

Notes on Semiclassical Gravity

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In this paper we investigate the different possible ways of defining the semiclassical limit of quantum general relativity. We discuss the conditions under which the expectation value of the energy-momentum tensor can act as the source for a semiclassical, c -number, gravitational field. The basic issues can be understood from the study of the semiclassical limit of a toy model, consisting of two interacting particles, which mimics the essential properties of quantum general relativity. We define and study the WKB semiclassical approximation and the gaussian semiclassical approximation for this model. We develop rules for finding the back-reaction of the quantum mode q on the classical mode Q . We argue that the back-reaction can be found using the phase of the wave-function which describes the dynamics of q . We find that this back-reaction is obtainable from the expectation value of the hamiltonian if the dispersion in this phase can be neglected. These results on the back-reaction are generalised to the semiclassical limit of the Wheeler–DeWitt equation. We conclude that the back-reaction in semiclassical gravity is $\langle T_{ik} \rangle$ only when the dispersion in the phase of the matter wavefunctional can be neglected. This conclusion is highlighted with a minisuperspace example of a massless scalar field in a Robertson–Walker universe. We use the semiclassical theory to show that the minisuperspace approximation in quantum cosmology is valid only if the production of gravitons is negligible. © 1989 Academic Press, Inc.

1. INTRODUCTION AND SUMMARY

This is the third in a series of papers [1, 2] in which we have attempted to find the conditions for the validity of semiclassical Einstein equations, by studying their relation to quantum general relativity. In the course of our investigation we came across a variety of methods for defining classical and semiclassical limits, apparently different, and all of which were possibly applicable to a quantum gravity. It then became necessary to compare these methods and to settle, once and for all, the relation of semiclassical gravity to quantum gravity. We hope to have accomplished this task in the present paper. In this section we discuss the reasons for studying semiclassical gravity, summarize previous work on this subject, and introduce the forthcoming sections.

It is useful to begin by understanding the relation between quantum and classical *mechanics*. A large class of quantum systems have a classical limit. The limit arises if the quantum system is in one of a special class of quantum states. The procedure

for obtaining such a limit is well understood. It may be obtained using Wigner functions for WKB states, through gaussian states or through path integrals. These different ways of defining a classical limit are briefly reviewed and compared in Section 2. All of them amount to taking an $\hbar \rightarrow 0$ limit of the quantum theory. (There is yet another way of obtaining the classical limit, which is conceptually very different and is applicable only to a macroscopic system. This—not so well-known result—is that a macroscopic quantum system can be effectively classical because of its interaction with the environment—a phenomenon which is often called “decoherence.” We give a brief discussion of decoherence at the end of this section.)

A entirely different kind of (intermediate) limit is possible in the quantum theory of a system with two interacting degrees of freedom, say Q and q . In this (semi-classical) limit the motion of Q is classical, whereas the motion of q is quantum mechanical. Just as there is a well-defined approximation under which classical mechanics is an (almost) exact description of nature, there is a well-defined approximation under which semiclassical mechanics is an (almost) exact description of nature. This is the limit in which the ratio (M/m) of the masses of the two interacting particles goes to infinity. Roughly speaking, for a heavy enough mass M , both Δx and $\Delta v \sim \hbar/M\Delta x$ can be small, and its trajectory can be classical, even if it is interacting with a quantum system. We emphasize that all quantum systems having a classical limit *do not* necessarily have a semiclassical limit—the existence of the semiclassical limit depends on the form of the Lagrangian. This is discussed in detail in Section 3.

Before we proceed further, we would like to clarify what we mean by the term “semiclassical.” Throughout our discussion, a semiclassical system is one in which quantum mechanical degrees of freedom interact with classical ones. Often in the literature, the word semiclassical is used to imply “nearly classical.” *We will not adopt this usage.* When we have to say nearly classical, we will say “quasi-classical,” “in the classical limit,” or just “classical.”

There are many examples of semiclassical systems in nature. After all, a quantum system interacting with a classical measuring apparatus belongs to this category. An oft-quoted example is from molecular physics—the motion of an electron of mass m_e in the field of nearly static nuclei of mass m_n ; the mass ratio (m_e/m_n) allows for the classical treatment of the nuclei.

Just as it is possible to define the semiclassical limit of some quantum theories, one can define the semiclassical limit for a certain class of interacting quantum fields—those in which a natural parameter of the kind (M/m) exists. Quantum general relativity with matter fields is definitely one such case, the analogue of (M/m) being the inverse of Newton’s gravitational constant. This is so because the gravitational part of the action is proportional to $(1/G)$, whereas the matter part of the action is independent of G . In the limit $G \rightarrow 0$, classical gravity interacts with quantum matter [1].

It should be noted that the semiclassical limit in, say, quantum electrodynamics, is structurally of a different kind. This is because of the form of the Lagrangian: there is a term for the electrons, a term for the electromagnetic field, and an interac-

tion term proportional to the charge e of the electron. In the limit $e \rightarrow 0$, we have free electrons and free electromagnetic fields. In this limit the quantum theory is a free field theory and the classical limit $\hbar \rightarrow 0$ may be taken either for the electrons, or for the electromagnetic field, or for both. Thus it is possible to treat either the electrons or the electromagnetic field classically, and the other one quantum mechanically. Such a procedure looks very artificial for quantum gravity; we do not know of an approximation to quantum general relativity in which quantum gravity interacts with classical matter fields.

Unlike the classical limit, the semiclassical limit of quantum theories is not well understood. Of particular importance are the following questions: (1) What is the correct mathematical procedure for obtaining the semiclassical limit of an exact quantum theory?; (2) Are the equations $G_{ik} = 8\pi G \langle T_{ik} \rangle$ the correct semiclassical limit to quantum gravity?

Section 3 is devoted to answering the first question and discusses the various possible constructions of the semiclassical limit and their comparison. The different ways of defining the classical limit—WKB state, gaussian state, and Wigner function—are adapted to the semiclassical context. The discussion is with the help of a general two-particle Lagrangian

$$L = \frac{1}{2} M \dot{Q}^2 - MV(Q) + \frac{1}{2} m \dot{q}^2 - u(q, Q). \quad (1.1)$$

The formal structure of quantum general relativity is analogous to that of the above quantum-mechanical system, and hence it proves useful to first work with this simple Lagrangian. The main conclusions on the semiclassical limit of the system (1.1) can be translated to the case of quantum general relativity.

The application of semiclassical methods in the context of quantum gravity is taken up in detail in Section 4. The major reason for studying semiclassical gravity is that we do not have a theory of quantum gravity. This is what led to the study of quantum fields interacting with classical gravity.

The dynamics of quantum fields in curved space has been studied extensively by De Witt [3] and others. In the functional language the propagation of a quantum field ϕ in an external, classical spacetime geometry g is described by the functional Schrodinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi(\phi, g) = \hat{H}(\phi, g) \Psi(\phi, g). \quad (1.2)$$

If the metric g is time-dependent, (1.2) will, in general, not have solutions which are stationary states. In particular, it will not be possible to define a state as the vacuum state, for all times. Another way of saying this could be that the effective action obtained by integrating over the field ϕ will have a real and an imaginary part. The real part can be absorbed in the gravitational action. The imaginary part can possibly be interpreted as the “production of particles” [4] by the time-dependent metric.

One may then ask: what is the effect of the evolution of the wave-functional, and

of “particle production,” on the geometry? A suggestion [5] was that the effect on the geometry should be found by solving the equations

$$G_{ik}(g) = 8\pi G \langle \psi_1(\phi, g) | T_{ik}(\phi) | \psi_2(\phi, g) \rangle \tag{1.3}$$

simultaneously with (1.2). That is, the source term for the Einstein tensor is the transition amplitude of T_{ik} between the states $|\psi_1\rangle$ and $|\psi_2\rangle$. Various possibilities have been suggested for $|\psi_1\rangle$ and $|\psi_2\rangle$. They may both be the “in-vacuum” state, or they may both be the “out-vacuum” state; provided the “in” and “out” vacua can be defined. The path integral derivation of the effective action suggests that $|\psi_1\rangle$ is the in vacuum, and $|\psi_2\rangle$ the out vacuum (see Ref. [1]). Another possibility [5] is that both $|\psi_1\rangle$ and $|\psi_2\rangle$ are the same state $|\psi\rangle$, where $|\psi\rangle$ is a solution of (1.2). The right-hand side is then the expectation value of the energy-momentum tensor in this state. *In this particular case, when the right-hand side of (1.3) is an expectation value, we call the Eqs. (1.3) as the “average energy equations.”*

It is well known [6] that the system of Eqs. (1.2) and (1.3)—classical gravity coupled to quantum fields—cannot be an exact description of nature. The average energy equations can at best only be approximately true. Of course, the important question is whether they are even approximately true—and if so—what is (are) the approximation(s) involved. The correct way to find these approximations is to start from an exact quantum theory. (We believe this to be the correct way because a semiclassical theory, unless related to a quantum theory, is ad hoc.) So, quantum general relativity is the proper place to start from, even if it is not perturbatively renormalizable, even if it does not turn out to be the correct quantum gravity, and even if we do not know how to solve the Wheeler–DeWitt equation.

Many attempts have been made earlier to relate quantum general relativity to semiclassical general relativity. Gerlach [7] was the first to show that if a WKB approximation is made for pure gravity (no matter fields) by expanding the wavefunctional $\Psi(g)$ in powers of \hbar , the Wheeler–DeWitt equation yields

$$G_{ik}(g) = 0. \tag{1.4}$$

When the matter fields ϕ are also present, the wave-functional $\Psi(\phi, g)$ can be expanded in powers of M_{pl} . In the limit $M_{pl} \rightarrow \infty$ we recover Eqs. (1.2) and (1.4). This was shown by Lapchinsky and Rubakov and by Banks [8]. In other words, they obtained free Einstein equations and an equation describing quantum fields in curved space, *but no back-reaction of matter on gravity.*

There is an apparent contradiction between the semiclassical system (1.2), (1.3), which is required for a self-consistent back-reaction, and the deduction of the authors of Ref. [8] who did not find a back-reaction. The reason is that $M_{pl} \rightarrow \infty$ implies $G \rightarrow 0$, which kills the right-hand side of (1.3). This also shows that (1.2) and (1.3) *cannot* be obtained from the quantum theory at the *same* order of approximation. Our investigation in Refs. [1, 2] and the present paper is concerned with finding higher order corrections to (1.4) and with finding the domain of validity of the system (1.2) and (1.3).

Earlier work in this direction [1, 2, 7–10] is summarized in Section 4.1. This includes a brief review of the work of Gerlach [7], Hartle [9], and Halliwell [10]. We then summarize our earlier results on the relevance of the phase of the matter wave-functional, so far as the semiclassical limit is concerned; and we recall the suggestion that the average energy equations are valid only for adiabatically varying metrics.

Section 4.2 is one of the major sections of the paper and illustrates the above results with examples from minisuperspace—a massless scalar field in a $k = +1$ Robertson–Walker universe [11]. Using explicit solutions to the Schrodinger equation we show how the back-reaction depends on the matter wave-function.

In Section 4.3 we re-examine the validity of our earlier work [12] on self-consistent solutions to (1.2) and (1.3), in view of the new results on the restricted applicability of these equations. We work with the minisuperspace model mentioned above and compare self-consistent solutions of the average energy equations with self-consistent solutions of the correct semiclassical equations. We find that the cosmological evolution can be different in the two cases.

We have said earlier that the gravitational field behaves classically in the limit $M_p \rightarrow \infty$. Since the action for the gravitational field scales as M_p^2 , it follows that *either all the gravitational modes are classical, or none is*. This raises the fundamental question, which, of course, has been asked by others before: is it correct to talk of quantized gravitational perturbations (gravitons) in a classical background metric? In our opinion, it is not, because no one has yet obtained such an approximation to quantum gravity. Thus the limit “quantum fields in curved space” is not the same as “gravitons in curved space.”

The easiest way of having gravitons in a classical background is to define something as a graviton! Suppose that the metric $g(x)$ is written as a perturbation on a background: $g(x) = g_0(x) + L_p h(x)$. If the gravitational action is rewritten, considering h as a perturbation, the part of the action corresponding to h appears independent of M_p^2 and is similar to the action for a matter field. One may now take the semiclassical limit for g_0 and retain h as a quantum field—the graviton. We believe that this is not the right thing to do and that when gravity is classical, all its modes are classical.

Even if we stick to the conventional viewpoint—that there can be gravitons in classical curved space—subtleties remain. So far in the discussion we have not emphasized an important difference between the semiclassical limit of non-linear field theories like general relativity and that of simple quantum-mechanical models. Though the limit $M_{\text{planck}} \rightarrow \infty$ leads from the Wheeler–DeWitt equation to quantum fields in (classical) curved space, it does not suppress production of gravitons in a time-varying classical background. If other particles are produced, so are gravitons; as noted by Duff [13]. In what sense then does the Eq. (1.3) hold? We obviously cannot have quantized gravitational perturbations (gravitons) on the left-hand side of this equation. A conventional suggestion [4] is to incorporate the $\langle T_{ik} \rangle$ for gravitons on the right-hand side, along with that for the other produced particles. However, our results so far indicate that (1.3) is valid only in the

adiabatic approximation, and in this limit production of gravitons and other particles will be strongly suppressed.

This indicates that the r.h.s. of (1.3), *in its domain of validity*, will possibly consist of the contribution from the real part of the effective action only, and not from the particles which are produced. Moreover, the semiclassical limit defined by $M_{\text{planck}} \rightarrow \infty$ is more general than (1.3), and appropriate graviton effects will have to be included therein. These points are discussed in detail in Section 4.4.

As explained in the same section, the above results relating to graviton production have a direct bearing on the validity of the minisuperspace approximation in quantum gravity. If a minisuperspace co-ordinate (say, the scale factor in a Robertson–Walker universe) excites a frozen mode (say, the gravitational wave mode), that would signal the breakdown of the minisuperspace approximation. Now that is precisely what happens when gravitons are produced by a time-dependent background, and the adiabatic approximation mentioned above appears to be a necessary condition for the validity of minisuperspace calculations.

We wish to mention that we have not concerned ourselves with the question of renormalization of $\langle T_{ik} \rangle$ nor with the renormalization of the source term in the semiclassical equations. If the source term is not $\langle T_{ik} \rangle$, renormalization becomes an even more difficult problem. Moreover, we believe that the current renormalization procedures for the divergences of $\langle T_{ik} \rangle$ are no more than stopgap methods. The final solution to the issue of divergences must be sought elsewhere.

In this paper we have aimed at developing an unambiguous mathematical scheme for finding the conditions of validity of the semiclassical limit of quantum gravity. This has been achieved by assuming that the gravitational field becomes classical because of a particular choice for its quantum state. One should also recall that the semiclassical equations are important, not only as a limiting case of quantum cosmology, but also as a limiting case of other gravitating systems like the evaporating black hole.

An entirely different way of studying the classical limit, which does not involve complete specification of the quantum state, is to be found in the work of Peres, Zurek, and Joos and Zeh [14], among others. It begins with the question: why are macroscopic objects found, almost always, in a classical state and never in a superposition of quantum states? If this is because they are in a WKB or a gaussian state, it is extremely surprising that they are never to be found in any other state, and do not obey the superposition principle.

Joos and Zeh explain that macroscopic objects are in continuous interaction with their environment, and hence are being continuously “measured” by the environment. The environment may consist, for example, of thermal radiation being scattered by the object. Alternatively, the environment may consist of those internal degrees of freedom of the object which we are not observing. If we are observing the object but not its environment, the state of the object is described by a density matrix, and the environment is traced out. It is then shown, with some examples, that if the number of degrees of freedom of the environment tends to infinity, the density matrix approaches a diagonal form. This is often referred to as the

“decoherence” of the density matrix. Moreover, it is known that a diagonal density matrix is indicative of classical behaviour.

