

Newtonian N -body calculations of the advance of Mercury's perihelion

J. V. Narlikar *Tata Institute of Fundamental Research, Bombay 400 005, India*

N. C. Rana[★] *Department of Physics, University of Durham, South Road, Durham DH1 3LE*

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Summary. The rate of advance of the perihelion of planet Mercury is calculated by a numerical integration of the Newtonian equations of motion and gravitation. It is found that the rate fluctuates but has a steady trend of ~ 528.95 arcsec cy^{-1} over a long-term period of about five centuries. Taking into account the observed precession we therefore find that the general relativistic correction explains all but about 2.3 arcsec cy^{-1} of the discrepancy. It may be necessary to invoke another cause like solar oblateness to account for the residual effect.

1 Introduction

Of the three classical tests proposed in the early days of general relativity, two (namely, the gravitational redshift and the bending of light) have received accurate confirmation in recent years because of experiments performed with modern technology. The third test by contrast relies on a comparison of the theoretical prediction and the astronomical data on the orbit of planet Mercury collected over a long period of time.

Thus, it was known from observation that the heliocentric longitude of the perihelion of Mercury's orbit advances in space at a rate of about 575 arcsec cy^{-1} . While the Newtonian theory predicts a steady Keplerian elliptic orbit for an isolated planet moving round the Sun, it is also known that the perturbation of Mercury's orbit by the other planets in the Solar System leads to a slow rotation of the major axis of the ellipse with time. The classic calculation of Newcomb (1895–98) using techniques of celestial mechanics led to the conclusion that Mercury's perihelion should advance at a rate of ~ 532 arcsec cy^{-1} . Thus there was a discrepancy of ~ 43 arcsec cy^{-1} between observations and the Newtonian prediction – a discrepancy which is almost *exactly* accounted for by general relativity.

Excellent though the agreement is between the general theory and observations, it is worthwhile re-examining the above test more closely, for three reasons. First, this is the only test which tells us about the parameter β measuring the non-linearity in the superposition law for

[★]On leave of absence from the Tata Institute of Fundamental Research, Bombay.

Table 1. Reciprocal masses of the principal planets ($M_{\odot}=1$).

Planet	Newcomb	Duncombe <i>et al.</i>
Mercury	6000000	5988000±3300
Venus	408000	408517±11*
Earth+Moon	329390	328900.12±0.20
Mars	3093500	3098709±9
Jupiter	1047.355	1047.366±0.007
Saturn	3501.6	3498.1±0.3

*Due to Laubscher (1981).

g_{00} in the PPN metric (*cf.* Misner, Thorne & Wheeler 1973):

$$ds^2 = - \left(1 - 2\alpha \frac{M_{\odot}}{r} + 2\beta \frac{M_{\odot}^2}{r^2} \right) + \left(1 + 2\gamma \frac{M_{\odot}}{r} \right) [dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \quad (1)$$

The gravitational redshift experiment gives $\alpha=1$ while the bending of light (and radar echo delay) experiments give $\gamma=1$. The perihelion test is necessary to conclude that $\beta=1$.

The second reason for re-examination springs from the fact that the agreement between theory and observation is so good that it does not leave room for any other small effect. The recent controversial claim of solar oblateness by Hill, Randal & Goode (1982) has revived the possibility that part of the perihelion precession might be due to the oblate Sun. Hence it is worth investigation whether the agreement between the theory and observations is really that precise.

The third and motivating reason for this paper is the question of actual computation of the discrepancy between the observed value and the theoretical Newtonian prediction. Newcomb's value was computed on the basis of the mass estimates of the planets then available in the 1890s. The dominant perturbation is caused by Venus, Earth and Jupiter. The masses of the last two were known fairly accurately because of their satellite systems. The mass of Venus was determined by indirect methods (by its effect on the orbits of Mars and Earth) and could not be known to better than two parts in 10^3 (Laubscher 1981). Table 1 shows for comparison the reciprocal masses of the principal planets in units of M_{\odot}^{-1} as used by Newcomb and as known a present (Duncombe, Klepeczynski & Seidelmann 1973; Laubscher 1981). It is clearly necessary to repeat the calculation with the new revised inputs. Moreover, rather than use the perturbation techniques of celestial mechanics we will employ numerical methods which are known to provide accurate results on fast computers.

Assuming that the perturbation expansion technique of celestial mechanics and our N -body calculation are equally accurate we expect a discrepancy between our result and the classical calculations of Newcomb to arise from the mass differences in Table 1. A mass discrepancy of two parts in 10^3 suggests a discrepancy of the order of $1 \text{ arcsec} \text{ cy}^{-1}$ in the rate of Mercury's perihelion, an effect large enough to justify the calculation. Moreover, a numerical integration maintaining an accuracy of 1 part in 10^{13} at each step would also allow us to check whether the second-order perturbation theory for Mercury is adequate enough.

