynman and Hibbs [4] discuss the motion of a Dirac particle in one space-like and one time-like dimension. Instead of giving a rule like (2) they give another which involves only those paths which are made of null segments. Briefly the rule may be described as follows.

Suppose the particle of mass m moves backward and forward in x-direction, starting at $x = 0$ at $t = 0$, and ending at $x = X$ at $t = T$, where $|X| \leqslant T$. Divide the interval $[0, T]$ into a large number n of small intervals of ϵ-duration, so that

$$n\epsilon = T. \tag{4}$$

The particle is allowed to move only with the speed of light, so that if at the end of rth interval it is at X_r, then $|X_{r+1} - X_r| = \epsilon$, for $1 \leqslant r < n$. Suppose in the entire interval $[0, T]$, the particle goes forward on n_1 occasions and backwards on n_2 occasions. Then

$$n_1 + n_2 = n = \frac{T}{\epsilon},\ n_1 - n_2 = \frac{X}{\epsilon}. \tag{5}$$

A typical path, shown in Fig. 1, will therefore have null segments meeting in sharp corners. The amplitude for a path with R corners is given to be

$$(i\, m\, \epsilon)^R. \tag{6}$$

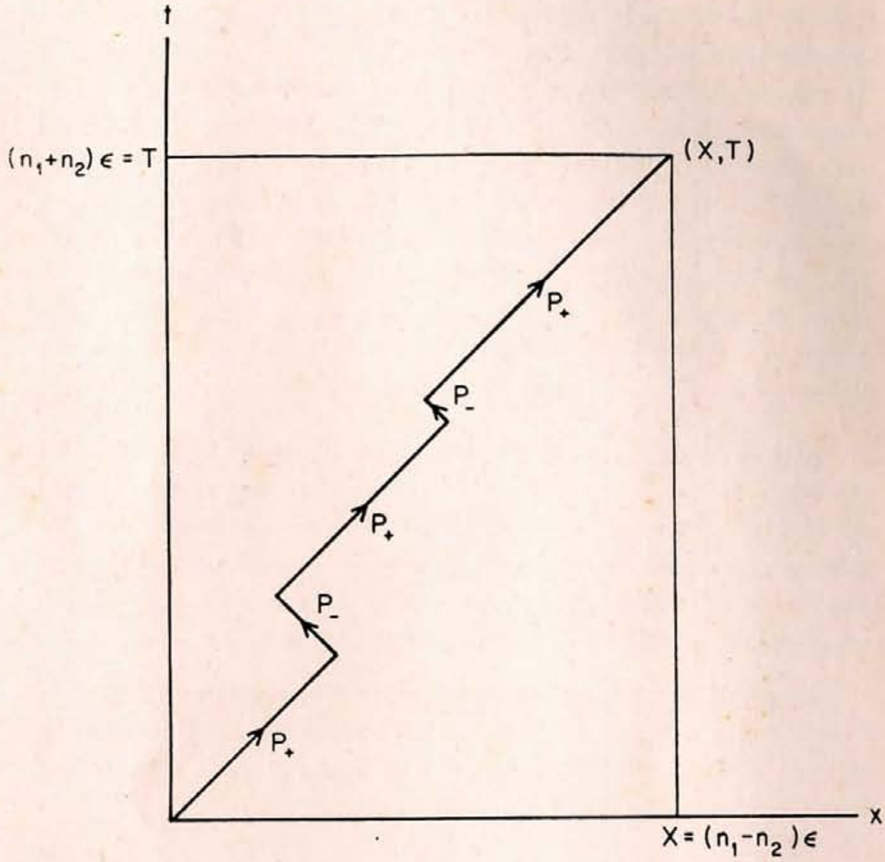

Fig. 1. Typical path of a particle moving backward and forward with the velocity of light. The amplitude for this path is given by $P_+\gamma_4\, P_-\gamma_4\, P_+\gamma_4\, P_-\gamma_4\, P_+$.

The propagator from $[0, 0]$ to $[X, T]$ is then obtained by summing over all paths an expression like (6).

As mentioned above this rule appears to differ from that given in (2) in a radical way and seems to have an ad-hoc character about it. We shall show how the connection between the two rules may be established, and how we can arrive at the Dirac propagator from this picture.

First, we note that just as $K(2, 1)$ in (3) is obtained from $P(\Gamma)$, we can obtain $P(\Gamma)$ from $K(2, 1)$. This is done as follows. Divide the path Γ into a large number of small segments, denoting the intermediate points of division by $X_1, X_2, \ldots, X_{n-1}$, with X_0, X_n standing for the end points 1, 2. Consider the product

$$\prod_{r=1}^{n} \frac{1}{A_r} K(X_r, X_{r-1}), \tag{7}$$

where A_r represents some measure factor.† For suitably chosen A_r, the product (7) tends to $P(\Gamma)$ as $n \to \infty$, *i.e.* as the division becomes finer and finer. Thus we can build up $P(\Gamma)$ from a chain of K's. We can get back to $K(2, 1)$ by summing (7) over all paths, and using the property

$$K(X_{r+1}, X_{r-1}) = \int K(X_{r+1}, X_r)\, K(X_r, X_{r-1})\, d^3\mathbf{x}_r, \tag{8}$$

where the integration is over the space coordinates $\mathbf{x}_r$ of X_r.

This method can be easily extended to the problem in question. We need $K(X_r, X_{r-1})$ for the case where X_r, X_{r-1} are close to each other. The $K(2, 1)$ in $1 + 1$ dimensions satisfies the inhomogeneous Dirac equation

$$\left[\gamma_4 \frac{\partial}{\partial t} - \gamma_1 \frac{\partial}{\partial x} + im \right] K = \delta_2(x, t), \tag{9}$$

where, for convenience we have taken the coordinates of 1 at $[0, 0]$ and of 2 at $[x, t]$. To solve (9) write

$$K = \left(\gamma_4 \frac{\partial}{\partial t} - \gamma_1 \frac{\partial}{\partial x} - im \right) I(x, t), \tag{10}$$

where

$$\frac{\partial^2 I}{\partial^2 t} - \frac{\partial^2 I}{\partial x^2} + m^2 I = \delta(x)\, \delta(t). \tag{11}$$

In analogy with the non-relativistic case, we want a solution that vanishes for $t < 0$. This has been worked out in Appendix A. The result is

$$I(x, t) = \tfrac{1}{2}\, \theta(t)\, \theta(S^2)\, J_0(mS), \quad s^2 = t^2 - x^2, \tag{12}$$

†This factor must have the dimensions $(\text{length})^{-3}$ to make (7) dimensionless.

where θ is the Heaviside function and J_0 the Bessel function of order zero.

