

INFLATION IN BIANCHI MODELS AND THE COSMIC NO HAIR THEOREM IN BRANE WORLD

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Abstract

In this paper, the cosmic no hair theorem for anisotropic Bianchi models which admit inflation with a scalar field is studied in the framework of Brane world. It is found that all Bianchi models except Bianchi type IX, transit to an inflationary regime with vanishing anisotropy. In the Brane world, anisotropic universe approaches the inflationary era much faster than that in the general theory of relativity. The form of the potential does not affect the evolution in the inflationary epoch. However, the late time behaviour is controlled by a constant additive factor in the potential for the inflaton field.

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Over the past couple of years there is a growing interest to study cosmological models in the framework of higher dimensional space-time motivated by the developments in superstring and M-theory [1,2]. In these theories, gravity is a higher dimensional theory which reduces effectively 4-dimensional at lower energy scale. Such higher dimensional theories open up the possibility of solving the hierarchy problem in particle physics by considering large compactified extra dimensions and make the string scale accessible to the future accelerators [3]. In such scenario our observed universe is described by a brane embedded in higher dimensional spacetime [4] and usual matter field and force except for gravity that are confined on the brane. The gravitational field may propagate through the bulk dimensions perpendicular to the brane. Randall and Sundrum [2] shown that even if the extra dimensions are not compact, four dimensional Newtonian gravity is recovered in five dimensional anti-de Sitter spacetime (ADS_5) in low energy limit.

Recently in the brane world scenario, homogeneous and isotropic cosmological models are studied [5] which describe the early universe satisfactorily. Maartens *et al.* [6] shown that chaotic inflation can be accommodated on the brane and found that the modified braneworld Friedmann equation leads to a stronger condition for inflation. The brane effects ease the condition for slow-roll inflation for a given potential. Maartens, Sahni and Saini [7] also explored the behavior of an anisotropic Bianchi type-I brane world in the presence of a scalar field. They found that a large anisotropy on the brane does not prevent inflation, moreover, a large anisotropy enhances more damping into the scalar field equation of motion, resulting greater inflation. In brane world, cosmological solutions with a singularity and without singularity in an anisotropic Bianchi type-I universe are obtained by the author [8] which accommodate inflationary regime. Toporensky [9] explored the shear dynamics in Bianchi type I cosmological model on a brane with perfect fluid obeying the equation of state $p = (\gamma - 1)\rho$ where ρ and p are energy density and pressure respec-

tively and γ is a constant. He found that for $1 < \gamma < 2$, the shear attains a maximum value during its transition from non-standard to standard cosmology i.e., when the matter energy density is comparable to the brane tension. Campos and Sopena [10] studied qualitatively two of the Bianchi universes, Bianchi-I and Bianchi type V in the Randall-Sundrum brane world scenario with matter on the brane obeying the barotropic equation of state. It is found that anisotropic Bianchi I and V braneworlds always isotropize, although there could be intermediate stages in which the anisotropy grows. The brane world scenario near the bigbang is found to differ from the general theory of relativity (henceforth, GTR). The anisotropy dominates for $\gamma \leq 1$ in the braneworld whereas in GTR it happens for $\gamma \leq 2$. Frolov [11] carried out geometrical construction of the Randall-Sundrum braneworld, without the assumption of spatial isotropy but by considering a homogeneous and anisotropic Kasner type solution of the Einstein-Ads equation in the bulk. Recently, Santos *et al.* [12] studied no hair theorem for global anisotropy in the brane world scenario following Wald [13]. It is found that the brane matter and bulk metric in the anisotropic brane under certain condition evolves asymptotically to a de Sitter space-time in the presence of a positive four dimensional cosmological constant. In this brief report cosmic no hair theorem in anisotropic Bianchi brane in the presence of matter described by a scalar field is studied. In the brane the effective four dimensional cosmological constant may be assumed to be zero by the choice of the brane tension which is also considered here.

Consider a five dimensional (bulk) space-time in which the Einstein's field equation is given by

$$G_{AB}^{(5)} = \tilde{\kappa}^2 \left[-g_{AB}^{(5)} \Lambda_{(5)} + T_{AB}^{(5)} \right] \quad (1)$$

with $T_{AB}^{(5)} = \delta(y)[- \lambda g_{AB} + T_{AB}]$. Here $\tilde{\kappa}$ represents the five dimensional gravitational coupling constant, $g_{AB}^{(5)}$, $G_{AB}^{(5)}$ and $\Lambda_{(5)}$ are the metric, Einstein tensor and the cosmological constant of the bulk space-time respectively, T_{AB} is the matter energy momentum tensor.