This suggests that classical behaviour is inevitable for macroscopic objects—there is no other alternative. However, microscopic objects (few degrees of freedom) may be in a wave-packet state, for which a classical description is approximately valid; more often they are in (quantum) states for which a classical description is impossible. Thus, the usual discussion of classical limit (through specification of quantum state) and the process of decoherence appear to be applicable under different circumstances.

It is indeed an issue of importance as to whether classical behaviour for the gravitational field can emerge because of its interaction with a quantum matter field, rather than because of a choice of quantum gravitational state. Attempts towards answering this question, in the context of *quantum cosmology*, have been made by several workers, including one of us [15]. However, whether or not decoherence is exclusively responsible for a semiclassical universe should still be considered an open question. One does not know, for example, about which classical trajectory the density matrix is peaked. At this stage, one may need to use the Wigner function to find this classical trajectory. In fact, it is conceivable that despite decoherence, the density matrix may not be peaked about classical behaviour. This could happen when the wave-function of the universe is exponential in form (i.e., non-oscillatory)—the environment could still destroy interference between different values of the three-metric, but the density matrix would not be peaked about a set of classical trajectories. (We thank an anonymous referee for bringing to our notice the questions raised in this paragraph). Decoherence is not discussed any further in this paper; the reader may see Refs. [14, 15] for further details. Moreover, the development of the semiclassical limit in this paper is completely independent of the process of decoherence.

2. CLASSICAL LIMIT TO QUANTUM THEORY

This section briefly reviews and compares different known methods for defining the classical limit of a quantum theory—WKB approximation, Gaussian approximation, and Wigner function. The purpose of discussing the classical limit is that the same methods can be used to define the semiclassical limit. The discussion is elementary and directly applicable to quantum gravity.

2.1. The WKB Approximation

The WKB approximation for a single particle with the Lagrangian

$$L = \frac{1}{2} M \dot{Q}^2 - V(Q) \quad (2.1.1)$$

and the Schrodinger equation

$$i\hbar\dot{\psi} = -\frac{\hbar^2}{2M} \frac{\partial^2 \psi}{\partial Q^2} + V(Q)\psi = E\psi \quad (2.1.2)$$

is obtained by first writing $\psi = R(Q) \exp(iS(Q)/\hbar)$, where both R and S are real functions of Q . The Schrodinger equation then becomes

$$(R^2 S')' = 0 \tag{2.1.3}$$

$$\frac{S'^2}{2M} + V(Q) - E = \frac{\hbar^2}{2M} \frac{R''}{R}. \tag{2.1.4}$$

If we assume that $S' \neq 0$ we get $R = \text{const}/\sqrt{S'}$ from (2.1.3), and (2.1.4) becomes

$$\frac{S'^2}{2M} + V(Q) - E = \frac{\hbar^2}{2M} \sqrt{S'} \left[\frac{d^2}{dQ^2} [1/\sqrt{S'}] \right]. \tag{2.1.5}$$

Next we *assume* that the factor $R''(Q)/R$ does not depend on \hbar , so that the right-hand side of (2.1.5) vanishes in the limit $\hbar \rightarrow 0$ (This assumption is *not true* for all quantum states; we mention one example below.) It then follows that in (2.1.5) the first quantum correction appears at order \hbar^2 . The classical limit is obtained by expanding S as a power series in \hbar^2 :

$$S(Q) = S_0(Q) + \hbar^2 S_1(Q) + \dots \tag{2.1.6}$$

In the lowest order Eq. (2.1.5) becomes

$$\frac{S_0'^2}{2M} + V(Q) - E = 0 \tag{2.1.7}$$

and $R(Q) = \text{const}/\sqrt{S_0'}$. The resulting wave-function

$$\psi_E^{(0)} = \frac{1}{\sqrt{S_0'}} \left[C_1 \exp \left[\frac{i}{\hbar} S_0(Q) \right] + C_2 \exp \left[-\frac{i}{\hbar} S_0(Q) \right] \right] \tag{2.1.8}$$

is the WKB wave-function. Equation (2.1.7) is the Hamilton–Jacobi equation for the classical action S_0 , and $S_0' = \sqrt{2M(E - V(Q))}$ can be identified with the classical momentum.

The validity of the above approximation requires that

$$\hbar \left| \frac{S''}{S'^2} \right| = \left| \frac{d}{dx} \left(\frac{\hbar}{S'} \right) \right| \ll 1 \tag{2.1.9}$$

which is the condition that the change in the particle’s deBroglie wavelength is small compared to itself. This condition is not satisfied at or near turning points. It is useful to express the constraint (2.1.9) in the alternate form

$$2M\hbar |V'| \ll (2M[E - V(Q)])^{3/2} \tag{2.1.10}$$

which implies that there is a restriction on the gradient of the potential when E is specified.

In the state (2.1.8) the probability $|\psi^{(0)}(Q)|^2$, of finding the particle between Q and $Q + dQ$, is inversely proportional to the classical momentum. This means that the probability is high near those values of Q where the particle spends more time. Thus the WKB wave-function (2.1.8) has properties of classical motion, as in (2.1.8) and (2.1.9), but it is not peaked about a specific classical trajectory. The precise sense in which WKB states provide a near classical description is through the Wigner probability distribution [Section 2.2], which, given a quantum state, can be used as a measure of its classical behaviour.

We mentioned above that there can be quantum states for which the right-hand side of Eq. (2.1.5) does not vanish in the limit $\hbar \rightarrow 0$. One such example is the gaussian wave-packet for a harmonic oscillator. It is elementary to show that the wave-packet which is of the form

$$\psi(Q, t=0) = N \exp \left[-\frac{M\omega}{2\hbar} (Q - a)^2 \right] \quad (2.1.11)$$

evolves into

$$\begin{aligned} \psi(Q, t) = N \exp \left[-\frac{M\omega}{2\hbar} (Q - a \cos \omega t)^2 \right. \\ \left. - i \left(\frac{\omega}{2} t + \frac{M\omega}{\hbar} aQ \sin \omega t - \frac{M\omega}{4\hbar} a^2 \sin \omega t \right) \right]. \end{aligned} \quad (2.1.12)$$

For this state it follows that

$$\frac{\hbar^2}{2M} \frac{R''}{R} = \frac{1}{2} M\omega^2 (Q - a \cos \omega t)^2 - \frac{1}{2} \hbar\omega. \quad (2.1.13)$$

There is an \hbar independent term in (2.1.13) which will not vanish as $\hbar \rightarrow 0$. This shows that one cannot always drop the right-hand side of (2.1.5) to write a Hamilton-Jacobi equation for the phase. In the literature this point is often overlooked.

The higher order correction to the WKB state may be found by returning to (2.1.5) and iterating $S(Q)$ as $S(Q) = S_0(Q) + \hbar^2 S_1(Q)$. Thus, S_1 satisfies the equation

$$S_1' = \frac{1}{2\sqrt{S_0'}} \left[\frac{d^2}{dQ^2} [1/\sqrt{S_0'}] \right]. \quad (2.1.14)$$

2.2. Wigner Function

Since position and momentum cannot be measured simultaneously for a quantum mechanical particle, we cannot talk of trajectories in phase-space. The closest one can get to trajectories is to define a joint probability distribution in the phase-space. Thus for any quantum state we can ask: what is the probability that the particle is at position Q with momentum p ? If a quantum state is to "represent"

classical motion, the corresponding distribution function should exhibit a strong correlation between position and momentum. The Wigner function [16] is one example of such a distribution function. As we shall see in Section 3, the Wigner function is very useful for finding conditions of validity of a semiclassical theory. (For other relevant discussions of Wigner function the reader may see Refs. [10, 17].)

The Wigner distribution function $F(Q, p, t)$ for a quantum state $\psi(Q, t)$ is defined as

$$F(Q, p, t) = \int_{-\infty}^{\infty} du \psi^*(Q - \frac{1}{2} \hbar u, t) e^{-ipu} \psi(Q + \frac{1}{2} \hbar u, t) \tag{2.2.1}$$

and has the properties

$$\int_{-\infty}^{\infty} dp F(Q, p, t) = |\psi(Q, t)|^2 \tag{2.2.2}$$

$$\int_{-\infty}^{\infty} dQ F(Q, p, t) = |\tilde{\psi}(p, t)|^2, \tag{2.2.3}$$

$\tilde{\psi}(p, t)$ being the Fourier transform of $\psi(Q, t)$. The Schrodinger equation (2.1.2) can be used to show that

$$\frac{\partial F}{\partial t} + \frac{p}{M} \frac{\partial F}{\partial Q} - \frac{dV}{dQ} \frac{\partial F}{\partial p} = \frac{\hbar^2}{24} \frac{d^3 V}{dQ^3} \frac{\partial^3 F}{\partial p^3} + \dots, \tag{2.2.4}$$

where dots indicate terms with higher powers of \hbar and higher derivatives of F and V .

Because of these properties, one would have liked to interpret F as a probability distribution in phase-space; unfortunately it is not positive definite for all quantum states. However, there are some states for which it is positive definite. This happens for example, for the WKB wavefunctions and for gaussian states. Another point to note is that the right-hand side of (2.2.4) vanishes for the WKB approximation ($O[\hbar]$) and for quadratic potentials. Then F satisfies the simple continuity equation and behaves like a classical probability distribution.

The Wigner function can be shown to be the unique construct from $\psi(Q, t)$ with the properties (2.2.2) and (2.2.3), [18], and hence the closest approach that one can make in defining correlations between Q and p .

We now calculate the Wigner function for the WKB state (2.1.8). Consider first the case with $C_2 = 0$, so that

$$\psi(Q) = (C_1 / \sqrt{S'_0}) \exp(iS_0/\hbar) \tag{2.2.5}$$

Using (2.2.1) we get

$$F(Q, p) = \frac{|C_1|^2}{S'_0(Q)} \delta\left(p - \frac{\partial S_0}{\partial Q}\right) + O(\hbar^2). \tag{2.2.6}$$

The Wigner function is peaked in momentum space, at one of the two possible values of momenta which a classical particle will have if it were at position Q ; and the probability of its having a position Q is inversely proportional to $S'_0(Q)$. In this sense, a WKB state is not peaked about any single trajectory, but gives a precise correlation between the position and momentum of the particle (also see Ref. [10]).

If we do not set either of the constants in (2.1.8) to zero, the corresponding Wigner function is

$$F(Q, p) = \frac{|C_1|^2}{S'_0(Q)} \delta\left(p - \frac{\partial S_0}{\partial Q}\right) + \frac{|C_2|^2}{S'_0(Q)} \delta\left(p + \frac{\partial S_0}{\partial Q}\right) + O(\hbar^2). \quad (2.2.7)$$

The Wigner function has a term which represents interference between the two WKB solutions, but this term is of order \hbar^2 and will vanish in the limit $\hbar \rightarrow 0$. This Wigner function is peaked at two different values of momenta: $p = \pm \partial S_0 / \partial Q$. These values correspond, respectively, to motion along the forward and backward direction, on a given path in configuration space. In the phase space, this F will be peaked on two families of trajectories—unlike the F in (2.2.6).

The Wigner function helps us decide whether the classical limit should be obtained using the general state (2.1.8) or the reduced state (2.2.5) in which one of the constants has been set to zero. This may be illustrated with the help of an example. Consider the motion of a particle in a potential $V(x)$; the potential is such that the particle executes bounded oscillatory motion. A snapshot of the particle cannot tell whether the instantaneous momentum is positive or negative, (i.e., whether it is $(+\partial S_0 / \partial Q)$ or $(-\partial S_0 / \partial Q)$.) Both possibilities exist; in this case the Wigner function evaluated at a given instant should not have information about the direction of motion. So, as the Wigner function (2.2.7) indicates, the correct WKB state to choose is the one in which neither of the constants has been set to zero. Moreover, it would be incorrect if the procedure for defining the *classical* limit picks out a classical trajectory but not the corresponding time-reversed trajectory. The original quantum system we began with was invariant under time-reversal and so should be the limiting system obtained from it.

This point may look trivial, but when we discuss the *semiclassical* limit, we will argue that the classical system should be in the reduced state (2.2.5). The reason for this choice is that when we couple the classical system to a quantum mechanical system, the evolution of the latter very much depends on the choice of the classical trajectory for the former. For instance, if the classical system was a Robertson–Walker universe with a scale-factor $a(t)$, the evolution of a scalar field will certainly depend on whether the universe is expanding (positive momentum) or contracting (negative momentum). Moreover, we certainly are not interested in finding how the field evolves if the universe is in a superposition of the expanding and contracting phase—this is irrelevant from the observational point of view. Hence one or the other of positive or negative momentum WKB state must be chosen a priori.

If one were to calculate F in the classically forbidden region where S_0 is imaginary, it turns out that F is of the form $F(Q, p) = F(Q) F(p)$, so that there is

no correlation between Q and p in the forbidden region. This reinforces the use of F as a classical distribution.

Thus, so far as the WKB state is concerned, the Wigner function is a very useful construct. In the next section we shall compute the Wigner function for a gaussian state.

2.3. Gaussian States and the Classical Limit

Besides the WKB state, another state may be used to define the classical limit. In contrast to a WKB state, one can construct a wave-packet, for which $|\psi|^2$ is actually peaked about a classical trajectory. The wave-packet can be obtained by a superposition of the WKB energy eigenstates, but it is instructive to directly use a gaussian wave-packet in the Schrodinger equation and carry out an \hbar expansion for the relevant parameters. An analogous procedure can be used for gaussian semiclassical states. At the end of the section we compare the WKB state with the gaussian state, in the context of the Wheeler-DeWitt equation.

The most general gaussian state

$$\psi(Q, t) = N(t) \exp \left[-B(t)(Q - \chi(t))^2 + \frac{i\theta(t)}{\hbar} \right], \tag{2.3.1}$$

where $N, B,$ and χ are complex functions of time, can be written as

$$\psi(Q, t) = N(t) \exp \left[-(Q - f)^2/4\sigma^2 + iPQ/\hbar + i\theta(t)/\hbar \right]. \tag{2.3.2}$$

Here $f(t) = \text{Re}(B\chi)/\text{Re } B, P(t) = 2\hbar |B|^2 \text{Im}\chi/\text{Re } B,$ and $\theta(t)$ are real, and $\sigma^2 = 1/4B$ is a complex quantity. Besides, f and P are respectively the expectation values of position and momentum in this state.

As we did for the function $S(Q)$ in the WKB state, we can now expand $f, \sigma^2,$ etc. in powers of \hbar . The leading order, however, is not \hbar^0 for all the parameters and has to be determined from the physical interpretation of these parameters. Thus the expectation value of the momentum, the phase $\theta(t)$ and the normalization factor $N(t)$ should be independent of $\hbar,$ and σ^2 and $(Q - f)^2$ should have the leading order $O(\hbar)$. It is useful to write the quantities which depend on \hbar in terms of new parameters which are independent of \hbar :

$$(Q - f) = \sqrt{\hbar} y + O(\hbar^{3/2}) \tag{2.3.3}$$

$$\sigma^2 = \hbar \Delta^2 + O(\hbar^2) \tag{2.3.4}$$

and

$$iPQ = iP(Q - f) + iPf = iP \sqrt{\hbar} y + iPf + O(\hbar^{3/2}). \tag{2.3.5}$$

Here, y and Δ^2 are independent of $\hbar.$ Next, we Taylor-expand the potential $V(Q)$ about $Q = f$:

$$V(Q) = V(f) + \sqrt{\hbar} y \left. \frac{dV}{dQ} \right|_{Q=f} + \frac{\hbar}{2} y^2 \left. \frac{d^2V}{dQ^2} \right|_{Q=f} + O(\hbar^{3/2}). \tag{2.3.6}$$

We assume that the third and higher derivatives of the potential can be neglected in the Taylor expansion. This quadratic approximation to the potential is essential, if the gaussian state has to be used to obtain the classical limit.