The method of calculation is outlined in Section 2 and the result is discussed in Section 3.

2 The method of calculation

As mentioned in Section 2 we have used an approach different from the traditional one employed by Newcomb to calculate the trajectory of Mercury. Treating the planetary system as an N -body system interacting via Newtonian gravitation, we employ the Bulirsch–Stoer method for our

numerical computation. This method has been coded by S. J. Aarseth for his program ORBITS (which had been tested by him extensively in other contexts). Our computations are based on this program run on the CYBER 170 computer.

We have taken the initial rectangular coordinates and velocities of the planets with respect to 1950.0 equinox and equator for the epoch JD 2444600.5 from the *Astronomical Almanac 1981* (page E4, H. M. Nautical Office, London). The planetary masses, with the exception of Venus are taken from Duncombe *et al.* (1973). For the mass of Venus we have used the estimate given by Laubscher (1981) based on all data. Externally we had specified an integration step of five ephemeris days. However, the program had also a built-in specification that determined step length such that an accuracy of one part in 10^{13} was maintained. The stability of the numerical integration procedure is checked by reversing the steps and comparing with the initial values. Thus integrating over a 10-year period and reversing back to the starting epoch introduces an accumulation of errors less than 10^{-11} for Mercury and less than 10^{-12} for the other planets. Hence for computations extending over, say, 500 yr, the calculated positions are expected to be accurate to well within 0.01 arcsec. The correctness of this procedure has been checked by computing Jupiter's orbit (which does not require any appreciable general relativistic corrections) for the past 10 years and finding that it tallies with the published ephemeris.

The initial coordinates and velocities of the planets are, however, always to be determined from observations. Geocentric positions of the centre of the disc of the planets are hardly known to an accuracy of better than 0.1 arcsec. This meaningfully limits us to use only seven significant digits for the initial coordinates and velocities of the planets. So the actual error in the orbit calculations due to this uncertainty in the initial conditions should amount to about 1 arcsec per decade. But we are interested here in the rate of change of the orbital elements, rather than in the orbits themselves. The orbital elements evolve very slowly, about a million times slower than the orbit. Hence the effect of the orbital uncertainty on the motion of the perihelion is negligibly small. The uncertainty due to the present day knowledge of the mass values of the planets imply an error no larger than $0.01 \text{ arcsec cy}^{-1}$ in the rate of perihelion precession of Mercury (Morrison & Ward 1975).

We next compute at 10 ephemeris days interval the positions and the velocities of all planets in rectangular Cartesian coordinates with reference to the equinox and ecliptic of B1950.0. Then we proceed to calculate the osculating heliocentric orbital elements using the formulae given in the *Explanatory Supplement to the Ephemeris 1961* (p. 115–117, H. M. Nautical Office, London). The osculating heliocentric longitude of the perihelion of Mercury is the element that we are interested in. The result for the data at the initial epoch exactly reproduces the ephemeris value of the osculating longitude of Mercury's perihelion. The first indication that the Newtonian inverse-square law is not sufficient to account for the fine details of the orbital motion of Mercury comes from the discrepancies which show up for a similar calculation at other epochs. Our computations extend from JD 2444600.5 to 2526600.5 in the forward direction and to JD 236200.5 in the backward direction, thus covering a total time span of about 450 yr (AD 1756.5 to 2205.5).

3 Results

Fig. 1 shows the computed trajectory of the perihelion of Mercury over a period of a few years. The motion of the perihelion is far from being smooth and regular; there are small as well as large-scale fluctuations in it whose origin can be understood as follows.

A large fluctuation, in the form of a rapid advance of the perihelion comes, for example, from the proximity of Venus to Mercury when the latter is close to the aphelion. Over a few weeks around this juncture the perihelion advances by as much as 10 arcsec. Such a jump had occurred around 1982 November 27 (JD 2445300.5) according to Fig. 1.