The $K(2, 1)$ obtained in this way satisfies a relation similar to (8):

$$K(X_{r+1}, X_{r-1}) = \int K(X_{r+1}, X_r)\, \gamma_4\, K(X_r, X_{r-1})\, d^3\mathbf{x}_r \qquad (13)$$

where the factor γ_4 is necessary to preserve spinor-covariance. For any path Γ, we can therefore form a product similar to that in (7), but with a γ_4 appearing between successive factors.

In forming this product we need $K(X_r, X_{r-1})$ when the points are close together. Since the only length scale appearing in this problem is m^{-1}, we need the approximate form of K for $|X_r - X_{r-1}| \ll m^{-1}$. We therefore look at K given by (10) and (12) in the case where $0 < t = \epsilon$ and $m\epsilon \ll 1$. Since $s < \epsilon$, we have $ms \ll 1$ and $J_0(ms) \approx 1$.

$$K \approx \left(\gamma_4 \frac{\partial}{\partial t} - \gamma_1 \frac{\partial}{\partial x} - im\right) [\tfrac{1}{2}\, \theta(t+x) - \tfrac{1}{2}\, \theta(x-t)]$$

$$= \tfrac{1}{2}(\gamma_4 - \gamma_1)\, \delta(t+x) + \tfrac{1}{2}(\gamma_4 + \gamma_1)\, \delta(t-x) - \frac{im}{2}[\theta(t+x) - \theta(x-t)] \qquad (14)$$

The two delta functions in (14) indicate that most of the amplitude is concentrated in the two directions $x = t = \epsilon$, $x = -t = -\epsilon$. To obtain the magnitude of this concentration we integrate K over $0 \leqslant x < \infty$ and over $-\infty < x \leqslant 0$ at $t = \epsilon$. We get respectively

$$P_+ = \tfrac{1}{2}(\gamma_4 + \gamma_1) - \frac{im\epsilon}{2},\ P_- = \tfrac{1}{2}(\gamma_4 - \gamma_1) - \frac{im\epsilon}{2}. \qquad (15)$$

Although the $-\, im\epsilon/2$ term really represents amplitude over $0 \leqslant x \leqslant \epsilon$, we may lump it all at the end $x = \epsilon$ and call P_+ as the amplitude along $x = +\epsilon$. P_- similarly represents amplitude along $x = -\epsilon$. Since $m\epsilon \ll 1$, the error involved is slight.

We now have passed from a continuous set of paths to a discrete set, as visualized in the beginning of this section. A typical path is made up of null segments like $x = \pm t$, and the amplitude along such a path is given by a series of factors P_+, P_- with γ_4 in between. The following types of combinations would appear in a typical product:

$$P_+ \gamma_4 P_+, P_- \gamma_4 P_-, P_+ \gamma_4 P_-, P_- \gamma_4 P_+. \tag{16}$$

From (15) we get to order $(m\epsilon)$,

$$\left.\begin{array}{ll} P_+ \gamma_4 P_+ = P_+, & P_- \gamma_4 P = P_-, \\ P_+ \gamma_4 P_- = (-im\epsilon)\, \gamma_4 P_-, & P_- \gamma_4 P_+ = (-im\epsilon)\, \gamma_4 P_+. \end{array}\right] \tag{17}$$

Thus whenever two consecutive segments are in the same direction the amplitude is unaffected. When they are in opposite direction, we get a product $(-im\epsilon)\,\gamma_4$. This explains why the rule given earlier in this section made use of paths with corners. The path described in Fig. 1 has the amplitude

$$\begin{aligned} P_+ \gamma_4 P_- \gamma_4 P_+ \gamma_4 P_- \gamma_4 P_+ &= (-im\epsilon)\, \gamma_4 P_- \gamma_4 P_+ \gamma_4 P_- \gamma_4 P_+ \\ &= (-im\epsilon)^2\, \gamma_4 \cdot \gamma_4 P_+ \gamma_4 P_- \gamma_4 P_+ \\ &= (-im\epsilon)^3\, \gamma_4 P_- \gamma_4 P_+ \\ &= (-im\epsilon)^4\, P_+. \end{aligned} \tag{18}$$

Using the product rules (17) it is easy to see that the paths can be divided into four classes $\{\Gamma_{++}\}, \{\Gamma_{--}\}, \{\Gamma_{+-}\}, \{\Gamma_{-+}\}$. The amplitude for a Γ_{++} path begins with P_+ and ends with P_+. The others are similarly defined. The Γ_{++} and Γ_{--} paths have even number of corners whereas Γ_{+-}, Γ_{-+} paths have odd number of corners. The compute the total amplitude we need the total number of paths with a given number of corners.

Let $N_{++}\,(2R), N_{--}\,(2R)$ respectively denote the number of Γ_{++} and Γ_{--} paths with $2R$ corners. Similarly let $N_{+-}\,(2R+1)$ and $N_{-+}\,(2R+1)$ denote the number of Γ_{+-} and Γ_{-+} paths with $2R+1$ corners. Then the total amplitude along paths with $R \geqslant 1$ corners is given by

$$\begin{aligned} Q &= \sum_{R\geqslant 1} \{N_{++}(2R)\, P_+ + N_{--}(2R)\, P_-\}\, (-im\epsilon)^{2R} \\ &= \sum_{R\geqslant 0} \{N_{+-}(2R+1)\, \gamma_4 P_- + N_{-+}(2R+1)\, \gamma_4 P_+\}(-im\epsilon)^{2R+1}. \end{aligned} \tag{19}$$