Substituting the gaussian state and the expansion of the potential in the time-dependent version of the Schrodinger equation (2.1.2) gives the following equations at various orders in \hbar :

$$O(\hbar^0): -\dot{\theta} = \frac{P^2}{2M} + V(f) + \dot{P}f \quad (2.3.7)$$

$$O(\hbar^{1/2}): P = M\dot{f}, \dot{P} = -dV/dQ|_{Q=f} \quad (2.3.8)$$

$$O(\hbar): \frac{i\dot{N}}{N} + \frac{iy^2}{2A^3} \dot{A} = \frac{1}{2} y^2 \frac{d^2V}{dQ^2} \Big|_{Q=f} + \frac{1}{4MA^2} - \frac{y^2}{8MA^4}. \quad (2.3.9)$$

The first equation determines the phase $\theta(t)$. A little manipulation shows that we can write $\theta(t)$ as

$$\theta(t) = \int dt L[f], \quad (2.3.10)$$

where L is the Lagrangian (2.1.1). The second equation shows that the mean value $f(t)$ satisfies the classical equation of motion, and hence that the probability

$$|\psi(Q, t)|^2 = |N|^2 \exp[-(Q-f)^2 \operatorname{Re}(1/2\sigma^2)] \quad (2.3.11)$$

is peaked about the classical trajectory. The third equation determines the time-evolution of the dispersion A^2 .

The Wigner function for the gaussian state (2.3.2) can be shown to be

$$F(Q, p) = |N|^2 \exp \left[-\frac{(P-p)^2}{\hbar^2 \operatorname{Re}(1/\sigma^2)} - \frac{8|B|^2}{\operatorname{Re}(1/\sigma^2)} (Q-f)^2 + \frac{(P-p)(Q-f) \operatorname{Im}(1/\sigma^2)}{2\hbar \operatorname{Re}(1/\sigma^2)} \right]. \quad (2.3.12)$$

The interpretation for this function is not as straightforward as in the WKB case. If P and f are interpreted as the classical momentum and classical position, and σ^2 is real, the Wigner function is peaked on the classical trajectory in phase space. One such example is the gaussian state for the harmonic oscillator. The dispersion σ^2 evolves with time as

$$\sigma^2(t) = \sigma^2(0) \operatorname{Cos}^2(\omega t) + \frac{(\hbar/2m\omega)^2}{\sigma^2(0)} \operatorname{Sin}^2(\omega t). \quad (2.3.13)$$

In general $\sigma^2(0)$ may be complex, but if we choose it to be real then $\sigma^2(t)$ always remains real and in this case the Wigner function shows a correlation between Q and P . Thus the Wigner function has a nice interpretation for the gaussian state with real initial dispersion. A particular case of this is the coherent state—it has a

constant, real dispersion $\sigma^2 = \hbar/2M\omega$. In general, Eq. (2.3.9) will have to be solved for σ^2 before interpreting the Wigner function.

We noted previously, in Eq. (2.1.10), that there is a constraint on the first derivative of the potential, if the WKB approximation is to be valid. We also saw above that a quadratic approximation to the potential is necessary for defining the classical limit using a gaussian state. These two constraints are similar, but not the same. Thus we may say that the WKB and the gaussian approximation are not exactly the same.

The crucial difference between these two approximations, so far as gravity is concerned, is the following. The WKB states are energy eigenstates, whereas the gaussian states are not. To write a gaussian solution we need an explicit $\partial\psi/\partial t$ term in the Schrodinger equation. Suppose we are not working with the Schrodinger equation (2.1.2) but with the Schrodinger equation for a system whose Hamiltonian is constrained to be identically zero. Then the evolution equation for the Q_i 's will be $H(Q_i)\psi(Q_i)=0$. We can still write (two) WKB solutions, each of zero energy. However, since there is no explicit time-dependence in the wave-function, a gaussian approximation is not possible right away. We can construct a wave-packet for this system if in the Hamiltonian we can identify a combination $f(Q_i)$ as a time-variable, so that the Schrodinger equation can be written as $H'(Q_i)\psi = i\partial\psi(Q_i)/\partial f(Q_i)$.

We conclude that if wave-packets can be constructed for a quantum system, they are the most natural way of defining a classical limit; otherwise the WKB approximation has to be used.

We would like to emphasize that our goal in this paper is to use a suitable semiclassical scheme to find the condition(s) of validity of Eqs. (1.3). The present discussion on classical limit is hence tuned to the development of such a scheme. One may alternatively wish to take the point of view that the definition of a classical (or semiclassical) limit is a more physical question, and that instead of choosing a particular state as being quasi-classical, one should ask for conditions under which any given state is peaked about classical behaviour. This may be done, for example, using the Wigner function. (For an alternative approach in terms of correlations, see Hartle [9]). However, it is not clear to us whether the semiclassical Einstein equations can be derived without restricting the gravitational wave-functional to a quasi-classical form. It is thus preferable to adhere to the latter choice.

The generalisation to a field theory, of the above definitions for classical limit, is straightforward. It involves using a WKB wave-functional $\Psi[\phi] = \exp[iS(\phi)/\hbar]$ or a gaussian wavefunctional

$$\Psi[\phi(\mathbf{x}), t] = N(t) \exp \left[- \int d\mathbf{y} d\mathbf{z} [\phi(\mathbf{y}, t) - f(\mathbf{y}, t)] B(\mathbf{y}, \mathbf{z}, t) [\phi(\mathbf{z}, t) - f(\mathbf{z}, t)] \right] \tag{2.3.14}$$

in the functional Schrodinger equation. A Wigner function for these states can be defined exactly as before (for an application to gravity, see Section 4).

We shall now make use of the results on classical limit to define the semiclassical theory. Before we go on to quantum gravity we study the semiclassical limit of a toy quantum mechanical model.

3. THE SEMICLASSICAL LIMIT TO QUANTUM THEORY

This section describes and compares the different methods of defining the semiclassical limit—in which limit Q obeys a classical equation of motion whereas q is quantum mechanical. These methods include the WKB approximation and the gaussian approximation.

We shall work with the two particle Lagrangian of Eq. (1.1) for which the classical equations are

$$\ddot{Q} = -\frac{dV}{dQ} - \frac{\partial u}{\partial Q} \quad (3.0.1)$$

$$\ddot{q} = -\frac{\partial u}{\partial q} \quad (3.0.2)$$

and the Schrodinger equation is

$$i\hbar\dot{\psi}(Q, q) = E\psi = -\frac{\hbar^2}{2M}\frac{\partial^2\psi}{\partial Q^2} + MV(Q)\psi - \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial q^2} + u(q, Q). \quad (3.0.3)$$

The form of the Lagrangian (1.1) is chosen so that it corresponds to a mini-superspace approximation of the Wheeler–DeWitt equation. One half of the Lagrangian scales with M , just as the gravitational action scales with the inverse of Newton's constant G . Thus Q is analogous to the metric g , and q is analogous to a matter field in a potential $u(q, Q)$ which depends on the “metric” Q .

The semiclassical limit is obtained when the parameter M tends to infinity. Note that for the Lagrangian $L = M(\dot{Q}^2/2 - V(Q))$ the WKB approximation is obtainable either through $\hbar \rightarrow 0$ or $M \rightarrow \infty$. This is because only the ratio (\hbar/M) appears in its Schrodinger equation. However, $\hbar \rightarrow 0$ and $M \rightarrow \infty$ have very different effects in the two-particle Schrodinger equation (3.0.3). The limit $\hbar \rightarrow 0$ is the classical approximation to the entire system; $M \rightarrow \infty$ is interpreted to be the classical approximation for Q .

3.1. The WKB Semiclassical Approximation

We write the wave function $\psi(Q, q)$ in (3.0.3) as

$$\psi(q, Q) = \exp[iS(q, Q)/\hbar], \quad (3.1.1)$$

where S is a complex function. The time-independent version of Eq. (3.0.3) now becomes

$$\frac{S'^2}{2M} + \frac{S_q^2}{2m} + MV(Q) + u(q, Q) - \frac{i\hbar^2}{2M} S'' - \frac{i\hbar^2}{2m} S_{qq} = E \quad (3.1.2)$$

($S' = \partial S/\partial Q$, $S_q = \partial S/\partial q$. While discussing the semiclassical limit it is convenient to work with the form of the wave-function given in (3.1.1). At times the form $\psi = R(q, Q) \exp iS(q, Q)/\hbar$, with R and S real, will be more useful; then we will switch to that form).

If at this stage we expand the function $S(q, Q)$ of (3.1.1) as a power series in \hbar we will obtain the Hamilton–Jacobi equation corresponding to the Lagrangian (1.1) at the leading order in \hbar . However, to obtain the semiclassical limit we have to expand $S(q, Q)$ as a power series in M :

$$S(q, Q) = MS_0(q, Q) + S_1(q, Q) + M^{-1}S_2(q, Q) + \dots \tag{3.1.3}$$

and separate the terms at different orders in M . Since one half of the Lagrangian scales with M , we assume that the energy E also scales with M and write it as $E = M\varepsilon$.

Substituting (3.1.3) in (3.1.2) gives, at order M^2 ,

$$\left(\frac{\partial S_0}{\partial q}\right)^2 = 0 \tag{3.1.4}$$

from which we conclude that S_0 does not depend on q . This result is crucial for the existence of the semiclassical limit. By comparing the terms at order M we conclude that

$$M\varepsilon = \frac{1}{2}MS_0'^2(Q) + MV(Q) \tag{3.1.5}$$

and that $S_0(Q)$ is real. At this order the wave-function is

$$\psi(Q) = [C_1 \exp(iS_0/\hbar) + C_2 \exp(-iS_0/\hbar)]. \tag{3.1.6}$$

Equation (3.1.5) is the Hamilton–Jacobi equation for the Q -mode and the defining equation for the classical momentum $MS_0'(Q)$ of the Q -mode. From this equation we can write down an equation of motion for Q . The validity of the H-J equation requires that

$$\left|\frac{d}{dx}\left(\frac{\hbar}{MS_0'}\right)\right| \ll 1 \tag{3.1.7}$$

which, like before, is a statement about the de Broglie wavelength of the Q -mode. If this condition is satisfied we can conclude that the Q -mode is behaving classically. Taking the $M \rightarrow \infty$ limit is equivalent to doing the WKB approximation for Q , while leaving q quantum mechanical. This can also be verified with the help of Wigner function, as discussed in Section 3.4.

To get information about the motion of q we look at the higher order terms in the Schrodinger equation. We begin by assuming that the leading order wavefunction is

$$\psi_0(Q) = C_1 \exp[iS_0/\hbar]. \tag{3.1.8}$$

That is, we have set $C_2 = 0$ in (3.1.6). It can then be shown that at order M^0 , (3.1.2) gives the equation

$$i\hbar S'_0 \frac{\partial f(q, Q)}{\partial Q} = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2} + u(q, Q) \right] f(q, Q). \quad (3.1.9)$$

Here we have defined the function $f(q, Q)$ by the relation

$$f(q, Q) = \sqrt{S'_0} \exp(iS_1/\hbar). \quad (3.1.10)$$

Thus at order M^0 the wave-function of the system is

$$\psi(q, Q) = \frac{C_1}{\sqrt{S'_0}} \exp[iMS_0(Q)/\hbar] f(q, Q). \quad (3.1.11)$$

The equation satisfied by the function f can be interpreted as the Schrodinger equation for the quantum mechanical mode q , in an external classical background Q , and in a "time-coordinate" τ defined by the relation

$$\frac{\partial}{\partial \tau} = S'_0 \frac{\partial}{\partial Q}. \quad (3.1.12)$$

This is a reasonable way of defining a time-coordinate because S'_0 defines the classical velocity $v(Q)$ for Q , and if $v(Q) \partial/\partial Q = \partial/\partial \tau$ then τ is nothing but the "time." Equation (3.1.9) is equivalent to Eq. (1.2) describing "quantum fields in curved space." The overall wave-function (3.1.11) has the form of a WKB wave-function for Q , times a wave-function describing the state of q .

At this stage we would like to comment on a few important points relating to the above derivation. The first has to do with the choice that we made in Eq. (3.1.8) for the leading order wave-function. Essentially we chose the wave-function at order M^0 as

$$\psi(q, Q) = C_1 \exp(i[MS_0 + S_1(q, Q)]/\hbar) \quad (3.1.13)$$

and *not* as the general solution

$$\psi(q, Q) = [C_1 \exp[iS_0/\hbar] + C_2 \exp[-iS_0/\hbar]] \exp[iS_1/\hbar]. \quad (3.1.14)$$

If we make the latter choice and retain terms up to order M^0 , the equation satisfied by $f(q, Q)$ can be shown to be

$$\begin{aligned} i\hbar S'_0 \frac{\partial f}{\partial Q} & \left[\frac{C_1 \exp[iMS_0/\hbar] - C_2 \exp[-iMS_0/\hbar]}{C_1 \exp[iMS_0/\hbar] + C_2 \exp[-iMS_0/\hbar]} \right] \\ & = -\frac{\hbar^2}{2m} f_{qq} + u(q, Q) f. \end{aligned} \quad (3.1.15)$$

As is apparent, this equation reduces to the Schrodinger equation for q if and only if the constant C_2 is zero.

We believe that the correct choice for the state is (3.1.13) and not (3.1.14). It may seem that this choice for the state contradicts the choice we made while discussing Wigner functions and the classical limit (see Section 2.2). There we explained that to obtain the correct classical limit we must not set either C_1 or C_2 as zero. However, in the semiclassical system that we are now discussing, we are talking of the evolution of q in a given background Q . We must choose this background to be a trajectory which is a solution of the equation of motion for Q . This is ensured if the WKB wave-function is taken as in (3.1.13). It would be meaningless to talk of the evolution of q while the background Q is in a state involving superposition of classical trajectories. And this is what will happen if the state is chosen to be (3.1.14). An example should make this clear. Suppose we are studying the evolution of a quantum field in a Robertson–Walker universe with scale-factor $a(t)$. In this case, the two choices for the “momentum” will correspond to expanding and contracting phases of the universe. We will be interested in studying the behaviour of quantum fields in either one of these two phases *but not in a universe described by a superposition of these two phases*. Clearly we must set one of the C 's to zero in this case. It would be right to say that the choice (3.1.13) is made on the basis of observation.

The second point is concerned with the introduction of the time parameter τ in (3.1.12). If in the Schrodinger equation for our toy model, we set E as zero (as is true for the Wheeler–DeWitt equation), there is no time-dependence left in the quantum system. Time dependence enters the system at the semiclassical level, through the definition of τ given above. Moreover, the correspondence between Q and time is unique.

Another interesting point regarding the semiclassical limit is the breakdown of superposition principle. For obtaining the semiclassical limit we require the wave-function $\psi(q, Q)$ to be of the type

$$\psi(q, Q) = \psi_{\text{WKB}}(Q) f(q, Q). \tag{3.1.16}$$

If ψ_1 and ψ_2 are of the form given in (3.1.16), the state $(\psi_1 + \psi_2)$ will in general not be of that form. In this sense the principle of superposition is lost at the semiclassical level.

The semiclassical system obtained at order M^0 is described by the H-J equation (3.1.5) for Q and the Schrodinger equation (3.1.9) for q . Note that in these equations there is no information as to how q affects the evolution of Q (no back-reaction). The reason is that the H-J equation is written at order M^1 . However, the system is now determined up to order M^0 , and so we must find out what the H-J equation is, at order M^0 . That will tell us what the back-reaction is. This is the non-trivial exercise towards which this investigation is aimed.

