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Mach's principle and the creation of matter*

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Accurate experiments have shown that the local inertial frame is the one with respect to which the distant parts of the universe are non-rotating. This coincidence, first noticed by Newton, later led to the formulation of Mach's principle. It is known that relativity theory by itself cannot explain this coincidence. The introduction of a scalar 'creation field' into the theory is likely to improve the situation. Calculation shows that the continuous creation of matter has the effect of smoothing out any irregularities in the universe as it expands, while rotation, if present, becomes less and less. This explains the observed remarkable degree of homogeneity and isotropy in the universe.

1. Introduction

The concept of inertia of matter dates back to Galileo. In stating that a body, if undisturbed, remains at rest or is in a state of uniform motion, Galileo must have realized that it is not the velocity but the acceleration of the body that owes its origin to external forces. The quantitative statement of the physical situation was, however, first given by Newton in his second law of motion. According to this law the force is proportional to the acceleration of the body. The constant of proportionality measures the inertia of the body and is called its inertial mass.

The law as stated above can only be applied to motions measured relative to inertial frames. These are idealized as constituted by bodies on which no forces act. If motion is measured relative to a non-inertial frame, additional forces are required to describe the situation. These are termed as inertial forces and can be detected experimentally. In his *Principia* (1686) Newton writes:

'The effects which distinguish absolute motion from relative motion are the forces of receding from the axis of circular motion. For there are no such forces in a circular motion purely relative, but in a true and absolute circular motion they are greater, or less, according to the quantity of the motion.'

He then proceeds to describe his famous experiment of a rotating water-filled bucket suspended from a twisted thread. The experiment shows that whenever rotation occurs relative to a specific reference frame, the surface of water becomes concave. This is an absolute effect and can be used to measure the amount of rotation relative to that frame.

This particular reference frame relative to which the inertial forces are observed turns out to be the frame in which the distant objects in the universe are non-rotating.

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This has been confirmed, within experimental error, by more accurate later experiments. Since the coincidence can scarcely be dismissed as accidental, it is necessary to explain it.

A suggestion, not an explanation, was made by Mach, to the effect that inertial forces are related to the distant parts of the universe for the very good reason that they are generated by the distant parts of the universe. Mach did not give any quantitative theory for this somewhat vague concept, which later came to be known as Mach's principle. Mach, however, did recognize that a crucial problem exists and requires explanation. In formulating his laws of motion Newton himself was faced with the same problem and asserted that this was 'a matter of great difficulty'. The success of his postulates was so great, however, that succeeding generations forgot the existence of the problem until it was raised again by Mach.

Einstein was much impressed by the ideas of Mach. He hoped at first that Mach's principle might be explained by the general theory of relativity. Various attempts have been made (see, for example, Davidson 1957) to relate, within the framework of general relativity, the local inertial properties of matter with the distribution of matter in the universe—but this hope has not been realized so far. Yet, the coincidence to be explained follows naturally in the normal forms of relativistic cosmology as will now be shown.

2. Normal relativistic cosmology

That the coincidence of the inertial frame of Newton's bucket with that in which the distant parts of the universe are non-rotating can be deduced from normal relativistic cosmology—but not from general relativity itself—may appear puzzling at first. The reason is that normal cosmology contains two postulates that are extraneous to the general theory of relativity. These postulates are:

(i) *The Weyl postulate.* The world lines of matter form a geodesic congruence normal to a spacelike surface. This leads to a line element of the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 + g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu, \tag{1}$$

where x^μ = constant along each world line. (Henceforth Greek suffixes will be restricted to the values 1, 2, 3, while the Latin ones take the full range 1, 2, 3, 4.) The t-co-ordinate is called the cosmic time.

(ii) *The cosmological principle.* This states that the subspaces t = constant are homogeneous and isotropic. This implies that an observer having any one of the world lines x^μ = constant will not be able to distinguish his position nor will he be able to distinguish one direction from another. Geometrically, this means that the space-time continuum admits a 6-parameter group of motions, and this leads after a considerable investigation to the result that the most general form of the line element (1) is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - S^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right], \tag{2}$$

where $k = 0, \pm 1$, and r can be chosen zero for any observer of the type mentioned above. The form (2) is the Robertson–Walker line element. The co-ordinates t, r, θ, ϕ are related to observed quantities by the well-known considerations of cosmology.