We are interested in Q as $n \to \infty$, $\epsilon \to 0$. However, to obtain the propagator from [0, 0] to [X, T] we must divide Q by 2ϵ, since the above amplitude is distributed over an interval $\pm\, \epsilon$ about [X, T]. Also, we must add the contribution from paths with *no* corners. We shall perform this calculation now. In the limit $n \to \infty$ we have

$$N_{++}(2R) = \frac{(n_1-1)!\,(n_2-1)!}{(n_1-R)!\,(n_2-R+1)!\,R!\,(R-1)}$$
$$\sim \frac{n_1^R n_2^{R-1}}{R!\,(R-1)!} = \frac{(T^2-X^2)^{R-1}(T+X)}{(2\epsilon)^{2R-1}} \cdot \frac{1}{R!\,(R-1)!},$$

$$N_{--}(2R) \sim \frac{(T^2-X^2)^{R-1}(T-X)}{(2\epsilon)^{2R-1}} \cdot \frac{1}{R!\,(R-1)!},$$

$$N_{+-}(2R+1) = \frac{n_1!\,n_2!}{(n_1-R)!\,(n_2-R)!\,R!\,R!}$$
$$\sim \frac{n_1^R\, n_2^R}{R!\,R!} \sim \frac{(T^2-X^2)^R}{(2\epsilon)^{2R}} \cdot \frac{1}{R!\,R!},$$

$$N_{-+}(2R+1) \sim \frac{(T^2-X^2)^R}{(2\epsilon)^{2R}} \cdot \frac{1}{R!\,R!}. \tag{20}$$

Using these approximations, and the following power series expansions for Bessel functions

$$J_0(\xi) = \sum_{R\geqslant 0} \frac{(-1)^R(\xi/2)^{2R}}{R!\,R!},\; J_0'(\xi) = \sum_{R\geqslant 1} \frac{\xi}{2} \cdot \frac{(-1)^R(\xi/2)^{2R-2}}{R!\,(R-1)!}, \tag{21}$$

we get

$$Q = \epsilon\left[m J_0'(mS).\,\frac{\gamma_4 T + \gamma_1 X}{S} - im\,\{J_0(mS)-1\}\right], \tag{22}$$

where

$$S^2 = T^2 - X^2. \tag{23}$$

(22) may be rewritten in the form

$$Q = 2\epsilon\left(\gamma_4\frac{\partial}{\partial T} - \gamma_1\frac{\partial}{\partial X} - im\right)\left[\tfrac{1}{2}\{J_0(mS)-1\}\right]. \tag{24}$$

Dividing by 2ϵ and adding (14) for $x = X, t = T$ as the contribution for paths with no corners, we get for $T > 0$

$$K[X, T; 0, 0] = \left(\gamma_4 \frac{\partial}{\partial T} - \gamma_1 \frac{\partial}{\partial X} - im \right) \{ \tfrac{1}{2} J_0(mS)\, \theta(S^2) \}. \quad (25)$$

Thus we have a self-consistent picture in which the finite amplitude along a path can be built out of a series of infinitesimal propagators and the finite propagator is then obtained by summing the amplitude over all paths. The main difference between the relativistic and the non-relativistic case is that in the former case paths making a significant contribution to the amplitude are built out of null segments. In the latter case this is not so. The relativistic picture is consistent with the fact that the eigenvalue of velocity of a Dirac particle is always ± 1.

3. Motion in 3 + 1 dimensions. The above picture can be generalized to $3+1$ dimensions. Given a path Γ we define amplitude along it in terms of a chain of infinitesimal propagators. The propagator is given by the retarded solution of the inhomogeneous Dirac equation,*

$$(\tilde{\nabla}_2 + im)\, K(2, 1) = \delta_4(2, 1) \quad (26)^*$$

where ∇_2 is with respect to the coordinates of point 2. As in the $1+1$ dimensional case we can write

$$K(2, 1) = (\tilde{\nabla}_2 - im)\, I(2, 1) \quad (27)$$

where

$$(\Box_2 + m^2)\, I(2, 1) = \delta_4(2, 1). \quad (28)$$

The retarded solution for $I(2, 1)$ is

$$I(2, 1) = \frac{\theta(t_2 - t_1)}{2\pi} \left[\delta(S_{21}^2) - \frac{m}{2S_{21}} J_1(mS_{21})\, \theta(S_{21}^2) \right], \quad (29)$$

where S_{21}^2 is the square of the invariant distance between the coordinates $(\mathbf{x}_1, t_1)$, $(\mathbf{x}_2, t_2)$ of points 1 and 2. J_1 is the Bessel function of order 1.

The delta-function in (29) again emphasizes the importance of null directions. As in the $1+1$ dimensional case we expect the 'important' paths to be made up of null segments. However, the null directions from a given point are not just two, but uncountably infinite. Hence it is not possible to look at the $3+1$ case in terms of

* For a vector A_i define $\tilde{A}$ as $\gamma^i A_i$.

counting paths with corners. The main features of the problem can, however, be described in terms of a perturbation expansion (cf. [4] for details). We write

$$K(2, 1) = \sum_{n \geqslant 0} K^{(n)}(2, 1), \tag{30}$$

where

$$\tilde{\nabla} K^{(0)}(2, 1) = \delta_4(2, 1), \tag{31}$$

and for $n \geqslant 1$

$$\tilde{\nabla} K^{(n)}(2, 1) = - im\, K^{(n-1)}(2, 1). \tag{32}$$

(31) can be solved in terms of the leading term of (29):

$$K^{(0)}(2, 1) = \tilde{\nabla}\, I^{(0)}(2, 1), I^{(0)}(2, 1) = \theta(t_2 - t_1)\, \frac{\delta(S_{21}^2)}{2\pi}. \tag{33}$$

A general $K^{(n)}(2, 1)$, $n \geqslant 1$ is given by

$$n = 2r\colon K^{(2r)} = (-im)^{2r}\, \tilde{\nabla}_2 \int \ldots \int I^{(0)}(2, P_r)\, I^{(0)}(P_r, P_{r-1}) \ldots \\ \ldots I^{(0)}(P_1, 1)\, d\tau_1 \ldots d\tau_r, \tag{34}$$

$$n = 2r+1\colon K^{(2r+1)} = (-im)^{2r+1} \int \ldots \int I^{(0)}(2, P_r)\, I^{(0)}(P_r, P^{r-1}) \ldots \\ \ldots I^{(0)}(P_1, 1)\, d\tau_1 \ldots d\tau_r, \tag{35}$$

where r is an integer. A typical $I^{(0)}$ describes propagation along a null segment, and (34), (35) represent summations over paths made up of null segments. An amplitude $(-im)^n$ is associated with the summation for $K^{(n)}$. This is the analogue of the $1 + 1$ dimensional case.