The classical equation for Q is equivalent to the H-J equation

$$\frac{1}{2} MS_0'^2 + MV(Q) + u(q, Q) = E. \tag{3.1.17}$$

In our semiclassical system Q is classical at order M^1 and possibly classical at order M^0 . This suggests that in the semiclassical system the H-J equation for Q may include a term like $\langle f | u(q, Q) | f \rangle$ —the expectation value of the potential $u(q, Q)$ in the quantum state for q . The idea is to see if we can get this result out of the full quantum theory. However, before we start looking at higher order corrections to the WKB semiclassical approximation, we want to consider another semiclassical approximation—the gaussian approximation. The reason for studying it is that it is not immediately obvious what the relation between these different approximations is. Also, it is important to decide whether one of them is more useful for studying higher order corrections, as compared to the other.

3.2. Semiclassical Limit Using Gaussian States

We saw earlier that it was possible to define the classical limit using WKB states or gaussian states. We now attempt to define a semiclassical limit for the two-particle Lagrangian by constructing a wave-packet for the Q -mode. We start with the time-dependent Schrodinger equation (3.0.3), and write the wave-function as

$$\psi(q, Q, t) = \chi(Q, t) \xi(q, Q). \quad (3.2.1)$$

Next, we choose $\chi(Q, t)$ to be a gaussian in Q , so that

$$\psi(q, Q, t) = N(t) \exp[-(Q - f)^2/4\sigma^2 + iMvQ/\hbar + iM\theta/\hbar] \xi(q, Q). \quad (3.2.2)$$

The gaussian part of the wave-function is written as in Eq. (2.3.2), with p there written as Mv here, and θ there as $M\theta$ here. $f(t)$, $v(t)$, and $\theta(t)$ are real, $\sigma^2(t)$ is complex—and they are all functions of time. The procedure is analogous to that for the WKB semiclassical limit, taking the limit $M \rightarrow \infty$ in the Schrodinger equation after expanding the parameters of the wave-function (3.2.2) in powers of M . However, in the WKB case, we had only one parameter, $S(q, Q)$, which we expanded in a power series in M . Here we have more than one parameter and it turns out that their leading powers in M should be chosen suitably to get a meaningful semiclassical limit. We can get a hint to this leading power by noting that in defining the semiclassical rather than the classical limit, an expansion in \hbar is replaced by an expansion in M^{-1} . Thus we recall the expansion of parameters used in Eqs. (2.3.3) to (2.3.5) and replace \hbar in these equations by M^{-1} . In this way we define new parameters y and A by the relation

$$(Q - f) \equiv \frac{1}{\sqrt{M}} y + O(M^{-3/2}) \quad (3.2.3)$$

$$\sigma^2 \equiv \frac{1}{M} A^2 + O(M^{-2}) \quad (3.2.4)$$

and

$$iMvQ/\hbar = \frac{iMv}{\hbar} \left(\frac{1}{\sqrt{M}} y + f \right). \tag{3.2.5}$$

Here $y(t)$ and $A(t)$ are assumed independent of M . It should be noted that this choice of leading powers is an assumption, which gives a meaningful semiclassical limit. Roughly, the idea is that the dispersion σ^2 scales as the expansion parameter M^{-1} , and the deviation $(Q - f)^2$ scales as the dispersion.

As in the earlier gaussian approximation we assume that the potentials have a Taylor expansion around the expected mean value:

$$MV(Q) = MV(f) + M(Q - f) V' + M \frac{(Q - f)^2}{2} V'' + O(Q - f)^3 + \dots \tag{3.2.6}$$

$$u(q, Q) = u(q, f) + (Q - f) u' + \frac{(Q - f)^2}{2} u'' + O(Q - f)^3 + \dots. \tag{3.2.7}$$

We also expand the function $\xi(q, Q)$ around $Q = f$,

$$\xi(q, Q) = \xi(q, f) + (y/\sqrt{M}) \xi'(q, f) + \dots. \tag{3.2.8}$$

Using the wave-function (3.2.2) and the above expansions in the Schrodinger equation yields the following relations at various orders in M :

$$O(M): -M\dot{\theta} = M\dot{f} + \frac{1}{2} Mv^2 + MV(f) \tag{3.2.9}$$

$$O(\sqrt{M}): v = \dot{f}, \dot{v} = -\left. \frac{dV}{dQ} \right|_{(Q=f)} \equiv -\frac{dV}{df} \tag{3.2.10}$$

$$\begin{aligned} O(M^0): i\hbar v \xi' / \xi + \frac{\hbar^2}{2m} \frac{\xi_{qq}}{\xi} - u(q, f) \\ = \frac{\hbar^2}{4A^2} - i\hbar \frac{\dot{N}}{N} - y^2 \left[\frac{\hbar^2}{8A^4} - \frac{1}{2} V'' + \frac{i\hbar}{2A^3} \dot{A} \right]. \end{aligned} \tag{3.2.11}$$

The first two of these equations are the same as we saw before: they are equations for the phase $\theta(t)$, for the time evolution of the mean $f(t)$, and finally for the evolution of the mean momentum $Mv(t)$, of the Q -mode. As before, the equation for θ can be integrated to get

$$M\theta(t) = \int dt M \left[\frac{1}{2} \dot{Q}^2 - V(Q) \right]. \tag{3.2.12}$$

This explains why we chose $\theta(t)$ to scale with M .

In view of the relations (3.2.10) the Q -mode may be said to behave classically in

the $M \rightarrow \infty$ limit. This is essentially because we chose part of the wave-function as a gaussian and expanded its parameters in powers of M .

The relation (3.2.11) is new—the right-hand side is the same as in (2.3.9); only, now it equals the Schrodinger operator acting on $\xi(q, f)$. The left-hand side is a function of q , while the right-hand side is not, hence the expressions on both sides must vanish independently. This recovers for us the Eq. (2.3.9) for the spread (Δ^2) of the gaussian, and the Schrodinger equation (3.1.9) for the q -mode. Once again, the same semiclassical system has been obtained as in the case of the WKB approximation; the term $(\partial u / \partial Q)$ still does not appear in the classical equation of motion (3.2.10) for Q (no back-reaction yet).

We compared the WKB and gaussian state at the end of Section 2.3, while discussing the classical limit. The same comparison continues to hold true here as well. The gaussian semiclassical approximation does not have more information than the WKB semiclassical approximation.

3.3. Higher Order Corrections to the WKB Approximation

Having satisfied ourselves that the gaussian semiclassical approximation gives the same result as the WKB, we return to where we left off the WKB approximation. So far we found that an order by order expansion in powers of M gives (i) the H-J equation (3.1.5) for the classical mode Q at order M^1 , with no coupling to q (ii) the Schrodinger equation (3.1.9) for the quantum mode q in an external background Q , at order M^0 . We now ask the central question: what is the coupling between the classical mode and the quantum mode, at order M^0 ?

Many reasons suggest that this coupling (back-reaction) can be found through the phase of the wave-function of q . We should recall that at order M^0 the system is described by the wave-function given in (3.1.11). The classical equation for Q was obtained by taking the limit $M \rightarrow \infty$, and by writing an H-J equation for the phase $S_0(Q)$. At this order there is no information about the phase of the function $f(q)$ in (3.1.11). That information is obtained at order M^0 . It then seems reasonable that at order M^0 we should identify the Q -derivative of the total phase— $S_0(Q)$ plus the phase of f —as the classical momentum of Q . There is one serious problem with this: the phase of f will in general depend on the quantum variable q , to which a definite value cannot be assigned.

This problem is related to an issue of principle: under what conditions can the Q -mode be called classical, at order M^0 ? In other words, what kind of quantum sources produce gravitational fields which can be considered classical? It is extremely difficult to provide a simple physical criterion as an answer to this. Here we will try to develop certain rules which will tell: when is a semiclassical description valid, and what are the semiclassical equations? As regards the second question, there seems to be a widespread belief that at the semiclassical level, the coupling between Q and q is through the expectation value of the hamiltonian for q . While this looks like a natural choice, it is not obvious how the semiclassical limit of the quantum theory will yield such an answer.

We return to Equation (3.1.11) for the wave-function at order M^0 and write f as

a real amplitude $R(q, Q)$ times a phase $\exp[i\beta(q, Q)/\hbar]$, so that the wave-function reads

$$\psi(q, Q) = \frac{1}{\sqrt{S_0}} \exp[(iMS_0(Q) + i\beta(q, Q))/\hbar] R(q, Q). \quad (3.3.1)$$

Note that $\beta(q, Q)$ does not involve M , and loosely speaking, we may think of this phase as a perturbation on the leading phase $S_0(Q)$. We shall consider various possible forms for $\beta(q, Q)$ and the possible semiclassical theories. Later on we shall study the Wigner function for this state.

If we go ahead and define the classical momentum (P) for Q as the derivative of the phase we get

$$P = M \frac{\partial S_0}{\partial Q} + \frac{\partial \beta(q, Q)}{\partial Q}. \quad (3.3.2)$$

The simplest way to get rid of the dependence on q is to replace $(\partial\beta/\partial Q)$ by its expectation value

$$\langle f | \left(\frac{\partial \beta}{\partial Q} \right) | f \rangle \quad (3.3.3)$$

in the state $f(q, Q)$ which satisfies the Schrodinger equation (3.1.9). It can be shown from the Schrodinger equation that if the wave-function ψ is a solution of the equation, then the phase $S(t)$ of ψ and the Hamiltonian of the system are related as $\langle \psi | \dot{S} | \psi \rangle = -\langle \psi | h | \psi \rangle$. In the present context this implies the relation

$$\langle f | \left(\frac{\partial \beta}{\partial Q} \right) | f \rangle = -\frac{1}{S_0'} \langle f | h(q, Q) | f \rangle; \quad h = \frac{1}{2} m\dot{q}^2 + u(q, Q) \quad (3.3.4)$$

and hence that

$$\frac{P^2}{2M} + MV(Q) + \langle f | h | f \rangle = E. \quad (3.3.5)$$

Here we have dropped a term of order M^{-1} and used the H-J equation (3.1.5) for S_0' .

Equation (3.3.5) is an H-J equation for Q ; it incorporates the back-reaction through $\langle h \rangle$, and is a possible description of the semiclassical theory. In gravity this corresponds to coupling the metric to the expectation value of T_{ik} . The trouble with this semiclassical theory, of course, is that it is ad hoc. It does not say why the derivative of the phase is replaced by its expectation value. Such a description neither tells when the semiclassical theory is valid, nor does it try to derive the semiclassical equations from the quantum theory. At best, one can try to find the conditions for validity of this particular semiclassical theory—Eq. (3.3.5) [19].

The above result suggests that the phase plays an important role in a semiclassi-

cal description. Another result which suggests the same involves rewriting the state (3.1.11) as [1]

$$\psi(q, Q) = \frac{1}{\sqrt{S'_0}} \exp[(1/\hbar)[iMS_0 + iR(Q)]] f(q, Q) \exp[-iR(Q)/\hbar]. \quad (3.3.6)$$

We define a new function f_n as f times $\exp(-iR)$ and we *choose* to define R through the relation

$$-S'_0 R'(Q) = \langle f_n | h | f_n \rangle. \quad (3.3.7)$$

If we now say that the classical momentum P should be defined as the derivative of $(MS_0 + R)$ we can show that

$$\frac{P^2}{2M} + MV(Q) + \langle f_n | h | f_n \rangle = E \quad (3.3.8)$$

and

$$i\hbar S'_0 \frac{\partial f_n}{\partial Q} = (h - \langle f_n | h | f_n \rangle) f_n. \quad (3.3.9)$$

Thus by making a suitable choice for R we can get $\langle h \rangle$ as the back-reaction term, though we have modified the Schrodinger equation for q . Of course, the result is spurious because we could have made any choice for R . It is just another indication that the phase may be relevant.

We now consider how the phase can provide higher order correction without our having to take an expectation value. To do that, we must first ensure that the phase gives the correct classical limit. That is, if we take the limit $\hbar \rightarrow 0$ in the Schrodinger equation (3.1.9), not only should we get the classical equation (3.0.2) for q , we should also get the classical equation (3.0.1) for Q . This is because our system is now exact upto order M^0 and the term $\partial u / \partial Q$ is of this order. This is also a check on use of the phase of f to define the semiclassical limit.

If we expand the phase β of f in a power series in \hbar , as

$$\beta = \beta_0(q, Q) + \hbar\beta_1(q, Q) + \dots \quad (3.3.10)$$

and use this expansion in (3.1.9), we will get at the leading order

$$S'_0 \beta'_0 + \frac{1}{2m} \left(\frac{\partial \beta_0}{\partial q} \right)^2 + u(q, Q) = 0 \quad (3.3.11)$$

which is the H-J equation for q . (Note that it becomes equivalent to the classical equation for q only if the auxiliary time-parameter τ defined in (3.1.12) is identified with the "classical time" t used in the Lagrangian (1.1). From the point of view of gravity this is a very important identification.) The classical limit for q is ensured by (3.3.11) and $\partial \beta_0 / \partial q$ is the classical momentum for q .

Thus at order M^0 and \hbar^0 the wave-function of the system is

$$\psi(q, Q) = \frac{1}{\sqrt{S'_0}} \exp(1/\hbar)[iMS_0 + i\beta_0]. \quad (3.3.12)$$

We define the classical momentum (P) for Q as the derivative of the phase in this equation. Then we can show that up to order M^0 ,

$$\frac{P^2}{2M} + V(Q) - S'_0\beta'_0 = E \quad (3.3.13)$$

and, using the H-J equation (3.3.11), we get

$$\frac{P^2}{2M} + V(Q) + \frac{1}{2m} \left(\frac{\partial\beta_0}{\partial q} \right)^2 + u(q, Q) = E. \quad (3.3.14)$$

This equation looks different from the H-J equation (3.1.15) for Q because it has an additional term. However, it does lead to the classical equation (3.0.1) as we show below.

We know that the H-J equation (3.3.11) is equivalent to the equation of motion

$$m \frac{d^2q}{dt^2} = - \frac{\partial u(q, \tau)}{\partial q} \quad (3.3.15)$$

with the auxiliary time defined as in (3.1.12). In this equation q is a function of τ and hence of Q . We can also write this equation as

$$S'_0 \frac{\partial}{\partial Q} \frac{\partial\beta_0}{\partial q} = - \frac{\partial u}{\partial q}, \quad (3.3.16)$$

where

$$\frac{\partial\beta_0}{\partial q} = mS'_0 \frac{\partial q}{\partial Q}. \quad (3.3.17)$$

Next note that in (3.3.14) we can define an effective potential V_{eff} as

$$V_{\text{eff}}(Q) = V(Q) + h(q, Q) = V(Q) + u(q, Q) + \frac{1}{2m} \left(\frac{\partial\beta_0}{\partial q} \right)^2. \quad (3.3.18)$$

The equation of motion for Q will involve the Q -derivative of this potential. In particular,

$$\frac{\partial h}{\partial Q} = \frac{1}{m} \frac{\partial\beta_0}{\partial q} \frac{\partial^2\beta_0}{\partial Q \partial q} + \frac{\partial u}{\partial Q} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial Q} = \frac{\partial u}{\partial Q} \quad (3.3.19)$$

as may be checked using Eqs. (3.3.16) and (3.3.17). Thus

$$\frac{\partial V_{\text{eff}}}{\partial Q} = \frac{\partial V}{\partial Q} + \frac{\partial u}{\partial Q} \quad (3.3.20)$$

which ensures that the H-J equation leads to the correct classical equation for Q . Essentially, the definition of time through Q creates an additional complication which has to be tackled carefully.

In this long digression we showed that if the phase of the wave-function (3.3.1) is used to define the classical momentum for Q , a sensible $\hbar \rightarrow 0$ limit is obtained. Of course, that is a necessary but not a sufficient condition for using the phase to obtain a semiclassical limit. So long as β depends on q , it cannot be directly used to define the semiclassical limit. One can also show that in general β must depend on q , otherwise it will not be possible to solve the Schrodinger equation for q . In such a case, one has to find a suitable prescription for the semiclassical limit. Of these, replacing $(\partial\beta/\partial Q)$ by its expectation value is too ad hoc. Let us consider alternatives.