Now a transformation
$$r' = rS(t), \quad t' = t + \tfrac{1}{2}(\dot{S}/S)\,r'^2 \tag{3}$$
reduces (2) to the form
$$\mathrm{d}s^2 = \mathrm{d}t'^2\left(1 - r'^2\frac{\ddot{S}}{S}\right) - \frac{\mathrm{d}r'^2}{\left(1 - \dfrac{kr'^2}{S^2} - r'^2\dfrac{\dot{S}^2}{S^2}\right)} - r'^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \tag{4}$$
for sufficiently small r' (neglecting r'^3 and higher powers of r'). In the case of the de Sitter line element, $S = \exp(Ht)$, $k = 0$, the form (4) is exact, with the following transformation law:
$$r' = r\exp(Ht), \quad t' = t - (1/2H)\ln(1 - r'^2H^2), \tag{5}$$
H being the Hubble constant. It is particularly to be noticed that the transformation (3) does not change the angular co-ordinates θ, ϕ.

In any local experiment $r' \to 0$, and the form (4) goes into the metric of special relativity. This means the co-ordinates r', θ, ϕ, t' represent an inertial frame for a local experiment; any rotation relative to this frame produces inertial forces. In other words, this is the special frame indicated by the bucket experiment.

These same considerations continue to hold when one adds in the effect of a local mass. The form (4) can then be regarded as giving the boundary condition at 'infinity'. The field in this case is described by a Schwarzschild solution near the mass concentration and by (4) at large distances from it. It is interesting to note that the terms in r'^2 are very small, even for large distances. For example, for the case of the sun the approximation involved in (4) is of the order 10^{-19} at distances of $\sim$ 1 pc. (To obtain this numerical result one must of course relate the function $S(t)$ to the observed Hubble constant, by $H = \dot{S}/S$).

Since θ, ϕ have not been changed by the transformation, the values of these co-ordinates to be associated with any particular distant galaxy are independent of t'—the distant parts of the universe are non-rotating, which is the result required by Mach's principle.

If we now consider Einstein's field equations,
$$R^{ik} - \tfrac{1}{2}g^{ik}R + \lambda g^{ik} = -\kappa T^{ik}, \tag{6}$$
and insert the g^{ik} from the form (2), we find that T^{ik} must be expressible by an equation of the type
$$T^{ik} = (\rho + p + \tfrac{4}{3}u)\frac{\mathrm{d}x^i}{\mathrm{d}s}\frac{\mathrm{d}x^k}{\mathrm{d}s} - (p + \tfrac{1}{3}u)\,g^{ik}, \tag{7}$$
where $\mathrm{d}x^i/\mathrm{d}s = \delta^i_4$ is the flow-vector of matter, and where the quantities u, p, ρ depend on $S(t)$. (Interpreted physically, u, p, ρ are the radiation pressure, the gas pressure and the matter density, respectively.) The point is that $T^{11} = T^{22} = T^{33}$ and that T^{ik} cannot have off-diagonal components that are non-vanishing in the co-ordinate system in question.

The procedure adopted here, typical of cosmological studies in general, is evidently the direct contrary of all that is implied by Mach's principle. Mach's principle requires us to read equation (6) from right to left. In other words, we should ask the following question; 'Given T^{ik} in the form (7), do the equations (6) lead uniquely to the line element (2)?' If the answer is affirmative then relativity theory can be said to have explained the observations concerning rotary inertial forces. However, it was shown by Gödel (1949) that, at any rate for the normal form of T^{ik} the answer is not affirmative. Gödel obtained an explicit solution in which the line element is of the form

$$ds^2 = dt^2 + 2e^{x^1} dt\, dx^2 - (dx^1)^2 + \tfrac{1}{2}e^{2x^1}(dx^2)^2 - (dx^3)^2 \tag{8}$$

and in which T^{ik} is given by (7) with $u = p = 0$, $\rho = 1/\kappa$, $\lambda = -\frac{1}{2}\kappa\rho$. This solution is fundamentally different from (2)—it cannot be obtained from it by a co-ordinate transformation. The importance of Gödel's solution from the present point of view is that it exhibits a vorticity of matter. If we again take a transformation that reduces (8) in the neighbourhood of any particular observer (i.e. particle) to the form of special relativity, then the distant matter possesses rotation, and Mach's principle is not satisfied. Mach's principle, as interpreted above, is therefore not incorporated within the usual formulation of the general theory of relativity.