The propagator obtained so far is useful only when considered along with the hole theory. This is because it describes only forward propagation of particles of positive and negative energies. To describe electrons and positrons we need the Feynman propagator

$$K_+(2, 1) = (\tilde{\nabla} - im)\, I_+(2, 1) = (\tilde{\nabla} - im) \times \\ \times \left[\frac{\delta(S_{21}^2)}{4\pi} - \frac{m}{8\pi S_{21}}\, H_1^{(2)}(mS_{21}) \right], \tag{36}$$

where $H_1^{(2)}$ is the Hankel function of the second kind. It was shown in an earlier paper [5] how K_+ arises from $K(2, 1)$ when we take into account the electromagnetic interaction of electrons and positrons. We will therefore not go into those details here.

4. The non-relativistic approximation. The relativistic propagator described above has been obtained by summing amplitude over piecewise continuous paths made up of null segments. The non-relativistic propagator, on the other hand, is obtained by summing over all paths from 1 to 2 which always go forward in time. Also, the amplitude in the latter case is given by (2), *i.e.* by

$$P(\Gamma) = (\text{constant}) \exp\left[+ i \int_{\Gamma} \tfrac{1}{2}\, m x^2 \, dt \right] \tag{37}$$

for a free particle. It is therefore not clear how the non-relativistic case can be obtained from the relativistic one by a suitable approximation. In this section we show how this transition may be made.

First we obtain the non-relativistic forms for the propagators $K(2, 1)$ and $K_+(2, 1)$. For convenience we write

$$T = t_2 - t_1,\, \mathbf{X} = \mathbf{x}_2 - \mathbf{x}_1,\, X = |\mathbf{X}|. \tag{38}$$

The non-relativistic approximation is given by

$$T \gg X,\ \ mS_{21} \gg 1. \tag{39}$$

We therefore use the asymptotic formulae for J_1 and $H_1^{(2)}$:

$$J_1(mS_{21}) \sim \left(\frac{\pi m S_{21}}{2}\right)^{-1/2} \cos\left(mS_{21} - \frac{3\pi}{4}\right), \tag{40}$$

$$H_1^{(2)}\,(mS_{21}) \sim \left(\frac{\pi m S_{21}}{2}\right)^{-1/2} \exp\left\{ - i \left(mS_{21} - \frac{3\pi}{4}\right)\right\}, \tag{41}$$

and also use the approximations for S_{21}:

$$S_{21} \approx T,\ S_{21} \approx T - \frac{X^2}{2T} \tag{42}$$

respectively in the first and second factors of (40) and (41).

It is convenient to use the Dirac representation

$$\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \tag{43}$$

and write $K(2, 1)$, $K_+(2, 1)$ explicitly in matrix form. [The $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$ is the set of Pauli matrices.] We then have

$$K(2,1) \sim \left(\frac{m}{2\pi T}\right)^{3/2} \times$$

$$\times \begin{bmatrix} i^{-3/2} \exp\left(-mT + \frac{mX^2}{2T}\right) i & -\frac{\boldsymbol{\sigma}\,.\,\mathbf{x}}{T} \sin\left(mT - \frac{mX^2}{2T} - \frac{3\pi}{4}\right) \\ \frac{\boldsymbol{\sigma}\,.\,\mathbf{x}}{T} \sin\left(mT - \frac{mX^2}{2T} - \frac{3\pi}{4}\right) & -(-i)^{-3/2} \exp\left(mT - \frac{mX^2}{2T}\right) i \end{bmatrix} \tag{44}$$

and

$$K_{+}(2,1) \sim \begin{bmatrix} 1 & -\frac{\boldsymbol{\sigma}.\mathbf{x}}{2T} \\ \frac{\boldsymbol{\sigma}.\mathbf{x}}{2T} & 0 \end{bmatrix} . \left(\frac{m}{2\pi i T}\right)^{3/2} \exp -i\left(mT - \frac{mX^2}{2T}\right), \tag{45}$$

for $T > 0$. $K_{+}(2, 1)$ for $T < 0$ can be written down similarly, while $K(2, 1)$ for $T < 0$ is zero.

The non-relativistic form of the Dirac equation separates the wave-function into a large part and a small part. The large part is propagated essentially by the top left-hand element of the propagator. This, we see, is

$$\left(\frac{m}{2\pi i T}\right)^{3/2} \exp -i\left(mT - \frac{mX^2}{2T}\right). \tag{46}$$

(46) is just the non-relativistic propagator for Schrödinger equation.

We now give a rule for computing path amplitudes which leads to $K(2, 1)$ or $K_{+}(2, 1)$. The rule is obtained in the following way. The action for a relativistic particle is given by

$$S = -\int_{\Gamma} m ds, \tag{47}$$

where

$$ds^2 = dt^2 - d\mathbf{x}^2. \tag{48}$$

If we use (47) in (2), we will not arrive at a description of the Dirac particle — because (47) does not contain spin. To include spin we have to take the square root of (48) in the space of 4×4 matrices:

$$dt^2 - d\mathbf{x}^2 = (v_4\, dt - \boldsymbol{\gamma} \,.\, d\mathbf{x})^2. \tag{49}$$

This is analogous to

$$\Box \equiv \tilde{\nabla}^2 \tag{50}$$

which led to the Dirac equation. Thus (47) is replaced by

$$S = -\int_\Gamma m\,(\gamma_4 - \boldsymbol{\gamma} .\ \mathbf{x})dt. \tag{51}$$

In the non-relativistic approximation Γ will be an arbitrary path going forward in time.