Consider a state for which β is of the form

$$\beta(q, Q) = \beta_a(Q) + \beta_b(q, Q), \quad (3.3.21)$$

where $\beta_b(q, Q)$ does not contain any part depending only on Q . That is, $\beta_b(q, Q)$ either depends only on q , or consists of functions which are products of q and Q . In such a case the semiclassical limit may be obtained using $\beta_a(Q)$ and by defining the classical momentum for Q as

$$P = \frac{\partial S_0}{\partial Q} + \frac{\partial \beta_a}{\partial Q}. \quad (3.3.22)$$

Such a definition has an important constraint on it: it must give the correct classical limit for Q at order M^0 . We have seen earlier that if we use the full phase β for defining P , the correct classical limit is obtained. In general the definition in (3.3.22) will not give the same classical limit as given by β . But in those cases where it does we may use it to define the semiclassical limit for Q .

There are examples in which (3.3.22) gives a sensible semiclassical equation for Q . These include stationary and quasi-stationary states for q . If the potential $u(q)$ is independent of Q or has an adiabatic dependence on Q , the phase β is of the form $(\beta_a(Q) + \beta_b(q))$, which permits the use of $\beta_a(Q)$ for defining a semiclassical theory. Another case where β_a can be used is when $(\partial\beta_b/\partial Q)$ vanishes in the classical limit.

There is yet another class of states for which the definition (3.3.22) can be used. These are the ones for which the phase β does not satisfy the H-J equation in the limit $\hbar \rightarrow 0$. We saw an example of this in Section 2.1 on the WKB approximation. Essentially, if the right-hand side of (2.1.4) does not vanish as $\hbar \rightarrow 0$ we say that the phase does not have a classical limit. In these cases the use of β_a in (3.3.22) is not

constrained by the requirement of classical limit. For such states we may use β_a to define the classical limit for Q at order M^0 . We give one such example below. This is the example of an oscillator with time-dependent frequency, which is in a gaussian state with zero mean position and mean momentum, while the dispersion evolves with time. We show below that in this case the phase does not have a classical limit. A second example is given in Section 4.2 using a minisuperspace model.

Before we go to the example, it is useful to note from (3.3.4) and (3.3.21) that

$$\frac{\partial \beta_a(Q)}{\partial Q} = -\frac{1}{S'_0} \langle f | h(q, Q) | f \rangle - \langle f | \frac{\partial \beta_b}{\partial Q} | f \rangle. \tag{3.3.23}$$

This equation tells how the back-reaction found using β differs from that found using β_a . If the second term on the right-hand side is negligible these two match.

We now consider the example of a harmonic oscillator whose frequency depends on Q , and the interaction potential is

$$u(q, Q) = \frac{1}{2} \omega^2(Q) q^2. \tag{3.3.24}$$

We shall consider a particular solution of the Schrodinger equation (3.1.9) in this potential and for convenience convert all functions of Q into functions of τ , using (3.1.12). We assume that the initial state is a gaussian of the form (2.3.2) but with mean position f and mean momentum P set as zero. Essentially, this state is the ground state of the oscillator. Since the potential is quadratic, it propagates this state to another gaussian. Thus at all times the state is of the form

$$f(q, \tau) = N(\tau) \exp[-B(\tau)q^2 + i\theta(\tau)/\hbar]. \tag{3.3.25}$$

Using the form (3.3.25) in the Schrodinger equation for q and comparing different powers of q gives the equations

$$\begin{aligned} \dot{\theta} &= -\frac{\hbar^2}{m} B_R \\ \dot{B}_R &= \frac{4\hbar}{m} B_R B_I \\ \dot{B}_I &= -\frac{2\hbar}{m} (B_R^2 - B_I^2) + \frac{\omega^2}{2\hbar} \end{aligned} \tag{3.3.26}$$

(dots indicative derivative w.r.t. τ .) One can show that for this state the phase does not have a classical limit. To do so, we write B in terms of a new variable as

$$B = -\frac{im}{2\hbar} \frac{\dot{\chi}}{\chi} \tag{3.3.27}$$

and find that

$$\ddot{\chi} + \frac{\omega^2}{m} \chi = 0. \quad (3.3.28)$$

This equation does not involve \hbar and we find that B and hence B_R scales as $1/\hbar$. It is then easy to show that the r.h.s. of (2.1.4) does not vanish in the limit $\hbar \rightarrow 0$.

For the state (3.3.25) the phase β_a is equal to $(\theta(\tau))$, and the classical momentum for Q , from (3.3.22), is

$$P = S'_0 + \frac{1}{S'_0} \dot{\theta} = S'_0 - \frac{\hbar^2}{m} \frac{B_R}{S'_0}. \quad (3.3.29)$$

To obtain the Q -dependence of B_R we should solve the coupled equations in (3.3.26) for a specific $\omega(Q)$. A more interesting exercise is to see when $\langle \dot{\beta}_b \rangle$ vanishes, so that the use of β_a by itself leads to the H-J equation (3.3.5). From the form of the gaussian in (3.3.25) we find that β_b is equal to $-B_1 q^2$, and the condition for the expectation value of its τ -derivative to vanish is

$$\omega^2 = \frac{4\hbar^2}{m} (B_R^2 - B_1^2) = \frac{4\hbar^2}{m} (R_0^2 \exp(8B_0\tau) - B_0^2). \quad (3.3.30)$$

Here, B_0 is the constant value of B_1 . In particular, if B_0 is zero, we get that ω is a constant. Equation (3.3.30) is a condition on the time-dependence of the frequency if the back-reaction term should equal $\langle h \rangle$.

The back-reaction equals $\langle h \rangle$ when \dot{B}_1 is exactly zero. It will be approximately equal to $\langle h \rangle$ if \dot{B}_1 is nearly zero, and B_1 is nearly constant at a value which is much smaller than B_R . In that case it follows from (3.3.26) that

$$\omega \approx R, \quad \frac{\dot{\omega}}{\omega} \ll \omega. \quad (3.3.31)$$

This equation says that if the frequency ω changes very slowly with time, the back-reaction is equal to $\langle h \rangle$. For gravity this suggests that the back-reaction is T_{ik} if the metric varies adiabatically. Otherwise the semiclassical source for Q differs from $\langle h \rangle$, in the manner indicated in (3.3.23).

We should note that the example we have discussed above is nothing but a harmonic oscillator which has been prepared in its ground state with some initial frequency ω_1 and whose frequency evolves due to an external perturbation Q . Our prescription for obtaining the semiclassical limit (use $\beta_a(Q)$) suggests that as the ground state evolves, it reacts back on Q , but the reaction is not in general equal to $\langle h \rangle$.

We can possibly define a semiclassical limit for Q if the phase of the state for q has a part depending only on Q . In the next section we consider the case when such a separation does not necessarily occur in the phase.

3.4. *Wigner Function in the Semiclassical Theory*

While discussing the WKB classical approximation we used the Wigner function to find a probability distribution on the phase space. We interpreted a strong correlation between position and momentum as classical behaviour. To find out whether the mode Q behaves classically when we take the limit $M \rightarrow \infty$ we can again use the Wigner function. (For previous discussions of Wigner function in the semiclassical theory see Halliwell [10] and Boucher and Traschen in [17].) The Wigner function will be defined as

$$F(Q, P, q, p) = \int_{-\infty}^{\infty} du dv \psi^*(Q - \frac{1}{2}\hbar u, q - \frac{1}{2}\hbar v) \times e^{-iPu - ipv} \psi(Q + \frac{1}{2}\hbar u, q + \frac{1}{2}\hbar v). \tag{3.4.1}$$

Before we put this function to any use we must ask about the relation of its definition to the purpose of its construction. The parameter \hbar appears in the Wigner function in this particular form because we have in mind the $\hbar \rightarrow 0$ limit. Recall, for example, that to interpret the Wigner function for the WKB state, we expanded the state in powers of \hbar . Since the semiclassical limit is the limit $M \rightarrow \infty$ and not the limit $\hbar \rightarrow 0$ it seems reasonable to construct a modified Wigner function which involves a parameter M^{-1} . For a single degree of freedom this function F_M may be defined as

$$F_M(Q, P) = \frac{1}{M} \int_{-\infty}^{\infty} du \psi^*\left(Q - \frac{\hbar}{2M}u\right) e^{-iPu/M} \psi\left(Q + \frac{\hbar}{2M}u\right). \tag{3.4.2}$$

(Since the equation satisfied by F_M is linear we could have dropped the constant $1/M$. However, it is retained in the definition to get a normalised F_M .)

This function is obtained by just a redefinition of the variable of integration u in Eq. (2.2.1) and hence preserves the properties (2.2.2) to (2.2.4) of the Wigner function. Thus it is really the same function as before. However, it appears useful to write it like this and take the limit $M \rightarrow \infty$ in the wave-function when the need arises. Formally, the limit $M \rightarrow \infty$ will coincide with the limit $\hbar \rightarrow 0$ in this definition. That should be expected from what we have seen earlier about the equivalence of $\hbar \rightarrow 0$ and $M \rightarrow \infty$ for a single particle. The difference is expected to arise when we try to define a modified Wigner function for two particles. Thus for the two-particle system we define the Wigner function as

$$F_M(Q, P, q, p) = \frac{1}{Mm} \int_{-\infty}^{\infty} du dv \psi^*\left(Q - \frac{\hbar}{2M}u, q - \frac{\hbar}{2m}v\right) \times \exp[-i(Pu/M) - i(pv/m)] \psi\left(Q + \frac{\hbar}{2M}u, q + \frac{\hbar}{2m}v\right), \tag{3.4.3}$$

where M and m are respectively the masses of the particles. We see that now the limit $M \rightarrow \infty$ and the limit $\hbar \rightarrow 0$ are not equivalent. However, this function is

obtained from (3.4.1) by a redefinition of u and v , and hence preserves the properties of the Wigner function.

We now try to calculate F_M for specific states and see if it gives meaningful results. Of course, our interest is in the states we have been using to define the semi-classical limit of the two-particle Lagrangian (1.1). Consider first the state (3.1.6), which is obtained at order M^1 in the expansion in powers of M . For reasons discussed earlier, we set C_2 as zero, and for convenience we take C_1 as one. This state does not depend on q and for it, F_M is equal to

$$F_M(Q, P, p) = \frac{1}{M} \delta(p) \int_{-\infty}^{\infty} du \exp[-i(Pu/M)] \\ \times \exp \left[iMS_0 \left(Q + \frac{\hbar}{2M} u \right) / \hbar - iMS_0 \left(Q - \frac{\hbar}{2M} u \right) / \hbar \right]. \quad (3.4.4)$$

By expanding the wave-function in powers of $(1/M)$ and retaining up to order M^0 we get

$$F_M = \delta(p) \delta(P - MS'_0(Q)) \quad (3.4.5)$$

which gives the expected correlation between the position and momentum of Q .

Now consider the wave-function at order M^0 , namely Eq. (3.1.11). It is convenient to write it as an amplitude times a phase, as in (3.3.1). The Wigner function is

$$F_M(Q, P, q, p) = \frac{1}{Mm} \int_{-\infty}^{\infty} du dv \frac{1}{\sqrt{S'_0(Q + \hbar u/2M)}} \frac{1}{\sqrt{S'_0(Q - \hbar u/2M)}} \\ \times R \left(Q - \frac{\hbar}{2M} u, q - \frac{\hbar}{2m} v \right) R \left(Q + \frac{\hbar}{2M} u, q + \frac{\hbar}{2m} v \right) \\ \times \exp[-i(Pu/M) - i(pv/m)] \\ \times \exp \left(\frac{i}{\hbar} \right) \left[\beta \left(Q + \frac{\hbar}{2M} u, q + \frac{\hbar}{2m} v \right) - \beta \left(Q - \frac{\hbar}{2M} u, q - \frac{\hbar}{2m} v \right) \right] \\ \times \exp \left(\frac{iM}{\hbar} \right) \left[S_0 \left(Q + \frac{\hbar}{2M} u \right) - S_0 \left(Q - \frac{\hbar}{2M} u \right) \right]. \quad (3.4.6)$$

In this function q is a quantum mechanical degree of freedom at all orders, and we do not expect a correlation between the position and momentum of q . On the other hand, we expect a correlation between Q and P , in some approximation. To check for such a correlation we should obtain a marginal probability distribution $F_M(Q, P)$ by integrating $F_M(Q, P, q, p)$ over q and p . That is, we restrict ourselves to the Q, P section of the phase-space. Integrating the F_M of (3.4.6) over p and q gives

$$\begin{aligned}
 F_M(Q, P) = & \frac{1}{M} \int_{-\infty}^{\infty} dq du \frac{1}{\sqrt{S'_0(Q + \hbar u/2M)}} \frac{1}{\sqrt{S'_0(Q - \hbar u/2M)}} \\
 & \times R\left(Q - \frac{\hbar}{2M} u, q\right) R\left(Q + \frac{\hbar}{2M} u, q\right) \\
 & \times \exp[-i(Pu/M)] \exp\left(\frac{i}{\hbar}\right) \left[\beta\left(Q + \frac{\hbar}{2M} u, q\right) - \beta\left(Q - \frac{\hbar}{2M} u, q\right) \right] \\
 & \times \exp\left(\frac{iM}{\hbar}\right) \left[S_0\left(Q + \frac{\hbar u}{2M}\right) - S_0\left(Q - \frac{\hbar u}{2M}\right) \right]. \tag{3.4.7}
 \end{aligned}$$

Next, we expand R , β , and S_0 in powers of M^{-1} and retain only the leading terms. Such an expansion is equivalent to taking the limit $M \rightarrow \infty$ in the wavefunction. The result is

$$F_M(Q, P) = \frac{1}{S'_0(Q)} \int_{-\infty}^{\infty} dq R^2(Q, q) \delta[P - MS'_0(Q) - \beta'(Q, q)]. \tag{3.4.8}$$

This probability distribution can be interpreted as follows. If the particle q were at a well-defined position, it would contribute an amount $\partial\beta/\partial Q$ to the momentum for Q . However, the probability for it to be at q is $R^2(q, Q)$, and hence we average over all positions for q . We can say that a semiclassical theory holds if this averaging yields a correlation between Q and P . It then also becomes clear that the existence of a semiclassical limit depends on the quantum state for q and that the phase determines the back-reaction. The averaging in (3.4.8) appears to be more natural than the one which has to be made in replacing the derivative of the phase β by its expectation value. Moreover, the averaging in (3.4.8) is not the same as saying that

$$P = MS'_0 + \langle f | \beta' | f \rangle. \tag{3.4.9}$$

It is easier to see the difference if we work with an example. Consider again the oscillator we talked of in the previous section, and assume that it is in a gaussian state with zero mean position and zero mean momentum, as in Eq. (3.3.25). (That is, the oscillator is initially in its ground state, defined for a particular frequency. The frequency evolves with time because of an external perturbation.) To compute the F_M of (3.4.8) for this state we use the representation

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \exp[-i\lambda x]. \tag{3.4.10}$$

It can then be shown that

$$\begin{aligned}
 F_M(Q, P) = & \frac{1}{2\pi S'_0} \int_{-\infty}^{\infty} d\lambda \left(1 + \frac{i\lambda B'_1}{2B_R} \right)^{1/2} \\
 & \times \exp[-i\lambda(P - MS'_0 - \theta')]. \tag{3.4.11}
 \end{aligned}$$

If B'_1 is zero, we find that the momentum P is peaked at the value $MS'_0 + \theta'$. Upon comparison with Eq. (3.3.23) we conclude that then the back-reaction is equal to $\langle h \rangle$. We saw a similar result in the previous section also—if we use only the Q -dependent part of the phase to define the back-reaction then putting B'_1 as zero is a necessary condition for the back-reaction to equal $\langle h \rangle$.