The most that can be done within the framework of the normal theory is the following. Take a space-like surface and on it define co-ordinates x^μ, t leading to the line element (1). This simply removes the arbitrariness of the co-ordinates. On this surface specify the matter and the kinematical situation and also the quantities

$$g_{\mu\nu}, \quad \frac{\partial g_{\mu\nu}}{\partial x^i}, \quad \frac{\partial^2 g_{\mu\nu}}{\partial x^\lambda \partial t}, \quad \frac{\partial^2 g_{\mu\nu}}{\partial x^\lambda \partial x^\sigma}, \tag{9}$$

consistently with (6). Then use (6) to calculate both the dynamical situation and the metric tensor off the initial surface.

Now, these specifications on the initial surface can be made in such a way that the line element turns out everywhere to be of the form (2), thereby giving the correct boundary condition for Newton's bucket, as well as for the solar Schwarzschild solution. The Newtonian concept of 'absolute space' has therefore been replaced in the relativity theory by initial boundary conditions on the matter and the metric tensors.

In the normal relativistic cosmologies the situation is dealt with by supposing that the required boundary conditions were imposed at the origin of the universe. It 'just happened' that out of an infinite number of possibilities (e.g. the Gödel solution) the solution chosen was one in which the line element was of the form (2). If this view is correct, the coincidence implicit in Newton's bucket experiment must be accepted without any further explanation. That it should be so was decided at the moment of creation of the universe, either by chance or by causes unknown.

3. The creation field

One of the main aims of the so-called steady-state cosmology is to dispense with the need for special initial conditions. It must be made clear at the outset that the steady-state theory we are concerned with here is that of Hoyle (1948, 1949, 1960), not that of Bondi & Gold (1948). In the latter, the 'perfect' cosmological principle is the basis for the theory. There are no equations telling us what happens when any departure from this situation occurs, for the reason that according to the postulate of the perfect cosmological principle, the universe, on a large scale, always remains in the steady state. The logical status of Mach's principle appears therefore to be much the same as it is in the relativistic cosmologies—the coincidence we observe must be there at all times because of the perfect cosmological principle—it is not for us to reason why it is so.

The logical situation is, however, quite different in Hoyle's approach, which depends on a modification of Einstein's field equations through the introduction of a new field. Two objections may be raised. First, the modification can be accomplished in an infinity of ways, each way giving a new field theory. Secondly, although the steady-state solution turns out to be a simple solution of the equations, departures from this solution are likely to be more difficult to handle than is the normal theory itself. The more important first objection can be countered by the remark that science would indeed be in a poor case if no new possibilities were ever to be considered. The case for doing so must of course rest with the relation of observation to theory. The curious situation surrounding Mach's principle in particular, and of the general role of boundary conditions in cosmology, would seem to us to justify some widening of perspective. At the same time, it is reasonable to confine oneself to modifications that possess a simple logical structure. The simplest field that one can introduce is evidently a scalar one—we denote it by C. Such a field was used by Hoyle, but a simpler and more elegant development of the theory was suggested by M. H. L. Pryce (private communication) and this will be used throughout the following argument.

And to the second objection it can be answered that the very complications introduced by the new field appear to be of great significance in that they may well be able—as we shall see later—to rescue the theory from the impasse of Mach's principle.

The field equations are derived from an action principle—the action function which we shall assume is

$$\mathscr{A} = \frac{1}{16\pi G}\int R\sqrt{-g}\,\mathrm{d}^4x - \Sigma m\int \mathrm{d}s + \tfrac{1}{2}f\int C_i\,C^i\sqrt{-g}\,\mathrm{d}^4x - \Sigma m\int C_i\frac{\mathrm{d}x^i}{\mathrm{d}s}\,\mathrm{d}s, \tag{10}$$

where f is a coupling constant. The conditions $\delta\mathscr{A} = 0$, taken with respect to the metric tensor and independently with respect to the field C yield the equations

$$R^{ik} - \tfrac{1}{2}g^{ik}R = -\kappa[T^{ik} - f\{C^iC^k - \tfrac{1}{2}g^{ik}C^lC_l\}], \tag{11}$$

$$C^i_{;\,i} = (1/f)j^i_{;i}, \quad T^{ik}_{;k} = fC^iC^k_{;\,k}, \tag{12}$$

where j^i is the mass current. It must be noted here that in either of the equations (12) it is not the case that both sides necessarily vanish separately, since in the present theory world lines of matter may end or begin at various points in space-time.

In the special case of the Robertson–Walker line element (2), these equations lead to

$$3\frac{\dot{S}^2+k}{S^2} = \kappa(\rho - \tfrac{1}{2}f\dot{C}^2], \tag{13}$$

$$2\frac{\ddot{S}}{S} + \frac{\dot{S}^2+k}{S^2} = \tfrac{1}{2}\kappa f\dot{C}^2, \tag{14}$$

$$\ddot{C} + 3\frac{\dot{S}}{S}\dot{C} = \frac{1}{f}\left[\rho + 3\frac{\dot{S}}{S}\rho\right], \tag{15}$$

provided that we use (7), with $p = 0$, $u = 0$, for T^{ik}, and provided that we argue that in the homogeneous-isotropic case the new field is a function of t only.