However, the amplitude along Γ is not given by

$$P(\Gamma) = (\text{constant}) \,.\, \exp\{-i\int m\,(\gamma_4 - \boldsymbol{\gamma} . \mathbf{x})\, dt\}. \tag{52}$$

The reason is that the right-hand side of (52) is independent of the path and depends only on end points. Such a prescription will not lead to a satisfactory quantum theory. Instead, we need a path-dependent amplitude. To achieve this we divide the path Γ into a large number of small segments and use (52) for each small segment. The $P(\Gamma)$ is then obtained by the ordered product of the amplitudes along the segments. If we write $\Gamma = \Gamma_1 + \Gamma_2 + \ldots + \Gamma_n$, and Γ_r is a typical segment, then

$$P(\Gamma) = \prod_r P(\Gamma_r) = \prod\, (\text{constant}) \,.\, \exp.\{-i\int_{\Gamma_r} m(\gamma_4 - \boldsymbol{\gamma} . \mathbf{x})dt\}. \tag{53}$$

Since for matrices A, B, the law

$$\exp\,(A + B) = (\exp A) \,.\, (\exp B) \tag{54}$$

does not hold, the expression (53) as $n \to \infty$ is dependent on the particular path Γ.

It should be emphasized that the structure associated with this law is cruder than that discussed in the earlier section. The path Γ here can be approximated by another made up of null segments on a much finer scale. The amplitude along Γ is thus the sum of all such finer scale paths computed according to the last section. The above rule is at best an approximation that will be shown to work well.

We shall take Γ to be between $0 \leqslant t \leqslant T$ and let $P(t)$ denote the amplitude along the section of the path from 0 to t. Then it is easy to see that (53) corresponds to

$$\frac{dP}{dt} = -im(\gamma_4 - \boldsymbol{\gamma} \cdot \dot{\mathbf{x}})\, P \tag{55}$$

where $\mathbf{x}(t)$ is the position of a typical point on Γ. The constant in (53) is taken to be unity.

It is easy to verify that (52) is not an integral of (55) unless $\mathbf{x}$ is constant. In that case the law (54) holds. We shall return to this special case later. We write P as a 4×4 matrix:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{56}$$

where $P_{ij}(i, j = 1, 2)$ are 2×2 matrices. Using the representation (43) and writing $\mathbf{v} = \dot{\mathbf{x}}$, (55) takes the form

$$\dot{P}_{11} = -im\,(P_{11} - \boldsymbol{\sigma} \cdot \mathbf{v}\, P_{21}),\ \ \dot{P}_{21} = im\,(P_{21} - \boldsymbol{\sigma} \cdot \mathbf{v}\, P_{11}), \tag{57}$$

$$\dot{P}_{12} = -im\,(P_{12} - \boldsymbol{\sigma} \cdot \mathbf{v}\, P_{22}),\ \ \dot{P}_{22} = im\,(P_{22} - \boldsymbol{\sigma} \cdot \mathbf{v}\, P_{12}). \tag{58}$$

Initially we take $P_{11} = 1$. The initial values of P_{12}, P_{21}, P_{22} also need to be specified in order to complete the problem. It turns out that these are crucial in determining whether we finally arrive at (44) or (45). We shall settle this question at a later stage. To solve (57), put

$$\xi = P_{11}\, e^{imt},\ \eta = P_{21}\, e^{-imt}. \tag{59}$$

Then we get

$$\dot{\xi} = ime^{2imt}(\boldsymbol{\sigma} \cdot \mathbf{v})\, \eta,\ \dot{\eta} = -\, ime^{-2imt}(\boldsymbol{\sigma} \cdot \mathbf{v})\, \xi. \tag{60}$$

(60) can be solved in terms of the expansions

$$\xi = \sum_{n=0}^{\infty} \xi_{2n},\quad \eta = \sum_{n=0}^{\infty} \eta_{2n+1}, \tag{61}$$

where $\xi_0 = 1$ and for $n \geqslant 1$,

$$\dot{\xi}_{2n} = ime^{2imt}(\boldsymbol{\sigma} \cdot \mathbf{v})\, \eta_{2n-1}, \tag{62}$$

$$\dot{\eta}_{2n+1} = -ime^{-2imt}(\boldsymbol{\sigma} \cdot \mathbf{v})\, \xi_{2n-2}. \tag{63}$$

Given the initial conditions, and the function $v(t)$, we can solve (62), (63) by iteration.

In the non-relativistic approximation we have

$$|\mathbf{v}| \ll 1, \; |\dot{\mathbf{v}}| \ll m\,|\mathbf{v}|. \tag{64}$$

The first inequality implies that motion is slow compared with the speed of light. The second inequality means that the time scale over which velocity changes significantly is large compared to m^{-1}. The latter inequality suggests that the solution of (57) and (58) will be somewhat similar to that for $\mathbf{v} =$ constant. In this case the general solution of (55) is given by

$$P(t) = \exp\{-im(\gamma_4 t - \boldsymbol{\gamma} \cdot \mathbf{x})\}.\, P_0 \tag{65}$$

where P_0 is an arbitrary matrix. Taking $P_0 = 1$ gives

$$P = \begin{bmatrix} \cos mt\sqrt{(1-v^2)} - \dfrac{i \sin mt\sqrt{(1-v^2)}}{\sqrt{(1-v^2)}} & \dfrac{i\,\boldsymbol{\sigma} \cdot \mathbf{v}}{\sqrt{(1-v^2)}} \sin mt\sqrt{(1-v^2)} \\ -\dfrac{i\,\boldsymbol{\sigma} \cdot \mathbf{v}}{\sqrt{(1-v^2)}} \sin mt\sqrt{(1-v^2)} & \cos mt\sqrt{(1-v^2)} + \dfrac{i \sin mt\sqrt{(1-v^2)}}{\sqrt{(1-v^2)}} \end{bmatrix} \tag{66}$$

However, this choice of initial conditions does not give either of K or K_+. As will be seen shortly, to obtain K we need

$$P_0 = \frac{\gamma_4 - \boldsymbol{\gamma} \cdot \mathbf{v}}{\sqrt{(1-v^2)}}, \tag{67}$$

whereas to obtain K_+, P_0 is given by

$$P_0 = \tfrac{1}{2}\left\{ 1 + \frac{\gamma_4 - \boldsymbol{\gamma} \cdot \mathbf{v}}{\sqrt{(1-v^2)}} \right\}. \tag{68}$$

We shall consider these two cases in detail.