We have $\tilde{\kappa} = \frac{8\pi}{M_P^3}$, where $M_P = 1.2 \times 10^{19}$ GeV. A natural choice of coordinates is $x^A = (x^\mu, y)$ where $x^\mu = (t, x^i)$ are space-time coordinates on the brane. The the upper case Latin letters ($A, B, \dots = 0, \dots, 4$) represents coordinate indices in the bulk spacetime, the Greek letters ($\mu, \nu, \dots = 0, \dots, 3$) for the coordinate indices in the four dimensional spacetime and the small case latin letters ($i, j = 1, 2, 3$) for three space. The space-like hypersurface $x^4 = y = 0$ gives the brane world and g_{AB} is its induced metric, λ is the tension of the brane which is assumed to be positive in order to recover conventional general theory of gravity (GTR) on the brane. The bulk cosmological constant $\Lambda_{(5)}$ is negative and represents the five dimensional cosmological constant.

The field equations induced on the brane are derived using geometric approach [14] leading to new terms which carry bulk effects on the brane. The modified dynamical equations on the brane is

$$G_{\mu\nu}^{(5)} = -\Lambda g_{\mu\nu} + \kappa^2 T_{\mu\nu} + \tilde{\kappa}^4 S_{\mu\nu} - E_{\mu\nu}^{(5)}. \quad (2)$$

The effective cosmological constant Λ and the four dimensional constant κ on the brane are given by

$$\Lambda = \frac{|\Lambda_5|}{2} \left[\left(\frac{\lambda}{\lambda_c} \right)^2 - 1 \right]$$

$$\kappa^2 = \frac{1}{6} \lambda \tilde{\kappa}^4, \quad (3)$$

respectively, where λ_c is the critical brane tension which is given by

$$\lambda_c = 6 \frac{|\Lambda_5|}{\kappa_5^2}. \quad (4)$$

However one can make the effective four dimensional cosmological constant zero by a choice of the brane tension. The extra dimensional corrections to the Einstein equations on the brane are of two types and are given by :

- $S_{\mu\nu}$: quadratic in the matter variables which is

$$S_{\mu\nu} = \frac{1}{12}TT_{\mu\nu} - \frac{1}{4}T_{\mu\alpha}T_{\nu}^{\alpha} + \frac{1}{24}g_{\mu\nu} \left[3T_{\alpha\beta}T_{\alpha\beta} - (T_{\alpha}^{\alpha})^2 \right]. \quad (5)$$

where $T = T_{\alpha}^{\alpha}$, $S_{\mu\nu}$ is significant at high energies i.e., $\rho > \lambda$,

- $E_{\mu\nu}^{(5)}$: occurs due to the non-local effects from the free gravitational field in the bulk, which enters in the equation via the projection $\mathbf{E}_{AB}^{(5)} = C_{ACBD}^{(5)}n^Cn^D$ where n^A is normal to the surface ($n^An_A = 1$). The term is symmetric and traceless and without components orthogonal to the brane, so $\mathbf{E}_{AB}n^B = 0$ and $\mathbf{E}_{AB} \rightarrow E_{\mu\nu}g_A^{\mu}g_B^{\nu}$ as $y \rightarrow 0$.

To analyse the cosmological evolution, we consider two components of the dynamical equation (2). First we consider the "initial-value" constraint equation

$$G_{\mu\nu}n^{\mu}n^{\nu} = \kappa^2T_{\mu\nu}n^{\mu}n^{\nu} + \tilde{\kappa}^4S_{\mu\nu}n^{\mu}n^{\nu} - E_{\mu\nu}n^{\mu}n^{\nu} \quad (6)$$

and the Raychaudhuri equation

$$R_{\mu\nu}n^{\mu}n^{\nu} = \kappa^2 \left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \right) n^{\mu}n^{\nu} + \tilde{\kappa}^4 \left(S_{\mu\nu} - \frac{1}{2}g_{\mu\nu}S \right) n^{\mu}n^{\nu} - E_{\mu\nu}n^{\mu}n^{\nu} \quad (7)$$

where n^{μ} is the unit normal to the homogeneous hypersurface. It may be pointed out here that both $G_{\mu\nu}n^{\mu}n^{\nu}$ and $R_{\mu\nu}n^{\mu}n^{\nu}$ are expressed in terms of the three geometry of the homogeneous hypersurfaces and the extrinsic curvature $K_{\mu\nu} = \nabla_{\nu}n_{\mu}$ respectively. The extrinsic curvature can be decomposed into its trace K and trace-free part $\sigma_{\mu\nu}$ which represents the shear of the timelike geodesic congruence orthogonal to the homogeneous hypersurface

$$K_{\mu\nu} = \frac{1}{3}Kh_{\mu\nu} + \sigma_{\mu\nu} \quad (8)$$

where $h_{\mu\nu} = g_{\mu\nu} + n_{\mu}n_{\nu}$, projects orthogonal to n_{μ} . We consider a scalar field theory to describe the energy momentum tensor which is given by