These equations have a remarkable property. In the case $C = 0$ they reduce to the usual cosmological equations. Even in the case $C^{i}_{;i} = 0$ no significantly different solution is obtained. For example, in the case $k = 0$ the solution is

$$S^3 = \tfrac{1}{2}A^2/(A-B) + \tfrac{1}{2}(A-B)(t-t_0)^2 \tag{16}$$

where A, B, t_0 are constants. This solution tends to the familiar Einstein–de Sitter solution as $t \to \infty$. This is to be expected since, with $C^{i}_{;i} = 0$, C is like any other scalar field.

If, however, we admit the possibility of a general infinitesimal perturbation in which $C^{i}_{;i} \neq 0$, i.e. we admit the possibility that matter can be created or destroyed, the S function is moved on to a new class of solutions of the type

$$S^3 = B + B\cosh 3H(t-t_0), \tag{17}$$

where $H = \sqrt{(\frac{1}{6}\kappa f)}$. As $t \to \infty$ this tends to the steady-state solution. Also the density of matter $\rho \to 3H^2/4\pi G$ and the creation rate tends to a positive steady value of $9H^3/4\pi G$ per unit volume. Thus in the asymptotic case creation and expansion are in exact balance—even though no such balance was initially there.

Further, in the case $C^{i}_{;i} \neq 0$ it turns out that $C = t$ at all t, not merely for the asymptotic steady state. This remarkable result would seem to give the strong hint that if we attempt to read (11) from right to left, returning to the project of deducing geometry from physics, the C field may turn out to play an essential role. Particularly, one might expect to read the equation $C = t$ from right to left. The suspicion aroused by the special case discussed above is that the surfaces $t = \text{const}$, the Weyl surfaces, are just the surfaces $C = \text{const}$. And the matter is observed to possess world lines that are approximately normal to these surfaces simply because it is created so.

Reverting then to Mach's principle, we can formulate the following problem. Set up a system of surfaces $C = \text{const}$ (even though the line element is not given by (2)) and define $t = C$. Suppose the distribution of matter and its velocity on a particular member of the C-surfaces is given. To permit a calculation of the situation off this particular surface it is still necessary to specify the metric tensor and its derivatives on the surface. But could it be that as time proceeds, i.e. as C increases, the free choices on the metric tensor gradually become irrelevant, in such a way that over any specified physical volume the line element tends to the form

$$ds^2 = dt^2 - e^{2Ht}\{(dx^1)^2 + (dx^2)^2 + (dx^3)^2\}? \tag{18}$$

If so, the Newtonian bucket experiment would be explained as the asymptotic condition to which all initial conditions lead.

We have not succeeded in answering this question in its most general form, because of the very difficult non-linearity of the equations. We have, however, been able to consider a more restricted problem in which the initial conditions are taken as a perturbation of (18). We take the metric tensor to be

$$g_{\mu\nu} = -(\eta_{\mu\nu} + h_{\mu\nu})\,e^{2Ht}, \quad g_{\mu 4} = 0, \quad g_{44} = 1 + h_{44}, \tag{19}$$

where $\eta_{\mu\nu} = 1$ if $\mu = \nu$ and $\eta_{\mu\nu} = 0$ if $\mu \neq \nu$, and where $|h_{ik}| \ll 1$. In the following calculation the squares and higher powers of h_{ik} and their derivatives are neglected; the expressions for R_{ik} can then be linearized. The co-ordinates x^μ are chosen in such a way that the lines $x^\mu =$ const are orthogonal to the surface $C = t =$ const. Hence terms in $g_{\mu 4}$ are absent. In this co-ordinate system the flow vector of matter need not be in the direction $u_0^i = (0, 0, 0, 1)$. The perturbed form of T^{ik} will be written as

$$T^{ik} = \rho_0 u_0^i u_0^k + \rho_1 u_0^i u_0^k + \rho_0 (u_0^i u_1^k + u_1^i u_0^k). \tag{20}$$