In the first case (65) and (67) give, when $|v| \ll 1$,

$$P_{11} = \frac{\cos mt\,\sqrt{(1-v^2)}}{\sqrt{(1-v^2)}} - i\sin mt\,\sqrt{(1-v^2)} \sim \exp[-i\,mt\,\sqrt{(1-v^2)}] \tag{69}$$

$$P_{21} = \frac{\boldsymbol{\sigma}\,.\,\mathbf{v}}{\sqrt{(1-v^2)}}\cos mt\,\sqrt{(1-v^2)} \sim \boldsymbol{\sigma}\,.\,\mathbf{v}\cos mt\,\sqrt{(1-v^2)}. \tag{70}$$

We now turn to the solution of (57) in the non-relativistic case. Guided by (69) and (70), but remembering that v now varies with t, we try

$$P_{11} \sim \exp - im\int_0^t \left\{1 - \frac{v^2(t_1)}{2}\right\} dt_1,$$

$$P_{21} \sim \boldsymbol{\sigma}\,.\,\mathbf{v}(t)\cos m\int_0^t \left[1 - \frac{v^2(t_1)}{2}\right] dt_1. \tag{71}$$

It is easy to verify that (57) is satisfied to within the non-relativistic approximation. P_{22}, P_{12} can be similarly obtained and we get

$$P(t) = \begin{bmatrix} \exp - im\int_0^t \left\{1 - \frac{v^2(t_1)}{2}\right\} dt_1 & -\boldsymbol{\sigma}.\,\mathbf{v}(t)\cos m\int_0^t \left\{1 - \frac{v^2(t_1)}{2}\right\} dt_1 \\ \boldsymbol{\sigma}.\,\mathbf{v}(t)\cos m\int_0^t \left\{1 - \frac{v^2(t_1)}{2}\right\} dt_1 & -\exp im\int_0^t \left\{1 - \frac{v^2(t_1)}{2}\right\} dt_1 \end{bmatrix}. \tag{72}$$

In the same way we can deal with the second case, and get

$$P_+(t) = \begin{bmatrix} 1 & -\frac{1}{2}\,\boldsymbol{\sigma}\cdot\mathbf{v}(t) \\ \frac{1}{2}\,\boldsymbol{\sigma}\,.\,\mathbf{v}(t) & 0 \end{bmatrix} . \exp - im\int_0^t \left\{1 - \frac{v^2(t_1)}{2}\right\} dt_1,\ t > 0. \tag{73}$$

Here we have written P_+ instead of P to distinguish between the two cases.

The propagator K or K_+ would now come out of summation of $P(T)$, $P_+(T)$ over all paths from $(\mathbf{0}, 0)$ to $(\mathbf{X}, T)$. We already know (cf. [3] for details) that the path integral

$$\int \exp -im \int_0^T \left\{ 1 - \frac{\mathbf{v}^2(t_1)}{2} \right\} dt\, \mathscr{D}^3\mathbf{x}(t) = \left(\frac{m}{2\pi i T}\right)^{\frac{3}{2}} \exp\left\{ \frac{imX^2}{2T} - imT \right\}. \tag{74}$$

The following results have been derived in Appendix B:

$$\int \mathbf{v}(T) \exp - im \int_0^T \left\{ 1 - \frac{v^2(t)}{2} \right\} dt\, \mathscr{D}^3\, \mathbf{x}(t)$$
$$= \frac{\mathbf{X}}{T} \left(\frac{m}{2\pi i T}\right)^{3/2} \exp\left\{\frac{imX^2}{2T}\right\} - im\, T \Big\}, \tag{75}$$

and

$$\int \mathbf{v}(T) \cos m \int_0^T \left\{ 1 - \frac{v^2(t)}{2} \right\} dt\, \mathscr{D}^3\, \mathbf{x}(t)$$
$$= \left(\frac{m}{2\pi T}\right)^{3/2} \frac{\mathbf{X}}{T} \sin\left(mT - \frac{mX^2}{2T} - \frac{3\pi}{4} \right). \tag{76}$$

Using these results it is easy to see that

$$K(2,\, 1) = \int P . \mathscr{D}^3\, \mathbf{x}(t), \tag{77}$$

$$K_+(2,\, 1) = \int P_+\, \mathscr{D}^3\, \mathbf{x}(t). \tag{78}$$

We therefore see that the rule for path amplitude given in (53) leads to the correct propagator. To obtain $K(2, 1)$ we use (53) only for forward going paths, with P_0, the initial value given by (67). For $K_+(2, 1)$ we must use (53) for forward as well as backward going paths but with initial condition given by (68). In the non-relativistic case all paths are time like and no problems such as given by pair creation or annihilation are present. To deal with such problems, which arise frequently in electrodynamics, we must use the methods of [5].

5. Conclusion. To summarize, the motion of a Dirac particle may be looked at from two different ends. In the extreme relativistic limit, the velocities are comparable to the velocity of light c, and time scales short compared to m^{-1}. Here the motion is described by paths made of null segments, with many changes of direction occurring in time m^{-1}. The mass of the particle is responsible for changing the direction from one null segment to another. Because of any such changes, the motion of a forward going particle is time like over times large compared to m^{-1}.