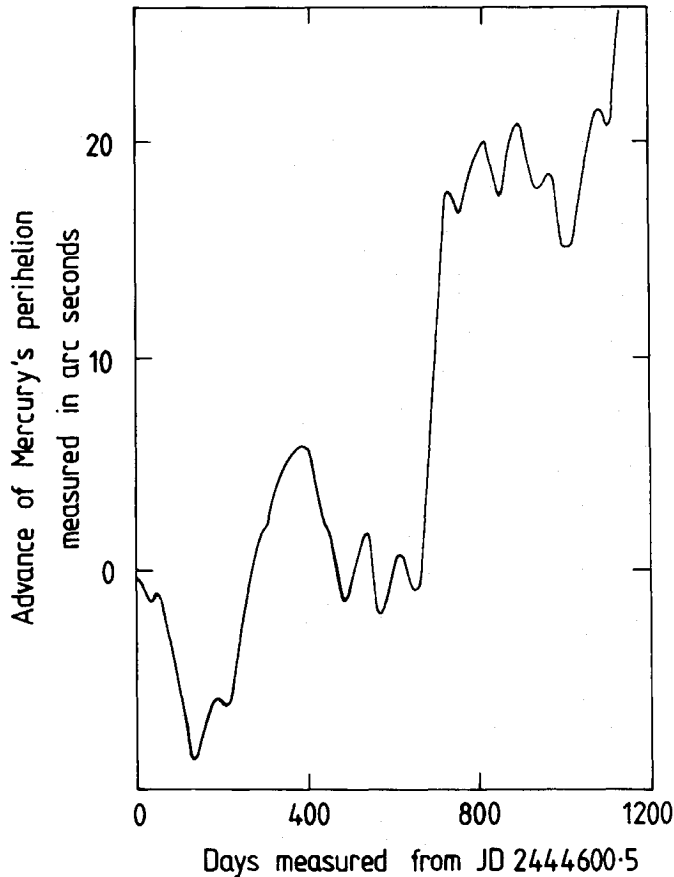


Figure 1. The osculating heliocentric longitude of the perihelion of Mercury is plotted against time from JD 2444600.5. The periodicity found in the perihelion precession is discussed in the text.

The nature of the fluctuations seems to be similar if the relative positional configuration of Mercury, Venus, Earth and Jupiter are the same, and to this extent periodicities may be detected in some fluctuations. The most important repeated phenomenon is of course, the Venus–Mercury configuration. The orbital periods of Venus and Mercury have the ratio ≈ 2.55433 , which may be approximated by ratios 5:2, 23:9, 212:83, 235:92 and 1387:543, of their orbital revolutions. Taking various combinations of these fundamental periods we have listed in Table 2 a few long periods of approximate duration of 50–60 yr that cover the entire period of our analysis.

In these calculations the zero epoch has been set arbitrarily at JD 2444600.5 and the entries in Table 2 are given backwards and forwards in time with respect to this epoch in a symmetrical manner so that the time elapsed between any two rows separated by five entries is 283.00 yr. This interval corresponds to the period after which the Earth, Venus and Mercury all come back (approximately) to the original orbital configuration. Jupiter, however, is off from the original configuration by an angle of about 50° .

The periodicities in Fig. 1 can be ‘eliminated’ by fitting a trend curve to the numerical data. We have done so by fitting a curve of the following type

$$y = a + bt + ct^2, \quad (2)$$

where y is the advance of the perihelion measured from the initial position at time t . The last column in Table 2 gives the numerically computed time average of y over the interval $(0, T)$ for

Table 2. Integral for the secular motion of the perihelion of Mercury.

Period ^a in days	Integral number ^b of revolutions of Mercury and Venus		The Julian day epoch JD	The integral $\bar{y}(T)$ in arcsec
	Venus	Mercury		
-80667.6 (-220.86)	-917	-359	2363932.9	-578.3955
-62017.8 (-169.80)	-705	-276	2382582.7	-443.2584
-41345.2 (-113.20)	-470	-184	2403255.3	-293.6128
-22695.9 (-62.14)	-258	-101	2421904.6	-158.4519
0	0	0	2444600.5	0
22695.9 (62.14)	258	101	2467296.4	169.9248
41345.2 (113.20)	470	184	2485945.7	305.2303
62017.8 (169.80)	705	276	2506617.3	454.7777
80667.6 (220.86)	917	359	2525268.1	589.8654

^aThe figure given in bracket against each entry in this column corresponds to the approximate equivalent in years.

^bTo bring out the periodicity in $\bar{y}(T)$ we have chosen the T -values to correspond to integral number of revolutions of Mercury and Venus, as closely as possible.

the different values of $T \geq 0$. For (2) this is given by

$$\bar{y}(T) = a + \frac{1}{2}bT + \frac{1}{3}cT^2. \quad (3)$$

$\bar{y}(T)$ may be taken to represent the mean advance of perihelion of Mercury, while the coefficients a , b and c can be determined from the comparison of numerical values with (3). Notice that b is immediately given by comparing two symmetrical entries:

$$b = 1/T[\bar{y}(T) - \bar{y}(-T)]. \quad (4)$$

Alternatively b may be estimated from the second differences of entries in Table 2 which eliminate c and a . This integration procedure for evaluating b is superior to the usual least-squares method which is better suited for random errors than periodic deviations. However, all the estimates of b lie fairly close to one another, giving the range of b as

$$b = (528.95 \pm 0.05) \text{ arcsec cy}^{-1}. \quad (5)$$

This is the Newtonian rate of precession of the perihelion of Mercury, accounting for the present work.