In this $\rho_0 = 3H^2/4\pi G$, while ρ_1, u_1^i are quantities of the first order of smallness. The vector u_1^i need not be in the direction of u_0^i. So long as departures from this direction are small, the last term in (20) can include the effect of streams in many directions. With this notation the vector j^i is given by

$$j^i = \rho_0 u_0^i + \rho_1 u_0^i + \rho_0 u_1^i. \tag{21}$$

At this stage, it is convenient to choose units such that $H = 1$, $\rho_0 = 1$. Then $\kappa = 6$. Also, $\kappa f = 6$. The following quantities are needed often in the calculations:

$$-g = e^{6t}(1+h), \quad h = h_{11}+h_{22}+h_{33}+h_{44}, \quad \ln\sqrt{-g} = 3t+\tfrac{1}{2}h,$$

$$\Gamma^4_{44} = \tfrac{1}{2}\dot{h}_{44}, \quad \Gamma^\mu_{44} = \tfrac{1}{2}e^{-2t}(\partial h_{44}/\partial x^\mu),$$

$$\Gamma^4_{4\nu} = \tfrac{1}{2}(\partial h_{44}/\partial x^\mu), \quad \Gamma^\nu_{4\mu} = \eta_{\mu\nu} + \tfrac{1}{2}\dot{h}_{\mu\nu},$$

$$\Gamma^4_{\mu\nu} = (\eta_{\mu\nu} + h_{\mu\nu} + \tfrac{1}{2}\dot{h}_{\mu\nu} - \eta_{\mu\nu}h_{44})\,e^{2t},$$

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\left(\frac{\partial h_{\lambda\mu}}{\partial x^\nu} + \frac{\partial h_{\lambda\nu}}{\partial x^\mu} - \frac{\partial h_{\mu\nu}}{\partial x^\lambda}\right).$$

By definition of t

$$C_i = (0, 0, 0, 1). \tag{22}$$

Substituting the expressions (20), (21) and (22) in the equations (12) and retaining terms of the first order only, leads to

$$\frac{\partial u_1^4}{\partial t} + 3u_1^4 + \frac{1}{2}\frac{\partial h_{44}}{\partial t} + 3h_{44} = 0, \tag{23}$$

$$\frac{\partial u_1^\mu}{\partial t} + 5u_1^\mu + \tfrac{1}{2}e^{-2t}\frac{\partial h_{44}}{\partial x^\mu} = 0, \tag{24}$$

$$\frac{\partial \rho_1}{\partial t} + 3\rho_1 + \frac{1}{2}\frac{\partial h_{44}}{\partial t} + \frac{\partial u_1^\mu}{\partial x^\mu} = 0. \tag{25}$$

The normalizing condition on the flow vector, however, requires

$$g_{ik}(u_0^i + u_1^i)(u_0^k + u_1^k) = 1, \tag{26}$$

which, in a first-order approximation, leads to

$$u_1^4 + \tfrac{1}{2}h_{44} = 0. \tag{27}$$

This, together with (23) implies $u_4^1 = 0$, $h_{44} = 0$. The remaining two equations (24) and (25) can then be integrated to give

$$u_1^\mu = \epsilon^\mu e^{-5t}, \quad \rho_1 = A\,e^{-3t} + \tfrac{1}{2}(\partial\epsilon^\mu/\partial x^\mu)\,e^{-5t}, \tag{28}$$

where A, ϵ^μ are independent of t. Hence the last term in T^{ik} decreases with time as e^{-5t}, a result not unexpected since a non-relativistic pressure decreases as S^{-5}.

Next, we note that the field equations (11) can be rewritten in the form

$$R_{ik} = -6T_{ik} + 3Tg_{ik} + 6C_i C_k. \tag{29}$$

With $T = 1 + A\,\mathrm{e}^{-3t} + \frac{1}{2}(\partial\epsilon^\mu/\partial x^\mu)\,\mathrm{e}^{-5t}$ and $R_{44} = 3 + \frac{1}{2}\ddot{h} + \dot{h}$ the (4, 4) component of equation (29) is

$$\tfrac{1}{2}\ddot{h} + \dot{h} = -(3A\,\mathrm{e}^{-3t} + \tfrac{3}{2}(\partial\epsilon^\mu/\partial x^\mu)\,\mathrm{e}^{-5t}),$$

which has the general solution

$$h = \alpha + \beta\,\mathrm{e}^{-2t} + \gamma\,\mathrm{e}^{-3t} + \delta\,\mathrm{e}^{-5t}. \tag{30}$$

Here α, β are arbitrary and γ, δ are given by

$$\gamma = -2A, \quad \delta = -\tfrac{1}{5}(\partial\epsilon^\mu/\partial x^\mu). \tag{31}$$

(Summation on the double suffix is assumed unless otherwise stated.)