In the non-relativistic limit we are concerned with this type of motion. Here paths are time like and do not change directions significantly over times of order m^{-1}. The amplitude in this case can be described by a relatively simple expression of the form $\exp(-im\tilde{q})$ where q^i denotes a small section of the path (compared to m^{-1}). The amplitude along a finite section of the path is given by a product of such factors in the correct order. The sum of amplitudes over all paths leads to the non-relativistic limit of the Dirac or Feynman propagator, *including spin.*

It is instructive to show explicitly the part played by the second inequality of (64), in the non-relativistic limit. This is seen from the iterative solution described in the last section. To fix ideas we will take the initial condition given by $P_0 = 1$. This corresponds to $\eta_1 = 0$ at $t = 0$. Hence for $n \geqslant 1$

$$\xi_{2n}(t) = m^{2n} \int_0^t e^{2imt_1} \boldsymbol{\sigma} . \mathbf{v}(t_1)\, dt_1 \int_0^t e^{-2imt_2} \boldsymbol{\sigma} . \mathbf{v}(t_2)\, dt_2 \ldots$$

$$\ldots \int_0^{t_{2n-2}} e^{2imt_{2n-1}} \boldsymbol{\sigma} . \mathbf{v}(t_{2n-1})\, dt_{2n-1} \int_0^{t_{2n-1}} e^{-2imt_{2n}} \boldsymbol{\sigma} . \mathbf{v}(t_{2n})\, dt_{2n}, \quad (79)$$

$$\eta_{2n}(t) = -im^{2n-1} \int_0 e^{-2imt_1} \boldsymbol{\sigma} . \mathbf{v}(t_1) dt_1 \ldots \int_0^{t_{2n-2}} \boldsymbol{\sigma} . \mathbf{v}(t_{2n-1})\, e^{-2imt_{2n-1}}\, dt_{2n-1}. \quad (80)$$

Consider the last two integrals of (79) taken together. If we integrate by parts,

$$\int_0^{t_{2n-1}} e^{-2imt_{2n}} \boldsymbol{\sigma}.\mathbf{v}(t_{2n})\, dt_{2n} = \frac{i}{2m}\left\{ e^{-2imt_{2n-1}} \boldsymbol{\sigma}.\mathbf{v}(t_{2n-1}) - \boldsymbol{\sigma}.\mathbf{v}(0) \right\} - $$
$$- \frac{i}{2m} \int_0^{t_{2n-1}} e^{-2imt_{2n}} \boldsymbol{\sigma}.\mathbf{v}(t_{2n})\, dt_{2n}. \qquad (81)$$

In this the integral on the right-hand side is less important than the first term because of the second inequality of (64). Further, when we take the first term of the right-hand side along with the integrand of the t_{2n-1} integral we get

$$\int_0^{t_{2n-2}} \frac{i}{2m}, \left\{ v^2(t_{2n-1}) - e^{2mt_{2n-1}} \left[\boldsymbol{\sigma}\mathbf{v}(t_{2n-1}) \right]\left[\boldsymbol{\sigma}.\mathbf{v}(0) \right] \right\} dt_{2n-1}. \quad (82)$$

Again, because of the rapid oscillations of $e^{2imt_{2n-1}}$ the second term of the integrand makes very little contribution and we can approximate (82) by

$$\frac{i}{2m} \int_0^{t_{2n-2}} v^2(t_{2n-1})\, dt_{2n-1}. \qquad (83)$$

Clearly, we can repeat this procedure for the subsequent integrals in $\xi_{2n}(t)$ and get, with redefinition of dummy variables,

$$\xi_{2n}(t) \sim m^{2n}. \left(\frac{i}{2m}\right)^n \int_0^t v^2(t_1)\, dt_1 \int_0^{t_1} v^2(t_2)\, dt_2 \ldots \int_0^{t_{n-1}} v(t_n)\, dt_n$$
$$= \left(\frac{im}{2}\right)^n \cdot \frac{1}{n!} \left\{ \int_0^t v^2(t)\, dt \right\}^n \qquad (84)$$

so that

$$P_{11} = e^{-imt} \sum_0^\infty \xi_{2n}(t)$$
$$\sim \exp\left[-im \int_0^t \left\{ 1 - \frac{v^2(t_1)}{2} \right\} dt_1 \right]. \qquad (85)$$

We can evaluate P_{12} similarly. With suitable account of initial conditions (72) and (73) can be obtained in this way.

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APPENDIX A

To obtain the solution of (11) that vanishes for $t < 0$, put

$$I(x, t) = \iint f(\omega, k)\, e^{+ikx-i\omega t} \cdot \frac{d\omega}{2\pi} \cdot \frac{dk}{2\pi}, \tag{A1}$$

where $-\infty < \omega < \infty$, $-\infty < k < \infty$.

Since

$$\delta(x)\ \delta(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ikx-i\omega t} \frac{d\omega}{2\pi} \frac{dk}{2\pi}, \tag{A2}$$

we get from (11),

$$f(\omega, k) = \frac{1}{m^2 + k^2 - \omega^2}. \tag{A3}$$

Hence

$$I(x, t) = \iint \frac{e^{ikx-i\omega t}}{m^2 + k^2 - \omega^2} \frac{d\omega}{2\pi} \frac{dk}{2\pi}. \tag{A4}$$

To perform the ω integral we use contour integration in the complex x-plane. The poles are at $\omega = \pm \sqrt{(m^2 + k^2)}$. To arrive at a solution which vanishes for $t < 0$, we integrate parallel to real axis with $\omega = \omega_R + i\epsilon$, where ω_R is the real part of ω, and $-\infty < \omega_R < \infty$. For $t < 0$ we can complete the contour by a semicircle at infinity in the upper half of the ω-plane. This contour has no poles and the integral along the semicircle vanishes. For $t > 0$ the contour is completed by a semicircle at infinity in the lower half of the ω-plane. This contour has poles with the residue

$$\frac{e^{ikx+i\sqrt{(m^2+k^2)}t}}{2\sqrt{(m^2+k^2)}} - \frac{e^{ikx-i\sqrt{(m^2+k^2)}t}}{2\sqrt{(m^2+k^2)}}. \tag{A5}$$

The integral along the semicircle vanishes. Hence we get

$$\begin{aligned} I(x, t) &= \frac{-i}{4\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{\sqrt{(m^2+k^2)}} \{e^{i\sqrt{(m^2+k^2)}t} - e^{-i\sqrt{(m^2+k^2)}t}\}\, dk \\ &= \frac{1}{\pi}\int_0^{\infty} \frac{\cos kx \sin \sqrt{(m^2+k^2)}\,t}{\sqrt{(m^2+k^2)}}\, dk. \end{aligned} \tag{A6}$$