What is more interesting is the case when B'_1 is not zero, but small compared to B_R , so that we may write

$$\left(1 + \frac{i\lambda B'_1}{2B_R}\right)^{-1/2} \approx \exp[-i\lambda B'_1/4B_R]. \quad (3.4.12)$$

In this case also, (3.4.11) implies that the momentum is peaked, but now at the value

$$P = MS'_0 + \theta' - \frac{B'_1}{4B_R}. \quad (3.4.13)$$

One can show that the expectation value of the hamiltonian in the state (3.3.25) is

$$\langle f | h | f \rangle = -\theta' + \frac{B'_1}{4B_R} \quad (3.4.14)$$

so that again the back-reaction in (3.4.13) is equal to $\langle h \rangle$. This is an interesting result because it tells: (i) the momentum P is peaked about a particular value only for $B'_1 \ll B_R$; (ii) when this peaking occurs, the back-reaction is $\langle h \rangle$. If this restriction on B'_1 does not hold, we will not get a delta function upon integration over q . We saw in the previous section that this restriction on B'_1 is equivalent to a slow variation of ω with time. This suggests that a semiclassical limit is definable only if the metric varies slowly. Also, it is encouraging to note that the Wigner function and the phase rule discussed earlier give similar results.

We are now in a position to conclude our analysis of higher order corrections to the WKB semiclassical approximation. We have reason to believe that the coupling between Q and q at order M^0 should be found using the phase of the wave-function. However, we have made only a partial attempt to answer the question: when can a semiclassical theory be defined? To answer that question, we must first settle what that semiclassical theory is. We tried alternatives to the conventional suggestion that the classical momentum for Q should be defined using the expectation value of the derivative of the phase. This conventional suggestion is in no sense derivable from the quantum theory.

The first alternative we suggested is that only that part of the phase should be used to define the classical momentum which depends on Q alone. This is a reasonable rule for a state like the ground state, and for states whose phase does not have a classical limit. The second alternative was to construct a modified Wigner function F_M by using M^{-1} rather than \hbar as the parameter. Here we find a criterion for the validity of a semiclassical theory. If the Wigner function is averaged

over q and the momentum for Q is sharply peaked around some value, we may say that Q is classical at order M^0 . (We must however remember the restrictions on the use of the Wigner function as a probability distribution.) In both these ways of defining the semiclassical limit we find that semiclassical Einstein equations with $\langle T_{ik} \rangle$ as a source have only a restricted validity.

We shall now try to bring out the connection between these two alternatives. In particular we will compare the two alternatives for our example of the oscillator. Let us return to Eq. (3.4.8) and rewrite it as

$$F_M(Q, P) = \frac{1}{2\pi S'_0} \int_{-\infty}^{\infty} d\lambda \exp(-i\lambda[P - MS'_0]) \times \int_{-\infty}^{\infty} dq R^2(q, Q) \exp i\lambda\beta'(q, Q). \quad (3.4.15)$$

It is easy to see that in general, the integral over q will not give the result $\exp[-i\lambda \langle \beta' \rangle]$. It is also easy to see that the condition for obtaining this result from the integration is

$$\langle \beta'^n \rangle = \langle \beta' \rangle^n; \quad n = 2, 3, \dots \quad (3.4.16)$$

Of course, this requirement can never be met exactly but only approximately. In other words the dispersion in β' should be small. The most relevant of these conditions is the leading one ($n = 2$), that is,

$$\langle \beta'^2 \rangle = \langle \beta' \rangle^2. \quad (3.4.17)$$

If these conditions hold the semiclassical equations will be determined by the momentum definition (3.4.9). We thus conclude that for obtaining the back-reaction as $\langle h \rangle$ the dispersion in the phase should be small. (Similar suggestions have been made earlier in Refs. [10, 19].)

It may be shown from the Schrodinger equation that (3.4.17) is equivalent to requiring

$$\langle h \rangle^2 - \langle h^2 \rangle = \hbar^2 \left\langle \frac{\ddot{R}}{R} \right\rangle, \quad (3.4.18)$$

where h is the Hamiltonian for q . This is not quite the requirement that the dispersion in $\langle h \rangle$ should be small, but a relation between the dispersion and the time-variation of R . The form of Eq. (3.4.18) has an interesting similarity with the equation satisfied by the phase of the wave-function in the sense that the term \ddot{R}/R is proportional to \hbar^2 . If we were to retain all the terms to order \hbar then the requirement of small dispersion in phase is same as requiring small dispersion in h . Moreover, there can be states in which R varies sufficiently slowly so that the right-hand side of (3.4.18) can be ignored. A very interesting case is that of the stationary state—in this case the amplitude R is a constant, and $\langle h^2 \rangle$ is identically equal to $\langle h \rangle^2$, so that the requirement (3.4.17) is naturally met. The same is true of a quasi-

stationary state. This confirms our earlier statement that a back-reaction can be defined in stationary and quasi-stationary states.

It can also be shown that for the ground state of the oscillator considered above, the condition (3.4.17) will hold if B'_1 is zero, and will hold approximately if B'_1 is nearly zero. To do that we go back to Eq. (3.3.25) which defines the ground state and use the phase to compute $\langle \beta' \rangle$ and $\langle \beta'^2 \rangle$. We find that

$$\langle \beta'^2 \rangle - \langle \beta' \rangle^2 = B_1'^2 [\langle q^4 \rangle - \langle q^2 \rangle^2] = \frac{B_1'^2}{4B_R^2}. \quad (3.4.19)$$

Thus we find that the adiabatic approximation for B_1 is a special case of the approximation of small dispersion. This is very interesting because it shows a simple connection between adiabaticity and small dispersion, both of which look like natural requirements for defining a semiclassical theory.

The Wigner function appears to provide the condition for the validity of the semiclassical theory. We may say that the semiclassical theory can be defined if there is a strong correlation between Q and P at order M^0 . To find out whether such a correlation exists we must average the Wigner function over q . This averaging procedure will define a semiclassical theory which in general will not have $\langle h \rangle$ as the back-reaction. If the dispersion in the phase is small, the classical momentum P is defined by (3.4.9), and a semiclassical theory with $\langle h \rangle$ as a source holds. Moreover, the example of the oscillator suggests that our first alternative (use the Q -dependent phase alone) will give $\langle h \rangle$ as source when the dispersion in the phase is small.

The Wigner function helps us find the general conditions under which $\langle h \rangle$ may be used as the source in a semiclassical theory. If these conditions do not hold we may define the semiclassical theory using our first alternative. We saw in the previous section that if the phase does not have a classical limit, it is possible to define a semiclassical theory using the Q -dependent phase alone. In particular, this may be done for the ground state (3.3.25). In this prescription the back-reaction will in general be different from $\langle h \rangle$. We also computed the Wigner function F_M for this state in Eq. (3.4.11) and found that in general F_M does not show a correlation between Q and P . In such a case our phase prescription may be used to define a back-reaction.

We can also find higher order corrections by using the gaussian semiclassical state for Q . We expect results similar to those obtained using WKB states, and hence we do not go into this analysis here.

Our approach to the semiclassical theory has been that of deriving it from a quantum theory. For two alternative approaches to semiclassical gravity the reader may see Boucher and Traschen [17], who *construct*, rather than derive semiclassical theories, and Hartle and Horowitz [20], who discuss the relation between semiclassical Einstein equations and a $1/N$ expansion of the quantum theory.

4. THE SEMICLASSICAL LIMIT TO QUANTUM GENERAL RELATIVITY

In Section 1 we introduced the equation of motion for a quantum field in curved space, Eq. (1.2), and the average energy equations (1.3). We also mentioned briefly the historical development of the semiclassical scheme. While working with the toy Lagrangian in Section 3 we often indicated what those results imply for gravity. In this section, we obtain the semiclassical limit of the Wheeler–DeWitt equation using the methods of Section 3 and compare our results with earlier work on the average energy equations.

4.1. *The Wheeler–DeWitt Equation*

The Einstein action

$$\mathcal{A} = (16\pi G)^{-1} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \mathcal{L}(\phi(x)) \tag{4.1.1}$$

can be rewritten using the metric

$$ds^2 = (N^2 - N_\alpha N^\alpha) dt^2 + 2N_\alpha dt dx^\alpha + h_{\alpha\beta} dx^\alpha dx^\beta \quad (\alpha, \beta = 1, 2, 3) \tag{4.1.2}$$

in the “3 + 1” form

$$\begin{aligned} \mathcal{A} = (16\pi G)^{-1} \int dt d^3\mathbf{x} [& \frac{1}{4} G_{\alpha\beta\gamma\delta} \dot{h}^{\alpha\beta} \dot{h}^{\gamma\delta} + \sqrt{|h|} {}^3R - NH - N_\alpha H^\alpha] \\ & + \int d^4x \sqrt{-g} [\frac{1}{2} \phi^i \phi_{,i} - V(\phi)]. \end{aligned} \tag{4.1.3}$$

Here,

$$G_{\alpha\beta\gamma\delta} = \frac{1}{2\sqrt{|h|}} [h_{\alpha\gamma} h_{\beta\delta} + h_{\alpha\delta} h_{\beta\gamma} - h_{\alpha\beta} h_{\gamma\delta}] \tag{4.1.4}$$

is the metric on superspace—the space of all metrics. H and H^α are respectively the superhamiltonian and supermomentum

$$\begin{aligned} H &= l^2 G_{iklm} \pi^{ik} \pi^{lm} - l^{-2} \sqrt{h} {}^3R \\ H_\alpha &= -2\pi_{\alpha;\beta}^\beta, \end{aligned} \tag{4.1.5}$$

where $l^2 = 16\pi G$ and $\pi_{\alpha\beta}$ is the momentum conjugate to $h_{\alpha\beta}$. The variation of the action w.r.t. N and N^α yields the constraint equations.

The action (4.1.3) has the same formal structure as the two-particle Lagrangian (1.1), except for the constraints. This is seen more easily if we write the superspace coordinates and superspace metric as

$$h_A = h_{\alpha\beta}; \quad G_{AB} = G_{\alpha\beta\gamma\delta}. \tag{4.1.6}$$

One then sees that the gravitational part of the action is a sum of kinetic energy and potential energy terms and it scales with the inverse of Newton's gravitational constant G . The analogy with the toy model is

$$Q \rightarrow g_A; \quad q \rightarrow \phi; \quad V(Q) \rightarrow \sqrt{\hbar} {}^3R(g_A); \quad u(q, Q) \rightarrow \sqrt{-g} V(\phi). \quad (4.1.7)$$

The analogy is incomplete to the extent that in the toy model the kinetic energy term ($m\dot{q}^2/2$) does not depend on Q , whereas the kinetic energy of the matter field ϕ certainly depends on the metric. However, this has no significant effect on the conclusions of Section 3.

In the quantum theory the constraints are imposed as operator constraints on the wave-functional $\Psi(\phi, h_{\alpha\beta})$. In particular, the Hamiltonian constraint ($H|\Psi\rangle = H_{\text{matter}}|\Psi\rangle$, also called the Wheeler-DeWitt equation) is

$$\begin{aligned} & \frac{1}{2} [-l^2 \nabla^2 - l^{-2} \sqrt{|\hbar|} {}^3R] \Psi(h_{\alpha\beta}, \phi) \\ & = \sqrt{|\hbar|} H_m \left(\phi, -i \frac{\delta}{\delta\phi} \right) \Psi(h_{\alpha\beta}, \phi). \end{aligned} \quad (4.1.8)$$

The operator ∇^2 is defined by

$$\nabla_x^2 = G_{ijkl} \frac{\delta^2}{\delta h_{ij}(\mathbf{x}) \delta h_{kl}(\mathbf{x})}. \quad (4.1.9)$$

There are factor-ordering ambiguities in the definition of ∇^2 when we replace the product $G_{AB}\pi^A\pi^B$ by its operator version. We are making the choice of replacing this product by the Laplacian corresponding to the supermetric G_{AB} . It has the advantage that it is invariant under the transformation $g_A \rightarrow g'_A$ in superspace.

The analog of the parameter M^{-1} of the toy model is l^2 and the limit $l^2 \rightarrow 0$ should define a semiclassical theory. However, since there is no $\partial\Psi/\partial t$ term in (4.1.8), one must proceed with caution in applying the methods of Section 3. The WKB approximation can be applied to the W-D equation, considering it as a zero energy state, but the construction of a wave-packet necessitates introduction of a clock.

Consider first the WKB approximation. We proceed as in Section 3.1 by writing the wave-functional $\Psi(\phi, h_{\alpha\beta})$ as $\exp[iS(\phi, h_{\alpha\beta})/\hbar]$ and expand S in powers of l^{-2} . Taking cue from our earlier result, we rewrite Ψ as

$$\Psi(h_{\alpha\beta}, \phi) = \frac{1}{\sqrt{\nabla S_0}} \exp[iS_0(h_{\alpha\beta})/l^2\hbar] f(\phi(x), h_{\alpha\beta}), \quad (4.1.10)$$

where $\nabla S_0 = \delta S_0 / \delta h_{\alpha\beta}$. If we substitute this wave-functional in Eq. (4.1.8) and take the limit $l^2 \rightarrow 0$, we get the following equations at successive orders

$$\frac{1}{2} l^{-2} (\nabla S_0)^2 - l^{-2} {}^3R = 0 \tag{4.1.11}$$

$$i\hbar \nabla S_0 \cdot \nabla f = H_m \left(\phi, -i \frac{\delta}{\delta \phi} \right) f \tag{4.1.12}$$

(the dot product is defined in superspace using the supermetric). The first equation is the H-J equation for free gravity, and we identify $l^{-2} \nabla S_0$ with the canonical momentum. From this equation we can derive the Einstein equations [7]. The second equation represents quantum fields in curved space and is the Schrodinger equation for the wave-functional f .

Time evolution is introduced in the theory at the semiclassical level, through the relations

$$\frac{d}{d\tau} = \nabla S_0 \cdot \nabla; \quad \frac{dh_{\alpha\beta}}{d\tau} = G_{\alpha\beta\gamma\delta} \frac{\delta S_0}{\delta h_{\gamma\delta}}. \tag{4.1.13}$$

Note that the definition of the time τ depends on the choice of a particular metric $h_{\alpha\beta}$ as the solution to (4.1.11). There are two levels of ambiguities here. First, we could choose for $h_{\alpha\beta}$ one of the inequivalent solutions to the free Einstein equations. Having done that we could choose any coordinate system whatsoever. The semiclassical theory does not exhibit preference for one of the possible time coordinates over the others. In particular, it does not show a preference for the Minkowski time coordinate over, say, the proper time in a uniformly accelerating frame of reference.

We know, however, that when one talks of a quantum field in a non-inertial frame or in curved space, different coordinate times obtained by coordinate transformations are inequivalent. The definition of a particle very much depends on the choice of the time coordinate. A different way of saying this is that the vacuum states defined by using different time coordinates are inequivalent. In flat spacetime the Minkowski time coordinate is special, because all inertial observers agree on the definition of vacuum and on the definition of a particle.

It is also special for a different reason, which has to do with Mach's principle [11]. This principle states that in a frame in which distant stars are fixed an unaccelerated particle does not experience a pseudo-force. Implicit in this statement is the assumption that the time coordinate being used is the Minkowski time coordinate. It is important to realise that if we were to make a transformation to another time coordinate, an unaccelerated particle can experience a pseudo-force, even though it did not experience a pseudo-force when we were using Minkowski time, and even though distant stars continue to remain fixed w.r.t. this new time coordinate. Such time transformations are of course forbidden in classical mechanics (time is absolute) but become important when we try to think of a quantum version of Mach's principle.

The quantum version may be formulated in the following way. We first recall how we take the limit (QFT) \rightarrow (classical mechanics) in flat spacetime. We start from QFT, take the non-relativistic limit of the transition element $\langle 1_{\mathbf{k}} | \phi(x) | 0 \rangle$ and identify the resulting quantity with the wave-function in the one particle

Schrodinger equation. In the next step we take the classical limit ($\hbar \rightarrow 0$) in the Schrodinger equation to obtain classical mechanics.