Consider next the $(\mu, 4)$ components of (29). $T_{\mu 4} = -\epsilon^\mu \mathrm{e}^{-3t}$ gives

$$\frac{\partial}{\partial t}\left[\frac{\partial h}{\partial x^\mu} - \frac{\partial h_{\mu\lambda}}{\partial x^\lambda}\right] = 12\epsilon^\mu \mathrm{e}^{-3t},$$

i.e.

$$\frac{\partial h}{\partial x^\mu} - \frac{\partial h_{\mu\lambda}}{\partial x^\lambda} = p_\mu - 4\epsilon^\mu \mathrm{e}^{-3t}, \tag{32}$$

where p_μ are functions of x^λ only. With the use of (30) a direct computation from the Christoffel symbols gives

$$\begin{aligned} R_{11} + R_{22} + R_{33} &= \frac{\partial p_\mu}{\partial x^\mu} - 4\frac{\partial \epsilon^\mu}{\partial x^\mu}\,\mathrm{e}^{-3t} - \mathrm{e}^{2t}[9 + 3h + 3\dot{h} + \tfrac{1}{2}\ddot{h}] \\ &= -(3\alpha + 9)\,\mathrm{e}^{2t} + \left(\frac{\partial p_\mu}{\partial x^\mu} + \beta\right) - 3A\,\mathrm{e}^{-t} - \left(4\frac{\partial\epsilon^\mu}{\partial x^\mu} - \frac{1}{10}\frac{\partial \epsilon^\mu}{\partial x^\mu}\right)\mathrm{e}^{-3t}. \end{aligned}$$

But $R_{11} + R_{22} + R_{33}$ can also be obtained from (29) in the form

$$\begin{aligned} &(3+h)\left(-3\,\mathrm{e}^{2t})\, 1 + A\,\mathrm{e}^{-3t} + \frac{1}{2}\frac{\partial\epsilon^\mu}{\partial x^\mu}\,\mathrm{e}^{-5t}\right) \\ &= -(3\alpha + 9)\,\mathrm{e}^{-2t} - 3\beta - 3A\,\mathrm{e}^{-t} - \frac{39}{10}\frac{\partial\epsilon^\mu}{\partial x^\mu}\,\mathrm{e}^{-3t}. \end{aligned}$$

Comparison of the two expressions gives

$$\beta = \tfrac{1}{4}(\partial p_\mu/\partial x^\mu). \tag{33}$$

The (1, 1) component of (29) gives

$$\begin{aligned} &\frac{\partial p_1}{\partial x^1} - 4\frac{\partial \epsilon^1}{\partial x^1}\,\mathrm{e}^{-3t} + \tfrac{1}{2}\nabla^2 h_{11} - \frac{1}{2}\frac{\partial^2 h}{(\partial x^1)^2} - \mathrm{e}^{2t}[\tfrac{1}{2}\ddot{h}_{11} + \tfrac{1}{2}\dot{h} + \tfrac{3}{2}\dot{h}_{11} + 3h_{11} + 3] \\ &\qquad = -3\mathrm{e}^{2t}\left[1 + A\,\mathrm{e}^{-3t} + h_{11} + \frac{1}{2}\frac{\partial \epsilon^\mu}{\partial x^\mu}\mathrm{e}^{-5t}\right]. \end{aligned}$$

This is satisfied if we put

$$h_{11} = \alpha_{11} + \beta_{11}\mathrm{e}^{-2t} + \gamma_{11}\mathrm{e}^{-3t} + \delta_{11}\mathrm{e}^{-5t}, \tag{34}$$

where α_{11}, β_{11}, γ_{11}, δ_{11} are functions of x^μ satisfying

$$\left.\begin{aligned} &\frac{\partial p_1}{\partial x^1}+\tfrac{1}{2}\nabla^2\alpha_{11}-\frac{1}{2}\frac{\partial^2\alpha}{(\partial x^1)^2}+\beta_{11}+\beta = 0,\\ &\tfrac{1}{2}\nabla^2\beta_{11}-\frac{1}{2}\frac{\partial^2\beta}{(\partial x^1)^2} = 0, \quad \tfrac{1}{2}\nabla^2\delta_{11}-\frac{1}{2}\frac{\partial^2\delta_{11}}{(\partial x^1)^2} = 0,\\ &\frac{\partial\epsilon^\mu}{\partial x^\mu}-4\frac{\partial\epsilon^1}{\partial x^1}+\tfrac{1}{2}\nabla^2\gamma_{11}-\frac{1}{2}\frac{\partial^2\gamma}{(\partial x^1)^2}-5\delta_{11} = 0, \end{aligned}\right\} \tag{35}$$

in which

$$\alpha = \alpha_{11}+\alpha_{22}+\alpha_{33}, \quad \beta = \beta_{11}+\beta_{22}+\beta_{33}, \text{ etc.}$$

Similar relations are satisfied by α_{22}, β_{22}, etc.