We now consider two cases separately: (i) $t^2 > x^2$ and (ii) $t^2 < x^2$. In the first case put

$$t = s\cosh\theta,\ x = s\sinh\theta,\ k = m\sinh\alpha. \tag{A7}$$

Then

$$\begin{aligned} I(x, t) &= \frac{1}{\pi}\int_0^{\infty} \cos\{ms\sinh\theta\sinh\alpha\}\sin\{ms\cosh\theta\cosh\alpha\}\, d\alpha \\ &= \frac{1}{2\pi}\int_0^{\infty} [\sin\{ms\cosh(\theta+\alpha)\} + \sin\{ms\cosh(\theta-\alpha)\}]\, d\alpha. \end{aligned} \tag{A8}$$

In the two terms of the integrand, put $\theta + \alpha = u$, $\theta - \alpha = u$ respectively to get

$$\begin{aligned} I(x, t) &= \frac{1}{2\pi}\int_{\theta}^{\infty} \sin(ms\cosh u)\, du + \frac{1}{2\pi}\int_{-\infty}^{+\theta} \sin(ms\cosh u)\, du \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \sin(ms\cosh u)\, du \\ &= \frac{1}{2} J_0(ms). \end{aligned} \tag{A9}$$

In the second case put

$$t = s\sinh\theta,\ x = s\cosh\theta,\ k = m\sinh\alpha. \tag{A10}$$

Then we get, by proceeding as above

$$I(x,\, t) = \frac{1}{\pi}\int\limits_{0}^{\infty} \cos\,\{ms \sinh\,\alpha \cosh\,\theta)\}\, \sin\,\{ms \cosh\,\alpha \sinh\,\theta\}\, d\alpha$$

$$= \frac{1}{2\pi}\int\limits_{0}^{\infty} [\sin\,\{ms \sinh\,(\theta + \alpha)\} + \sin\, ms\,\{\sinh\,(\theta - \alpha)\}]\, d\alpha$$

$$= \frac{1}{2\pi}\int\limits_{\theta}^{\infty} \sin\,(ms \sinh\, u)\, du + \frac{1}{2\pi}\int\limits_{-\infty}^{\theta} (ms \sinh\, u)\, du$$

$$= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \sin\,(ms \sinh\, u)\, du$$

$$= 0.$$

Hence the result follows :

$$I(x,\, t) = \frac{1}{2} J_0(ms)\; \theta(s^2)\, .\, \theta(t). \tag{A11}$$

APPENDIX B

Here we derive the results quoted in equations (75) and (76) of the main text. Consider a typical path $\mathbf{x}(t)$ from $(0,\, 0)$ to $(x,\, T)$. For small ϵ, suppose $x(t)$ intersects the hyperplane $t = T - \epsilon$ at $\mathbf{y}$ in

$$\mathbf{y} = \mathbf{x}(T - \epsilon). \tag{B1}$$

We can approximate $\mathbf{v}(T)$ by

$$\mathbf{v}(T) \approx \frac{\mathbf{x} - \mathbf{y}}{\epsilon} \tag{B2}$$

Since

$$\exp\, im \int\limits_{0}^{T} \frac{\mathbf{v}^2(t)}{2}\, dt = \left\{\exp\, im \int\limits_{0}^{T-\epsilon} \frac{\mathbf{v}^2(t)}{2}\, dt\right\} . \left\{\exp\, im \int\limits_{T-\epsilon}^{T} \frac{\mathbf{v}^2(t)}{2}\, dt\right\}, \tag{B3}$$

we have

$$\int \mathbf{x}(T) \exp \left\{ -im \int_0^T \left[1 - \frac{\mathbf{v}^2(t)}{2} \right] dt \right\} \mathscr{D}^3\, \mathbf{x}(t)$$

$$= e^{-imT} \int \left(\frac{\mathbf{x}-\mathbf{y}}{\epsilon} \right) \frac{1}{A} \cdot \exp \left[\frac{im(\mathbf{x}-\mathbf{y})^2}{2\epsilon} \right] d^3y \times$$

$$\times \exp \int \left\{ + im \int_0^{T-\epsilon} \frac{\mathbf{v}^2(t)}{2}\, dt \right] \mathscr{D}^3 \mathbf{x}(t). \qquad \text{(B4)}$$

The path integral from 0 to $T-\epsilon$ gives the nonrelativistic free particle propagator, and (B4) becomes

$$e^{-imT} \int \frac{\mathbf{x}-\mathbf{y}}{\epsilon} \cdot \frac{1}{A} \cdot \exp \left[\frac{im(\mathbf{x}-\mathbf{y})^2}{2\epsilon} \right] \cdot \left(\frac{m}{2\pi i(T-\epsilon)} \right)^{3/2} \times$$

$$\times \exp \left[\frac{imy^2}{2(T-\epsilon)} \right] d^3\mathbf{y}. \qquad \text{(B5)}$$

In (B4) and (B5) A is the measure constant, and is given by (*cf.* Feynman and Hibbs [3])

$$A = \left(\frac{2\pi i \epsilon}{m} \right)^{3/2}. \qquad \text{(B6)}$$

Substituting for A, we can evaluate (B5) to get

$$\left(\frac{m}{2\pi i T} \right)^{3/2} \cdot \frac{\mathbf{x}}{T} \cdot e^{-imT + im\mathbf{x}^2/(2T)} \qquad \text{(B7)}$$

This is the same as the right-hand side of (75).

Equation (76) is the real part of (75) and hence we need the real part of (B7). Writing $i = \exp\left(\frac{i\pi}{2}\right)$, we get the real part of (B7) as

$$\mathrm{Re} \left[\left(\frac{m}{2\pi i T} \right)^{3/2} \cdot \frac{\mathbf{x}}{T}\, e^{-imt + imx^2/(2T)} \right]$$

$$= \left(\frac{2\pi T}{m} \right)^{3/2} \cdot \quad \frac{\mathbf{x}}{T} \cos \left(mT - \frac{m\mathbf{x}^2}{2T} + \frac{3\pi}{4} \right)$$

$$= \left(\frac{m}{2\pi T} \right)^{3/2} \cdot \frac{\mathbf{x}}{T} \sin \left(mT - \frac{m^2\mathbf{x}}{2T} - \frac{3\pi}{4} \right) \qquad \text{(B8)}$$

This is the required result.

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