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu} \left[\frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} + V(\phi) \right] \quad (9)$$

with $V(\phi) = V_o + a\phi^4$, where a and V_o are constant. For a homogeneous scalar field the dynamical equation (6) and (7) can now be written as

$$K^2 = 3\kappa^2 \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) + \frac{\tilde{\kappa}^4}{4} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right)^2 + \frac{3}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} - \frac{3}{2} {}^{(3)}R - 3E_{\mu\nu}n^\mu n^\nu, \quad (10)$$

$$\dot{K} = \kappa^2 \left(-\dot{\phi}^2 + V(\phi) \right) - \frac{\tilde{\kappa}^4}{4} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) \left(\frac{5}{2}\dot{\phi}^2 - V(\phi) \right) - \frac{1}{3}K^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + E_{\mu\nu}n^\mu n^\nu \quad (11)$$

and the wave equation for the scalar field is given by

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi} \quad (12)$$

with ${}^{(3)}R$ as the scalar curvature of the homogeneous hypersurface. In fact ${}^{(3)}R$ can be expressed in terms of the structure constants tensor $C_{\beta\mu}^\alpha$ of the Lie algebra of the spatial group (of the space-time model) as

$${}^{(3)}R = -C_{\alpha\beta}^\alpha C_{\cdot\mu}^{\beta,\mu} + \frac{1}{2}C_{\beta\mu}^\alpha C_{\cdot\alpha}^{\mu,\beta} - \frac{1}{4}C_{\alpha\beta\mu} C^{\alpha\beta\mu} \quad (13)$$

(in this case raising and lowering of indices is done by $h_{\alpha\beta}$). Now, introducing the three form $\epsilon_{\alpha\beta\mu}$ (totally antisymmetric tensor) on the Lie algebra and using the antisymmetric property of the structure constants ($C_{\alpha\beta}^\mu = -C_{\beta\alpha}^\mu$) one can write [13]

$$C_{\alpha\beta}^\mu = M^{\mu\nu} \epsilon_{\nu\alpha\beta} + \delta_{[\alpha}^\mu A_{\beta]} \quad (14)$$

with $M^{\alpha\beta}$ as the symmetric tensor and A_α the dual vector. If we consider the Jacobi identity

$$C_{\nu[\alpha}^\mu C_{\beta\mu]}^\nu = 0$$

then one gets

$$M^{\alpha\beta} A_\beta = 0. \quad (15)$$

Consequently one obtains the three space curvature which is given by

$${}^{(3)}R = -\frac{3}{2}A_\beta A^\beta - h^{-1} \left(M^{\alpha\beta} M_{\alpha\beta} - \frac{1}{2}M^2 \right) \quad (16)$$

We note that ${}^{(3)}R > 0$ only if $M^{\alpha\beta}$ is either positive definite or negative. But from equation (15) we have $A_\beta = 0$, which is the symmetry for Bianchi type IX spacetime. Hence, except for Bianchi type IX we always get

$${}^{(3)}R \leq 0 \quad (17)$$

Thus using the inequality it is evident from equation (10) that the constraint equation leads to the inequality $K^2 > 0$ i.e., $K > 0$ (i.e., it will expand for ever) if the space-time is initially expanding and satisfying the constraint given by

$$E_{\mu\nu}n^\mu n^\nu \leq 0. \quad (18)$$

So, from equation (10) we have

$$K^2 > 3\kappa^2 \left[\frac{1}{2}\dot{\phi}^2 + V(\phi) \right] + \frac{\tilde{\kappa}^4}{4} \left[\frac{1}{2}\dot{\phi}^2 + V(\phi) \right]^2 \quad (19)$$

for all time t . Let us now define (using the ideas of Wald's [13] and the previous paper in GTR [15])

$$\begin{aligned} K_\phi &= K^2 - 3\kappa^2 \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right) + \frac{\tilde{\kappa}^4}{4} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi) \right)^2 \\ &= \frac{3}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} - \frac{3^{(3)}}{2}R - 3E_{\mu\nu}n^\mu n^\nu. \end{aligned} \quad (20)$$