The estimate of c is not accurate since its value is of order of the error terms in b averaged over $T \sim 283$ yr. At best we can assert that

$$|c| < 0.10 \text{ arcsec cy}^{-2}. \quad (6)$$

The constant a does not have a significant role. It merely gives the correction to be applied to the osculating longitude of the perihelion at the zero epoch in order to get the mean longitude of the perihelion. We find

$$a = 5.74 \pm 0.03 \text{ arcsec}. \quad (7)$$

Having applied this correction, the mean longitude of the perihelion of Mercury at the zero epoch tallies with its published ephemeris value to within 1 arcsec.

The results (5) and (6) are found to be independent of the choice of coordinate systems. Moving to the FK5 system and using the initial coordinates, velocities and mass values as given in the *Astronomical Almanac 1984*, we have essentially obtained the same result.

4 Discussion

Table 3 lists a few estimated rates of advance of Mercury's perihelion and the various contributions to it from known and established theoretical causes. On the whole the estimates are remarkably consistent so long as the Newtonian component is about $531.2 \text{ arcsec cy}^{-1}$. So the discrepancy is $2.3 \text{ arcsec cy}^{-1}$. In view of this discrepancy we suspect that the second-order perturbation theory is perhaps not adequate for computing the orbit of Mercury at least. We also feel that the analysis of the observed data on Mercury might have been biased by the theory, which was defective. Clemence (1943) and later Morrison & Ward (1975) had used the same old perturbation theory of Newcomb. Lestrade & Bretagnon (1982) also did not use sufficiently accurate equations of motion of the planets, which are found, for example in *Gravitation*, by Misner *et al.* (1973). The present work thus urges a re-analysis of the observational data on Mercury's position with adequate care.

Since the residual found by us is positive rather than negative, it could not be accounted for by the Brans–Dicke theory which predicts a precession rate

$$\eta = \frac{3\omega + 4}{3\omega + 6} \quad (8)$$

times the general relativistic value. For the positive effect required here we need $\omega \cong -17$. While a large negative ω would make the Brans–Dicke theory consistent with the findings of the bending of light test, its Machian interpretation becomes somewhat unphysical.

The other alternative is solar oblateness which does give a positive effect in a natural way. For the effect to be explained here the oblateness parameter is required to be

$$J_2 = 1.8 \times 10^{-5} \quad (9)$$

Table 3. Various estimates of the rate of perihelion advance of Mercury^a.

I. By Clemence (1943), with reference to J1900.00	
Total observed rate	5600.73 ± 0.41
Precession of the equinox	5026.50
	574.23 ± 0.41
II. By Clemence (1947), with reference to J1850.0	
Total observed rate	5599.74 ± 0.41
Precession of the equinox	5025.65 ± 0.50
	574.09 ± 0.65
III. By Bretagnon (1982), with reference to J2000.0	
Total observed rate ^b	5603.33 ± 2.11T
Precession of the equinox ^b	5029.10 ± 2.22T
	574.24 – 0.11T
General Relativity effect ^b	42.98 + 0.00T
	531.26 – 0.11T
Newtonian component (present work)	528.95 ± 0.05
Residual	2.31

^a All entries are in units of arcsec per Julian Century.

^b T corresponds to time interval in Julian centuries between J2000.0 and any arbitrary epoch.

which is ~ 3 times larger than the recent estimate of Hill *et al.* (1982). A more recent claim by Duvall *et al.* (1984) suggests even smaller solar oblateness: $J_2 = (1.7 \pm 0.4) \times 10^{-7}$. Indeed, the reduction of data on the orbit of Mercury will have to be done even more carefully than hitherto if the residual effect found by us cannot be explained by solar oblateness.

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Note added in proof

A satisfactory explanation has now been obtained for the above residual of $2.31 \text{ arcsec cy}^{-1}$. Since the ecliptic is not an absolutely fixed plane, it rotates very slowly (about $47 \text{ arcsec cy}^{-1}$) about an instantaneous axis which also advances at a slow rate ($870 \text{ arcsec cy}^{-1}$). Even such a slow motion of the ecliptic has an effect on the apparent rate of advance of Mercury's perihelion when referred to the equinox and ecliptic of date, in excess of the adopted general precession of the equinox. This extra effect amounts to $2.30 \text{ arcsec cy}^{-1}$, the details of which and the motion of the node of Mercury as well will hopefully be given in a separate communication. We now therefore conclude that the motion of the perihelion of Mercury is in excellent agreement with the general theory of relativity and that the solar oblateness, if any, should be negligibly small.