The (1, 2) component of (29) reduces to

$$\frac{1}{2}\left(\frac{\partial^2 h}{\partial x^1\,\partial x^2}-\frac{\partial^2 h_{\lambda 2}}{\partial x^\lambda\,\partial x^1}-\frac{\partial^2 h_{\lambda 1}}{\partial x^\lambda\,\partial x^2}+\frac{\partial^2 h_{12}}{\partial x^\lambda\,\partial x^\lambda}\right)-\mathrm{e}^{2t}(3h_{12}+\tfrac{3}{2}\dot{h}_{12}+\tfrac{1}{2}\ddot{h}_{12}) = -3h_{12}\mathrm{e}^{2t},$$

i.e.

$$\tfrac{1}{2}\nabla^2 h_{12}+\frac{1}{2}\frac{\partial}{\partial x^1}(p_2-4\epsilon^2\mathrm{e}^{-3t})+\frac{1}{2}\frac{\partial}{\partial x^2}(p_1-4\epsilon^1\mathrm{e}^{-3t})$$
$$-\frac{1}{2}\frac{\partial^2 h}{\partial x^1\partial x^2}-\mathrm{e}^{2t}(\tfrac{1}{2}\ddot{h}_{12}+\tfrac{3}{2}\dot{h}_{12}) = 0.$$

Substituting

$$h_{12} = \alpha_{12}+\beta_{12}\mathrm{e}^{-2t}+\gamma_{12}\mathrm{e}^{-3t}+\delta_{12}\mathrm{e}^{-5t}, \tag{36}$$

and equating coefficients of e^{-t}, e^{-2t}, etc., gives

$$\left.\begin{aligned} &\frac{1}{2}\left(\frac{\partial p_2}{\partial x^1}+\frac{\partial p_1}{\partial x^2}\right)+\tfrac{1}{2}\nabla^2\alpha_{12}-\frac{1}{2}\frac{\partial^2\alpha}{\partial x^1\partial x^2}+\beta_{12} = 0,\\ &\tfrac{1}{2}\nabla^2\beta_{12}-\frac{1}{2}\frac{\partial^2\beta}{\partial x^1\,\partial x^2} = 0, \quad \tfrac{1}{2}\nabla^2\delta_{12}-\frac{1}{2}\frac{\partial^2\delta}{\partial x^1\,\partial x^2} = 0,\\ &\tfrac{1}{2}\nabla^2\gamma_{12}-2\left(\frac{\partial\epsilon^1}{\partial x^2}+\frac{\partial\epsilon^2}{\partial x^1}\right)-\frac{1}{2}\frac{\partial^2\gamma}{\partial x^1\partial x^2}-5\delta_{12} = 0. \end{aligned}\right\} \tag{37}$$

Similarly for α_{23}, β_{23}, etc.

We therefore see that the $h_{\mu\nu}$ are of the form

$$h_{\mu\nu} = \alpha_{\mu\nu}+\beta_{\mu\nu}\mathrm{e}^{-2t}+\gamma_{\mu\nu}\mathrm{e}^{-3t}+\delta_{\mu\nu}\mathrm{e}^{-5t}, \tag{38}$$

where $\alpha_{\mu\nu}$, $\beta_{\mu\nu}$, $\gamma_{\mu\nu}$, $\delta_{\mu\nu}$ are functions of x^μ, but not of t, that satisfy certain consistency relations, (35), (37), demanded by the field equations. The consistency relations are of little importance, however. The last three terms in $h_{\mu\nu}$ die away as t increases. The first term remains; but over any given proper volume the expansion makes the variation of x^μ less and less, the $\alpha_{\mu\nu}$ become effectively constant over the proper volume in question. The line element in any specific finite proper volume therefore tends to the homogeneous-isotropic form (18), since constant values of $\alpha_{\mu\nu}$ can be absorbed trivially into x^μ.

This disposes of the case in which the metric tensor differs initially from (19) only by small quantities. Although our work is restricted by the requirement that the second- and higher-order terms in the h_{ik} can be neglected, it is worth noting that the present considerations are applicable to a state of the universe very appreciably more anisotropic than the observed state appears to be. Observation suggests that the $|h_{ik}|$ are in actuality of order 10^{-4} whereas our work could reasonably be taken as applying to initial conditions in which the $|h_{ik}|$ were of order 10^{-1}.