The same procedure may be followed for defining the classical limit of a quantum field in curved space, and in particular for defining the classical limit of a quantum field in the real universe. Depending on the choice of the time coordinate different one-particle states can be defined. The quantum version of Mach's principle could be the following: the one particle state for a quantum field in the real universe should be defined in such a way that in the non-relativistic limit these particles do not experience a pseudo-force in the frame of fixed stars.

This has a bearing on quantum cosmology. We saw above that a specific one-particle state for the matter field is not picked up when we take the semiclassical limit of the W-D equation. Clearly, additional input is required; we must select a particular solution of the W-D equation. The "wave-function of the universe" must be so chosen that in the semiclassical limit it picks out that vacuum state which is consistent with Mach's principle, in the sense discussed above. This issue is currently under investigation.

We next come to the question of back-reaction and higher order corrections to the free Einstein equations. The rules obtained in Section 3 can in principle be generalised to a field theory. The phase rule, when applied to semiclassical gravity, says that if the matter wave-functional $f(\phi, h_{\alpha\beta})$ has the phase $\exp[i\beta(h_{\alpha\beta}, \phi)]$, and this phase does not have a classical limit, there can be a higher order correction (back-reaction) to the Einstein equation (4.1.11). This back-reaction will be determined by that part $\beta_\alpha(h_{\alpha\beta})$ of the phase which depends on $h_{\alpha\beta}$ alone, through the definition

$$\pi^{\alpha\beta} = l^{-2} \frac{\delta S_0}{\delta h_{\alpha\beta}} + \frac{\delta \beta_\alpha}{\delta h_{\alpha\beta}}. \quad (4.1.14)$$

This rule will be applicable, for example, if the matter field is in the ground state. In general, $\beta_\alpha(h_{\alpha\beta})$ will not be equal to $\langle f | h(\phi, h_{\alpha\beta}) | f \rangle$ —the expectation value of the matter Hamiltonian. The equality will approximately hold if the metric varies adiabatically.

The results obtained from the semiclassical Wigner function for the toy model suggest that $\langle T_{ik} \rangle$ will be the source for the semiclassical theory if the dispersion in the phase of the matter wave-functional is negligible. We must remember that the phase will in general involve divergent sums and we are not addressing the question of regularisation here. Because of this it is convenient to work, not with the fields, but with their minisuperspace models.

The conclusion is that a semiclassical limit can be defined for quantum gravity by using a special class of states; the leading term in a $(1/l^2)$ expansion in these states depends only on the metric. Such an expansion yields Eq. (1.4) in the leading order; at the next order it does not yield Eq. (1.3), except under the circumstances mentioned above. Thus (1.4) and (1.2) do not form a self-consistent system, but are obtained at successive orders.

4.2. Illustrations from Minisuperspace

In this section we apply the results of the semiclassical theory to quantum cosmological models. In many ways this is like working with a special case of the toy model—only now the model will be more realistic. We shall study the evolution of a massless, homogeneous scalar field in a $k = +1$ Robertson–Walker metric

$$ds^2 = dt^2 - a^2(t) \left[\frac{dr^2}{1-r^2} + r^2 d\Omega^2 \right] \tag{4.2.1}$$

for which the Einstein action (4.1.1) reduces to

$$\mathcal{A} = \int dt [M(-a\dot{a}^2 + a) + \pi^2 a^3 \dot{\phi}^2], \quad M = 3\pi/4G. \tag{4.2.2}$$

This is similar in form to the two-particle action of the toy model, except that the coupling between matter and gravity is through the kinetic energy term, and there is no potential for ϕ . It is the simplest non-trivial model we can think of. The classical equations of motion are

$$M(a\ddot{a} + \dot{a}^2) = -(M + 3\pi^2 a^2 \dot{\phi}^2) \tag{4.2.3}$$

$$\frac{d}{dt} (a^3 \dot{\phi}) = 0 \tag{4.2.4}$$

and the constraint equation is

$$Ma\dot{a}^2 + Ma = \pi^2 a^3 \dot{\phi}^2. \tag{4.2.5}$$

The equation of motion (4.2.3) can be derived from the constraint equation and Eq. (4.2.4). The conjugate momenta $p_a = -2Ma\dot{a}$, $p_\phi = 2\pi^2 a^3 \dot{\phi}$ imply that the Hamiltonian is

$$H = -\frac{p_a^2}{4Ma} + \frac{p_\phi^2}{4\pi^2 a^3} - Ma = 0 \tag{4.2.6}$$

and the Schrodinger equation for $\psi(a, \phi)$ is

$$-\frac{\hbar^2}{4Ma^2} \frac{\partial}{\partial a} \left(a \frac{\partial \psi}{\partial a} \right) + Ma\psi - \frac{\hbar^2}{(4\pi^2 a^3)} \frac{\partial^2 \psi}{\partial \phi^2} = 0. \tag{4.2.7}$$

We have chosen the operator ordering corresponding to the invariant Laplacian. This equation can be solved exactly and has solutions of the form

$$\psi(a, \phi) = N_q \exp(-iq\phi) K_\mu(v),$$

where $\mu = iqM/2\pi^2\hbar^2$, $v = 2M^2 a^2/\hbar^2$, and $K_\mu(v)$ is the MacDonald function.

Consider next the semiclassical theory. If at the semiclassical level the source

term for gravity is $\langle T_{ik} \rangle$ then the Eq. (4.2.5) will be replaced by the average energy equation

$$\frac{p^2}{8Ma} + \frac{1}{2} Ma = \frac{1}{2} \pi^2 a^3 \langle \phi^2 \rangle. \quad (4.2.8)$$

To obtain the semiclassical limit we write ψ of (4.2.7) as

$$\psi(a, \phi) = \frac{1}{\sqrt{S'_0(a)}} \exp[iMS_0(a)/\hbar] f(\phi, a) \quad (4.2.9)$$

and in the limit $M \rightarrow \infty$ find that

$$MS_0'^2 + 4Ma^2 = 0. \quad (4.2.10)$$

$$-i\hbar a^2 S'_0 \frac{\partial f}{\partial a} = -\frac{\hbar^2}{2\pi^2} \frac{\partial^2 f}{\partial \phi^2} \quad (4.2.11)$$

(prime denotes derivative w.r.t. a).

Equation (4.2.11) is a free-particle Schrodinger equation in the time coordinate defined by the relation $-a^2 S'_0 \partial/\partial a = \partial/\partial t$, and the energy eigenstates are plane waves

$$f(\phi, t) \sim \exp(ik\phi - i\omega t) \quad (4.2.12)$$

with $\hbar\omega = (\hbar k)^2/2\pi^2$. We now consider the back-reaction in various quantum states. If ϕ is in a plane-wave mode, the phase of the state is $(k\phi - \omega t)$. The part of the phase which depends only on the metric is $(-\omega t)$. Using the rule that the momentum for a will be determined by this part of the phase we write

$$p_a = MS'_0 - \hbar\omega t' = MS'_0 + \hbar\omega/a^2 S'_0 \quad (4.2.13)$$

which implies that

$$\frac{p_a^2}{8Ma} + \frac{1}{2} Ma = \frac{\hbar\omega}{4a^3} = \frac{1}{2} \pi^2 a^3 \langle f | \phi^2 | f \rangle. \quad (4.2.14)$$

This equation is the same as the constraint equation (4.2.5), except that ϕ^2 has been replaced by its expectation value. Thus we recover the average energy equation in the semiclassical limit, using our phase rule. It is easy to see why this happens. We noted in Section 3.3 that the expectation value of the derivative of the phase will give a back-reaction $\langle \hbar \rangle$. In the plane-wave the derivative of the phase w.r.t. a comes only from the term ωt , and hence the expectation value gives the same result as our phase-rule. Another way of seeing this is from the Wigner function. Since the plane-wave is a stationary state there is no dispersion in the derivative of the phase, and the back-reaction has to be $\langle \hbar \rangle$.

Having considered the back-reaction in an energy eigenstate, we consider another tractable case. Assume that ϕ is in a gaussian state, which is a solution to (4.2.11) obtained by superposing the plane-wave states:

$$f(\phi, t) = N(t) \exp[-B(\phi - f)^2 + ip(t)\phi/\hbar + i\varepsilon(t)/\hbar]. \tag{4.2.15}$$

This state differs from the ground state for the oscillator which we considered in Section 3.3, in the sense that now we are allowing for a non-zero mean position f and a non-zero mean momentum p . When used in Eq. (4.2.11) this state gives the relations

$$\begin{aligned} \dot{B} &= -2i\hbar B^2/\pi^2, & \dot{N} &= \hbar N B_1/\pi^2, \\ \dot{\varepsilon} &= -\frac{1}{\pi^2} \left(\hbar^2 B_R + \frac{1}{2} p^2 \right), & p &= \pi^2 \dot{f}, & \dot{p} &= 0, \end{aligned} \tag{4.2.16}$$

where $B = B_R + iB_I$. The equation for B may be solved to get

$$B(t) = \frac{B_0}{[1 + (2i\pi^2 B_0 t/\hbar)]}. \tag{4.2.17}$$

We now ask about the back-reaction in this state. First, we note from (4.2.16) that B is proportional to $1/\hbar$, and so is B_R . Hence the phase in this state will not have a classical limit, a result similar to the one we saw in Section 3.3 (i.e., the phase does not satisfy the H-J equation in the limit $\hbar \rightarrow 0$). Thus the phase-rule—use only the a dependent part of the phase to determine back-reaction—is applicable in this state. The gaussian state (4.2.15) provides a phase $\beta(t) = (-\hbar B_I f^2 + \varepsilon)$ at order M^0 , and the momentum p_a for the scale factor will be $MS'_0 + \beta'$. Recall that $\beta' = -\dot{\beta}/a^2 S'_0$. One can show, using Eq. (4.2.10), that

$$\frac{p_a^2}{8Ma} + \frac{1}{2} Ma = \frac{1}{4\pi^2 a^3} \left[\frac{1}{2} p^2 + 2\hbar B_I f p + \hbar^2 B_R + 2\hbar^2 f^2 (B_R^2 - B_I^2) \right]. \tag{4.2.18}$$

This is the semiclassical Einstein equation at order M^0 . To compare it with the average energy equation we note that for the gaussian state (4.2.15) we have

$$\frac{1}{2} \pi^2 a^3 \langle \dot{\phi}^2 \rangle = \frac{1}{8\pi^2 a^3} [p^2 + \hbar^2 |B|^2/B_R]. \tag{4.2.19}$$

Upon comparing (4.2.18) and (4.2.19) we find that the source term for the semiclassical Einstein equation differs from the source term $a^3 \langle \dot{\phi}^2 \rangle/2$ in the average energy equation. In the limit $\hbar \rightarrow 0$ the average energy equation and the semiclassical equation match, but not otherwise. This result is different from what we saw for the plane-wave state. We can also show that the dispersion of the derivative of the phase does not vanish for the gaussian state (4.2.15). Hence the modified Wigner function will not exhibit a correlation between p_a and a for this choice of the quan-

tum state, in the limit $M \rightarrow \infty$. On the other hand, we may define a semiclassical limit by applying the phase rule to this state.

Let us compare the back-reaction obtained here with that obtained for the ground state of the harmonic oscillator in Sections 3.3 and 3.4. We found there that using the phase-rule gives $\langle h \rangle$ as the back-reaction only in the adiabatic approximation. Out here the phase rule gives $\langle h \rangle$ as the back-reaction if ϕ is in an energy eigenstate, but not if ϕ is in the gaussian state. These results are all special cases of the general result that the phase will give $\langle h \rangle$ as the back-reaction if the dispersion in the phase is negligible.

The examples of the plane wave and the gaussian state show that the back-reaction very much depends on the choice of the quantum state for the matter variable. Because the semiclassical evolution of the scale-factor is not always governed by $\langle \dot{\phi}^2 \rangle$, the self-consistent solution to the semiclassical equations differs from those to the average energy equations. This is illustrated in the next section.

The various results on back-reaction can be summarised in the context of this minisuperspace model. The most general condition under which the source term is $\langle T_{ik} \rangle$ is when the dispersion in the phase of the matter wave-function is negligible. A special case of this is the adiabatic approximation—slow variation of the metric. We saw this while working with the example of a harmonic oscillator in its “ground state.” (It is useful to note that if in the gaussian state (4.2.15) we set $f = p = 0$, the state will be like the ground state of ϕ . We can again show, as we did for the oscillator, that the source for gravity will be $\langle h \rangle$ in the adiabatic approximation.) If the phase does not have a classical limit, part of it can be used to obtain the back-reaction; this will in general differ from $\langle h \rangle$, unless the condition on dispersion, mentioned above, is satisfied.

So far we considered the WKB semiclassical approximation for the gravity scalar system. Alternatively we can apply the phase rule by choosing a in a gaussian semiclassical state and obtain similar results for the back-reaction.

4.3. Comparison with an Exact Solution of Average Energy Equations

The gravity scalar system (4.2.2) introduced in the previous section can be solved exactly at the classical and quantum mechanical level. It also exhibits self-consistent solutions for the scale factor a and the quantum variable ϕ at the semiclassical level, when we use average energy equations. It is of interest to compare the self-consistent solutions of the average-energy equations with those of the semiclassical Einstein equations, in the cases in which they differ.

To write the classical solution to this system we note from (4.2.4) that

$$\dot{\phi} = \frac{p_0}{a^3}; \quad p_0 = \text{const} \quad (4.3.1)$$

and substitute for $\dot{\phi}$ in (4.2.5) to get an equation for the time evolution of $a(t)$

$$M\dot{a}^2 + M = \pi^2 p^2 / a^4. \quad (4.3.2)$$

The solution for a looks simpler when written in the conformal time $\tau = \int dt/a(t)$

$$a^2(\tau) = a_0^2 \text{Sin } 2(\tau + \tau_0). \tag{4.3.3}$$

This shows that the solution is an oscillatory universe in the conformal time.

If we were to convert this system into one satisfying the average energy equations, we would replace (4.2.4) by the Schrodinger equation

$$i\hbar \frac{\partial}{\partial t} f(\phi, t) = - \frac{\hbar^2}{4\pi^2 a^3} \frac{\partial^2 f}{\partial \phi^2} \tag{4.3.4}$$

and $\dot{\phi}^2$ by $\langle \psi | \dot{\phi}^2 | \psi \rangle$ in the other two classical equations. This system has interesting self-consistent solutions for a and ϕ which were studied by us in an earlier work [12]. Here, we briefly summarize this solution to the extent that it is relevant.

We begin by assuming that the wave-function f in (4.3.4) is a gaussian state. This is allowed because we are dealing with a "free particle." We choose this gaussian state to be of the form (4.2.15). Using the expression for $B(t)$ from (4.2.11), and for $\langle \dot{\phi}^2 \rangle$ from (4.2.17), it is easy to show that for the gaussian state we have

$$\langle f | \dot{\phi}^2 | f \rangle = \frac{1}{4\pi^4 a^6} [p^2 + \hbar^2 B_0], \tag{4.3.5}$$

where both p and B_0 are constants. The constant p_0 of (4.3.1) can be identified with the constant p , as both are proportional to the classical momentum of ϕ .

When one goes to the semiclassical level through $(\dot{\phi}^2) \rightarrow \langle \dot{\phi}^2 \rangle$, then, as (4.3.5) indicates, the only change that takes place in the solution (4.3.1) is in the value of the constant: $p^2 \rightarrow (p^2 + \hbar^2 B_0)$. This implies that the only change in the solution (4.3.3) is in the radius of oscillation: $a_0^2 \rightarrow a_0^2 + \hbar^2 B_0$. In particular, if p were zero, there will be no evolution for $a(t)$ at the classical level, but the term $\hbar^2 B_0$ will provide evolution at the semiclassical level. Quantum corrections sustain an oscillatory universe.