Using the inequalities (17) and (18) it is found that K_ϕ is always a positive definite. The constraint equation in Brane gets modified from that of GTR [15], due to the presence of the term quadratic in energy momentum. The time differentiation of equation (20) can be written as

$$\dot{K}_\phi = -\frac{2}{3}KK_\phi - 2K(\sigma_{\mu\nu}\sigma^{\mu\nu}) + 2K(E_{\mu\nu}n^\mu n^\nu). \quad (21)$$

Since we have $E_{\mu\nu}n^\mu n^\nu \leq 0$, one obtains

$$\dot{K}_\phi \leq -\frac{2}{3}KK_\phi. \quad (22)$$

Using the Hubble parameter (H) we rewrite the inequality as

$$\dot{K}_\phi \leq -2HK_\phi. \quad (23)$$

On integrating the above inequality (we note that $K_\phi \geq 0$) one obtains

$$0 \leq K_\phi \leq K_{\phi_o} e^{-2 \int H dt}. \quad (24)$$

where K_{ϕ_o} is an arbitrary constant. During inflation (exponential expansion) H can be assumed to be a constant ($= H_o$) and we get

$$0 \leq K_\phi \leq K_{\phi_o} e^{-2H_o t}. \quad (25)$$

The expression K_ϕ falls of exponentially with time and goes to zero. However, if we consider a case where ($H \sim \frac{H_o}{t}$) then one obtains

$$0 \leq K_\phi \leq K_{\phi_o} t^{-2H_o}. \quad (26)$$

In this case K_ϕ decreases with time which follows a power law expansion mode. Thus the behavior of K_ϕ is not affected by the nature of the potential. In the brane-world scenario the quadratic term in energy density dominates. Thus at very high energy, we get

$$\dot{K} + \frac{1}{3}K^2 \leq \frac{\tilde{\kappa}^4}{12} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) \right)^2 \quad (27)$$

which is different from that obtained in GTR [15]. During slow-roll inflation, the scalar field decreases and the potential term behaves as a cosmological constant. Consequently one can write equation (11) as

$$\dot{K} + \frac{1}{3}K^2 \leq \kappa^2 V_o \left[1 + \frac{V_o}{2\lambda} \right]. \quad (28)$$

Now integrating the above inequality once we get

$$K \leq \frac{3\eta}{\tanh(\eta t)} \quad (29)$$

where $\eta = \sqrt{\frac{\kappa^2 V_o}{3} \left(1 + \frac{V_o}{2\lambda}\right)}$. Thus K , the expansion rate, approaches 3η exponentially over the time scale $\frac{1}{\eta}$. In the Brane world i.e., at very high energy scale ($\frac{V_o}{\lambda} \rightarrow \infty$) we have $\eta = \eta_{Brane} = \frac{\kappa V_o}{\sqrt{6\lambda}}$ whereas in GTR i.e., in the low energy scale i.e., ($\frac{V_o}{\lambda} \rightarrow 0$) it becomes $\eta = \eta_{GTR} = \kappa \sqrt{\frac{V_o}{3}}$. Thus $\frac{\eta_{Brane}}{\eta_{GTR}} = \sqrt{\frac{V_o}{2\lambda}}$, which implies that the time scale of approach of the inflationary regime in Brane world is faster than that in GTR.

Now using equation (29) as the upper limit of K we have

$$\frac{1}{2} \sigma_{\mu\nu} \sigma^{\mu\nu} \leq \frac{1}{3} (K^2 - 9\eta^2) \leq \frac{3\eta^2}{\text{Sinh}^2(\eta t)}. \quad (30)$$

Finally, as $K \rightarrow 3\eta$, the shear $\sigma_{\mu\nu}$ approaches zero at late times. The time dependence of the spatial metric can be approximately written as

$$h_{\mu\nu}(t) \sim e^{2(t-t_o)\eta} h_{\mu\nu}(t_o) \quad (31)$$

where t_o is the initial time. The spatial curvature ${}^{(3)}R$ scales away to zero. Thus the Bianchi universes (except Bianchi type IX) after inflation appear to be matter free with nearly flat spatial sections for time $t > \frac{1}{\eta}$ (i.e., isotropized) and the constant rate of isotropic expansion is $K \rightarrow 3\eta$.

To conclude, it is found that in the Brane world scenario, for all the Bianchi models (except type IX) it is possible to have either exponential or power law inflation with a scalar field with arbitrary potential. It is also noted that in the brane world scenario the universe isotropizes faster than that in GTR. The potential considered here is ϕ^4 -type with a constant additive part which influences the dynamics of the late universe.

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