We did not explicitly consider rotation in the above calculation. This can be easily computed by using Gödel's definition for the local angular velocity vector

$$\omega^i = \frac{\epsilon^{iklm}}{12\sqrt{-g}} a_{klm}, \tag{39}$$

where a_{klm} is a tensor given in terms of the flow vector v^i by

$$a_{klm} = v_k\left(\frac{\partial v_l}{\partial x_m} - \frac{\partial v_m}{\partial x^l}\right) + v_l\left(\frac{\partial v_m}{\partial x^k} - \frac{\partial v_k}{\partial x^m}\right) + v_m\left(\frac{\partial v_k}{\partial x^l} - \frac{\partial v_l}{\partial x^k}\right). \tag{40}$$

A simple calculation shows that angular velocity arising out of the perturbations u^i_1 in the fluid motion decays exponentially with time scale of the order H^{-1}.

It is however of considerable interest to note that the case of rotation can be treated without the need for restrictions on the initial $g_{\mu\nu}$, provided Weyl's postulate is satisfied, i.e. provided co-ordinates can be chosen in such a way that the flow vector of matter is given by $v^i = \delta^i_4$. A simple argument shows that rotation must die away if the universe continues to expand.

Following the notation of Raychaudhuri (1955) we define rotation for the line element

$$ds^2 = dt^2 + 2g_{\mu 4}\,dt\,dx^\mu + g_{\mu\nu}\,dx^\mu dx^\nu \tag{41}$$

by means of the anti-symmetrical tensor

$$\omega_{ik} = \tfrac{1}{2}(u_{i;k} - u_{k;i}). \tag{42}$$

Because the matter follows geodesics, it can readily be shown that the $g_{\mu 4}$ are independent of t, and that the expression (42) can be simplified to

$$\omega_{ik} = \frac{1}{2}\left(\frac{\partial g_{i4}}{\partial x^k} - \frac{\partial g_{k4}}{\partial x^i}\right). \tag{43}$$

It is convenient to define a three-vector $\boldsymbol{\omega}$ with components $\omega_1 = \omega_{23} = -\omega_{32}$, $\omega_2 = \omega_{31} = -\omega_{13}$, etc. And the scalar angular velocity ω is defined by

$$2\omega^2 = g^{ik} g^{lm} \omega_{il} \omega_{km}. \tag{44}$$

Now if the universe continues to expand indefinitely we may neglect the $g_{\mu 4}$ terms in comparison with $g_{\mu\nu}$ terms; since, as we have just noted, the $g_{\mu\nu}$ are independent of t, whereas the $g_{\mu\nu}$ continue to increase. Then (44) becomes

$$\omega^2 \cong \frac{1}{2g} g_{\mu\nu} \omega_\mu \omega_\nu. \tag{45}$$

Provided that the expansion is not highly anisotropic, we can take $g_{\mu\nu}/g \to 0$. Hence $\omega \to 0$, since the ω_μ are independent of t. In the case of an almost isotropic expansion with the expansion factor $S(t)$ we simply recover the familiar Newtonian law $S^2\omega = \text{const.}$ with respect to time.

The above work includes only gravitational effects and therefore is not inconsistent with the fact that irregularities actually develop in the universe—galaxies form—provided the irregularities arise from non-gravitational effects such as pressure or cooling.

4. Conclusion

If the arguments of the previous section are accepted we have the following picture. Provided the continuous creation of matter is allowed, the creation acts in such a way as to smooth out an initial anisotropy or inhomogeneity over any specified proper volume. Rotation, in the sense of Gödel is never destroyed, but it is made arbitrarily small over the given proper volume. In other words, any finite portion of the universe gradually loses its 'memory' of an initially imposed anisotropy or inhomogeneity.

The situation has some analogy in the case of electrical circuit containing a generator. The circuit may have currents flowing at a moment when the generator is switched on. These currents may be arbitrary; but once the generator is in operation a new current with a periodicity equal to that of the generator will gradually build itself, whereas thermal losses will gradually destroy the arbitrary initial currents. The steady state solution in our case corresponds to the driven current—the driving agency being the C-field. If the C-field is not present, the universe itself is simply a 'transient' and the observed regularity is just 'chance'. If the C-field is present with $C^{i}_{;i} \neq 0$, it seems that the universe attains the observed regularity irrespective of initial boundary conditions.

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