Another interesting self-consistent solution to the average energy equations is obtained if we add a constant potential V_0 to the Lagrangian of the scalar field. In this case the constraint equation can be shown to be

$$Ma\dot{a}^2 + Ma = \pi^2 a^3 \dot{\phi}^2 + 2\pi^2 a^3 V_0. \tag{4.3.6}$$

We can rewrite this equation, by first replacing for $\dot{\phi}$ and then using the definition of conformal time, as

$$\frac{1}{2} M \frac{da(\tau)^2}{d\tau} + V_{\text{eff}}(a) = 0, \tag{4.3.7}$$

where

$$V_{\text{eff}}(a) = \frac{1}{2} Ma^2 - \frac{\pi^2 p^2}{2a^2} - \pi^2 a^4 V_0. \tag{4.3.8}$$

By using the expression for $\langle \phi^2 \rangle$ from (4.3.5) we can see that the average energy equation will be almost the same as Eq. (4.3.7) except that p^2 will be replaced by $(p^2 + \hbar^2 B_0)$.

We are interested in the shape of the effective potential V_{eff} as a function of p . If p is non-zero the classical and semiclassical evolution will be nearly the same—the only change is in the constant p^2 . However, if p is zero, V_{eff} is zero at $a=0$, and the point $a=0$ is a local minimum—the evolution is oscillatory about $a=0$. At the semiclassical level, a new term $\hbar^2 B_0/2a^2$ is added to the potential— V_{eff} is now infinite at $a=0$, and the point $a=0$ is no longer a stable minimum. Quantum correction destroys the stability of the origin.

Having seen how the self-consistent classical solutions get modified if we use the average energy equations, let us see how the solutions to the average energy equations differ from those of the semiclassical theory obtained if we use the phase rule. That is, we want to study the semiclassical equation (4.2.18) and compare its solution with that of the average energy equation (4.2.8). For simplicity, consider the case $f=0$, so that (4.2.18) may be rewritten as

$$Ma\dot{a}^2 + Ma = \frac{1}{4\pi^2 a^3} \left[p^2 + \frac{2\hbar^2 B_0}{[1 + (4B_0^2 \pi^2 t^2 / \hbar^2)]} \right]. \quad (4.3.9)$$

The source term is no longer of the form (const/a^3) , as contrasted to the average energy equation, but evolves with time. In other words, the motion is in a time-dependent effective potential. By studying the form of the potential, we can say that the motion is bounded, but the maximum allowed value for the scale-factor shrinks with time. Moreover, the solution for $a(\tau)$ in the conformal time is no longer the same as (4.2.3), which is a sinusoid of constant amplitude. However, it is interesting to note that for sufficiently small times, the source term in (4.3.9) will be the same as $\langle T_{00} \rangle$.

Consider next the semiclassical constraint equation when a constant potential V_0 for the scalar field is included in the Lagrangian. This involves redoing the calculation of the previous section and one can finally show that the use of the phase-rule gives, instead of (4.3.9), the equation

$$Ma\dot{a}^2 + Ma = \frac{1}{4\pi^2 a^3} \left[p^2 + \frac{2\hbar^2 B_0}{[1 + (4B_0^2 \pi^2 t^2 / \hbar^2)]} + 4\pi^2 a^6 V_0 \right]. \quad (4.3.10)$$

(We have again set $f=0$.) We have to compare the evolution in this equation to that in (4.3.7). Like before, we can define an effective potential for $a(t)$. Again the interesting case is the one with $p=0$. By examining the form of this effective potential we can again show that quantum corrections destabilise the origin $a=0$ even though the potential is now time-dependent. In this respect the semiclassical equation gives the same conclusion as the average energy equation.

In general we expect the phase rule to provide different self-consistent solutions as compared to the average energy equations. It will be of interest to study other cosmological models from this point of view.

4.4. Graviton Production and the Minisuperspace Approximation

In Section 1 we mentioned an argument against the average energy equations: vacuum polarization effects for gravitons occur at the same order as for a matter field in curved space. If a time-dependent classical gravitational field produces particles of a quantum field it will also produce gravitons. Hence it is apparently wrong to talk of classical gravity coupled to quantum fields [13].

However, the above statement is not in contradiction with our claim that semiclassical gravity is a correct approximation. When the $M \rightarrow \infty$ limit is taken in the Wheeler–DeWitt equation, at order M^1 free Einstein equations appear. The gravitational field in these equations must be interpreted classically. No particles or gravitons are produced at this order of approximation of the theory.

At order M^0 we obtain the equation for quantum matter fields in curved space, we observe particle production, and by the phase rule, the Einstein equations at this order have a source term determined by the phase of the matter wave-functional. Only under special circumstances is this source term equal to the expectation value $\langle T_{ik} \rangle$.

Is it correct to consider the gravitational field to be classical at order M^0 ? To answer this, recall the Wigner function which tells us it is alright to use $\langle T_{ik} \rangle$ as the source if the dispersion in the phase can be neglected. In particular the T_{ik} may be that of the produced gravitons. We know that at order M^0 we can talk of linearized metric perturbations (gravitons) around the classical metric g_0 obtained at order M^1 . These perturbations h_{ik} will be described by a wave-functional $\Psi(h_{ik})$. For gravity to be classical at order M^0 the state $\Psi(h_{ik})$ should have a phase whose dispersion is negligible; alternatively the phase should not have a classical limit, and the part of the phase depending only on g_0 may be used to define a classical gravitational field at order M^0 .

Our analysis suggests that particle production in time-dependent gravitational fields by itself does not imply a back-reaction. We must first ascertain the correct semiclassical theory. The back-reaction arises only in a special class of matter and graviton states. Even then it may not always be of the form $\langle T_{ik} \rangle$.

Graviton production also has a bearing on the question of validity of the minisuperspace approximation in quantum cosmology. This approximation means that in a space-time, which has certain symmetries at the classical level, only a finite number of metric degrees of freedom are quantized, while the rest are frozen, that is, set equal to zero. It is important to realise that this freezing of unquantized modes cannot be exact and has to be consistent with the uncertainty principle.

The condition for the validity of this approximation is that the frozen modes should not get excited by a mechanism similar to particle production. To understand this, let us think of the minisuperspace quantization of the scale-factor $S(t)$ of the Robertson–Walker space-time, driven by a homogeneous scalar field $\phi(t)$. Consider next, the linearized metric perturbations $h_{ik}(x, t)$ around the Robertson–Walker metric. In the Fourier space these are a set of harmonic oscillators $h_{ik}(\mathbf{k}, t)$. We assume that they are all initially set in their ground states.

We very well know that when $S(t)$ evolves due to the scalar source, it can cause a transition of an h_{ik} mode from the ground state to an excited state. The transition probability depends on the rate of expansion, and the validity of the minisuperspace approximation requires this rate to be small, so that the transitions can be neglected. A related result was worked out earlier by Kuchar and Ryan [21] using quantum mechanical examples. We briefly recall below the results of Ref. [21] and then mention a calculation of Parker [22] on graviton production, in support of the above claim.

For the sake of convenience we adapt the results of Kuchar and Ryan [KR] to our two particle Lagrangian (1.1). Let us now think of Q and q as two degrees of freedom of the same particle, instead of thinking of them as two interacting particles. For example, if we were to describe the motion of a particle in cylindrical coordinates, Q and q could respectively be the radial and z coordinate (if we consider motions with zero angular momentum then θ can be set constant). We will think of Q as the minisuperspace coordinate and q as the external coordinate. Consider those classical motions for which $\dot{q} = 0$ initially. Such motions will always retain $\dot{q} = 0$ and hence are minisuperspace motions.

Consider next the quantum theory, described by the Schrodinger equation (3.0.3). Let us define the Hamiltonians

$$H(Q, P) = \frac{P^2}{2} + V(Q); \quad h(q, p; Q) = \frac{p^2}{2} + u(q, Q), \quad (4.4.1)$$

where $P = \dot{Q}$, $p = \dot{q}$. The evolution of $\psi(Q, q)$ cannot in general be reduced to the evolution of the state $\xi(Q)$ in minisuperspace. However, there may be conditions under which this reduction holds approximately. To find these conditions [KR] proceed as follows. They first solve the eigenvalue problem of the Hamiltonian h , considering Q as a parameter

$$h(q, p; Q) \chi_N(q; Q) = E_N(Q) \chi_N(q; Q). \quad (4.4.2)$$

Assume next that the state $\psi(Q, q, t)$ can be expanded in the basis $\chi_N(q; Q)$ as

$$\psi(Q, q, t) = \sum_N \xi_N(Q, t) \chi_N(q; Q). \quad (4.4.3)$$

Substituting this form for ψ in the Schrodinger equation [KR] deduce an equation for the time evolution of $\xi_N(Q, t)$. They then conclude that the minisuperspace states ξ_N can be treated in isolation from the external modes $\chi_M(q; Q)$ if the coefficients

$$\begin{aligned} A_{NM}(Q) &\equiv \int dq \chi_N^*(q; Q) \frac{d}{dz} \chi_M(q; Q) \\ B_{NM}(Q) &\equiv \frac{1}{2} \int dq \chi_N^*(q; Q) \frac{d^2}{dz^2} \chi_M(q; Q) \end{aligned} \quad (4.4.4)$$

are negligible. If they are not negligible, these coefficients cause transitions between the states $\xi_N(Q, t)$ and hence interfere with the minisuperspace evolution. Furthermore, the equation satisfied by ξ reduces to the Schrodinger equation with Hamiltonian h only if $E_N \ll V(Q)$. These are the conditions found by [KR] for the validity of the minisuperspace approximation.

In the language of gravity this may roughly be taken to mean the following. Let the minisuperspace coordinate Q stand for the FRW scale factor and the external coordinate q for the graviton mode in the external spacetime Q . Then the minisuperspace approximation is valid if the gravitons do not react back on the state $\xi(Q)$ describing the evolution of the scale factor. Thus [KR] take the point of view that if the minisuperspace approximation is to be valid, the external modes should not interfere with the evolution of the minisuperspace mode.

It seems to us, however, that this may not be a sufficient requirement. The external mode is frozen in the classical theory and is frozen in the quantum theory to the extent allowed by the uncertainty principle. One way of saying this could be that the external mode has been set in its ground state. Now it is quite plausible that the evolution of the minisuperspace coordinate Q can throw the external mode q out of the ground state. That should signal the breakdown of the minisuperspace approximation. This is a viewpoint complimentary to that of [KR]. In general we have no way of formulating this problem. This is because we want to find out whether the (quantum mechanical) mode Q causes transitions of the q mode from its ground state to another state. We can at the most write the eigenvalue equation for q treating Q as a parameter, but in general we have no reasonable way of asking how this equation is affected if Q is a quantum variable satisfying a Schrodinger equation.

One exception is the semiclassical theory we have been studying so far. We assume that in the Schrodinger equation (3.0.3) the limit $M \rightarrow \infty$ has been taken, so that the minisuperspace variable Q is in a WKB state, and the external variable q is a solution of the Schrodinger equation (3.1.9). In this approximation the minisuperspace states $\xi(Q)$ are WKB states, where Q is the solution of the H-J equation (3.1.5). The external variable q satisfies an equation with Q as a parameter. The point is that if the mode q has been frozen, say in its ground state, but gets excited even if Q is in a specific state (say the WKB state), we should consider that as the breakdown of the minisuperspace approximation. Thus we shall use the semiclassical limit of gravity to find the validity of this approximation. The minisuperspace variable will be the Robertson-Walker scale factor $a(t)$ and the external variable will be the linearized perturbation h_{ik} on the R-W metric.

We start with the action for free gravity (no matter fields) and write the metric g_{ik} as

$$g_{ik}(x, t) = g_{ik}^{\text{FRW}}(t) + h_{ik}(x, t), \tag{4.4.5}$$

where h_{ik} is a linearized perturbation on the metric. We shall treat g_{ik}^{FRW} as the minisuperspace variable and h_{ik} as the external variable. By substituting this form of the metric in the Einstein equations one can deduce the equation of motion

for h_{ik} . (The following calculation on graviton production is borrowed from Parker [22].) The h_{ik} 's satisfy eight coordinate conditions, so that there are two independent degrees of freedom left, corresponding to the two independent polarizations of the gravitational wave. If we separate h_{ik} into space and time parts as

$$h_i^j = \psi(t) G_i^j(\mathbf{x}) \quad (4.4.6)$$

then the equation satisfied by $\psi(t)$ is

$$a^{-1} \frac{d}{dt} \left(a^3 \frac{d\psi}{dt} \right) + (\kappa^2 + 2\varepsilon) \psi = 0, \quad (4.4.7)$$

where κ is determined by the eigenvalues of the Laplacian on the three surface and ε depends on the spatial curvature. Henceforth we assume the spatial sections to be flat, for which $\varepsilon = 0$. A further transformation to the variable

$$\tau = \int dt a^{-3}(t) \quad (4.4.8)$$

brings this equation to a simpler form.

We next consider the semiclassical limit where a is in a WKB state, and the variables h_{ik} are quantum mechanical. Since there are an infinity of them we will have to write a Schrodinger equation for the wave-functional $\Psi(h_{ik}; a)$. For once, we will switch to the Heisenberg picture for convenience, consider the h_{ik} as operators, and expand them in Fourier space. The creation and annihilation operators will be time-dependent and of the form $a_{\mathbf{k}} \psi_{\mathbf{k}}(t)$, where $\psi_{\mathbf{k}}(t)$ will satisfy (4.4.7).

To study graviton production we proceed as follows. Assume that the scale factor is nearly constant as ($\tau \rightarrow -\infty$) and that $\psi(\tau)$ has the asymptotic form

$$\psi = (2a^3\omega)^{-1/2} \exp \left[-i \int d\tau \omega a^3 \right] \quad (4.4.9)$$

as $\tau \rightarrow -\infty$, where $\omega = k/a$. This is the WKB form of the positive frequency solution. We also assume that all the graviton modes are initially set in their ground state, as defined w.r.t. this positive frequency. To calculate the production rate we must specify a particular form for the expansion factor. Parker uses the form for $a(\tau)$,

$$a^4(\tau) = a_1^4 + \exp(\tau s^{-1}) [(a_2^4 - a_1^4)(\exp(\tau s^{-1}) + 1) + b] (\exp(\tau s^{-1}) + 1)^{-2}. \quad (4.4.10)$$

Here a_1, a_2, b , and s are positive, adjustable parameters and they determine the shape of the curve and the rate of expansion. The rate of expansion is proportional

to $1/s$. It can be shown that in this form of the expansion the initial state (4.4.9) evolves into a mixture of positive and negative frequency solutions as $\tau \rightarrow \infty$:

$$\psi = (2a^3\omega)^{-1/2} \left[\alpha_k \exp\left(-i \int dt \omega a^3\right) + \beta_k \exp\left(i \int dt \omega a^3\right) \right]. \quad (4.4.11)$$

The rate of particle production is proportional to $|\beta_k/\alpha_k|^2$ and for sufficiently large frequencies k it can be shown

$$|\beta_k/\alpha_k|^2 \cong \exp(-4\pi s a_1^2 k). \quad (4.4.12)$$

Our interest in this result is to demonstrate the relation between the rate of expansion and the rate of particle production, both of which are determined by the parameter s . Another way of looking at graviton production is to think of it as the excitation of gravitons from its ground state. From the form of (4.4.12) we see that the production rate tends to zero when the rate of expansion (which is proportional to $1/s$) tends to zero. On the other hand, the frozen graviton mode gets strongly excited by a rapid expansion of the scale-factor. It is no longer in its ground state—the maximum excitation prescribed for a frozen mode in the minisuperspace models. In this case we should not consider the minisuperspace approximation to be valid.

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