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Signal analysis of the gravitational waveform of pulsars

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Abstract

We present signal analysis for the detection of gravitational radiation from pulsars. Typically the observation times will be of the order of a few weeks or months. Due to the rotation and orbital motion of the Earth, a monochromatic signal gets frequency and amplitude modulated. The effect of both these modulations is to smear out the monochromatic signal into a small bandwidth about the signal frequency of the wave. However, the effect on the Fourier transform of frequency modulation is much more severe compared to amplitude modulation in that the height of the peak is reduced drastically. The spread in the frequency, $\Delta f \approx \frac{2\pi f_0 R}{c}$, where f_0 is the frequency of the signal, R the radius of the Earth and c the velocity of light. Further, we develop analytical approximations to Fourier transform of the pulsar taking into account the rotation of the Earth and then the orbital correction to the signal. Inspection of the expression shows that for a 1kHz signal and one day observation time, the signal is spread into about 100 bands. These calculations are performed for arbitrary orientations of the detector and the arbitrary wave parameters.

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1. INTRODUCTION

The direct detection of gravitational radiation (GR) from astrophysical sources is probably the most important problem in experimental gravitation today. The rapid variation of space-time curvature due to, for example, collisions, pulsations or the rapid orbital motion should emanate curvature ripples or gravitational waves(GW) that travel at the speed of light and carry the identity of the graviton. The advent of the LIGO[1)](Laser Interferometric Gravitational Wave Observatory) project is opening a new window for the study of a vast and rich variety of non-linear curvature phenomena. The LIGO can in fact confirm that GW exist and by monitoring gravitational waveforms give important information on their amplitudes, frequency and other important physical parameters. Therefore, the study of different radiative processes from astrophysical sources and their GW luminosities and dimensionless amplitudes is important.

We can broadly classify the astrophysical sources of GR as continuous, burst type and stochastic. A prototype of continuous source is a pulsar. If the axis of rotation of a pulsar makes an angle α with the direction of the angular momentum of the pulsar, the resultant time dependent mass-quadrupole then becomes the source of GR from such a pulsar. The amplitudes of GR from these pulsars are probably very weak($\leq 10^{-26} - 10^{-28}$). The GR signal will be buried deep within the noise of the detector system. The detection of a GR signal therefore warrants the urgent need of careful data analysis with development of analytical methods and problem oriented algorithms.

In section 2, we briefly outline the nature of the GW signal from a pulsar, frequency modulation(FM) and amplitude modulation(AM) of the GW signal. The Doppler shift due to rotation and orbital motion of the Earth in the Solar System Barycentre(SSB) frame and its effect on the total phase of the received GW signal and the Fourier transform(FT) of the GW signal are described in section 3. The final section deals with the conclusions.

2. THE NATURE OF PULSAR SIGNAL

The amplitude h_0 of the GW is given by the formula[2)]

$$h_0 = 7.7 \times 10^{-24} \left(\frac{I_{zz}}{10^{45} g\ cm^2}\right)\left(\frac{10Kpc}{r}\right)\left(\frac{f}{1kHz}\right)^2\left(\frac{\epsilon}{10^{-4}}\right)$$

1 and 2.

The total response is a function of the position of the source, the orientation of the detector on Earth, orientation of the spin axis of the Earth and the orientation of the orbital plane. Since the signal is weak, long integration times $\approx 10^7$ secs will be needed to extract the signal out of the noise. Since the detector along with the Earth moves in this time, the frequency of the wave emitted by the source is Doppler shifted. Also since the detector has an anisotropic response, the signal recorded by the detector is both frequency and amplitude modulated. We discuss now the two modulations appearing in the response, namely, (a) frequency modulation, (b) amplitude modulation.

2a. Frequency Modulation: It arises due to translatory motion of the detector acquired from the motion of the Earth. We have only considered two motions of the Earth namely, its rotation about the spin axis and the orbital motion about the Sun. Hence the response is doubly frequency modulated with one period corresponding to 1 day and the other period corresponding to a year. The FM smears out a monochromatic signal into a small bandwidth around the signal frequency of the monochromatic waves. It also redistributes the power in a small bandwidth. The study of FM due to rotation of the Earth about its spin axis, for one day's observation shows that the Doppler spread in the bandwidth for 1kHz signal will be 0.029Hz. The Doppler spread in the bandwidth due to orbital motion for one day observation will be 1.74×10^{-3}Hz[7]. The consequence of the Doppler spread in signal detection will be discussed elsewhere separately. Since any observation is likely to last longer than a day it will be very important to incorporate this effect into the data analysis algorithms.

2b. Amplitude Modulation: The detector possesses a quadrupole antenna pattern. For a given incident wave, a detector in different orientations will record different amplitudes in the response. The functions F_+ and $F_\times$ appearing in the expression of the response completely characterise AM for the two polarizations. Since the expressions for F_+ and $F_\times$ are quite complicated, we will consider some special case to get some idea of AM. For the ideal case when the wave is optimally incident on the detector F_+ and $F_\times$ can individually have maximum value of unity. For a special case when the detector is situated on the equator with arms symmetrically placed about the North-South direction, *i.e.* $\alpha = \frac{\pi}{2}$, $\beta_2 = 0$, $\gamma = 0$ and the wave given by the parameters $\theta = \frac{\pi}{2}$, $\phi = 0$, $\psi = 0$, the antenna pattern functions

F_+ and $F_\times$ are given by,

$$F_+ = \frac{1}{2}\sin 2\varepsilon \cos(\omega_{rot}t), \qquad (4a)$$

$$F_\times = -\cos\varepsilon \sin(\omega_{rot}t) \ . \qquad (4b)$$

where ω_{rot} is the rotational frequency of the Earth about its spin axis. For the above case the AM results in about 40% drop in amplitude of the signal as compared to optimal incidence($i.e.F_+$ or $F_\times = 1$).

3. Effect of the Doppler Shift on the GW signal

3a. FM of a monochromatic signal : The monochromatic source is frequency modulated by the relativistic Doppler shift:

$$f_{rec} = f_o\gamma_o\left(1 + \frac{\vec{v}\cdot\vec{n}}{c}\right), \qquad (5)$$

where $\gamma_o = 1/\sqrt{1 - v^2/c^2} \approx 1$, f_o is the emitted frequency, $\vec{n}$ is unit vector from the antenna to the source, $\vec{v}$ is the relative velocity of the source and antenna(depends directly on source location) and f_{rec} is the frequency received. Consider the source of constant frequency f_o, the FM would be of the form

$$S(t) = \cos\left(2\pi f_o t + \Delta f \cos(2\pi f_m t)\right). \qquad (6)$$

The effect of FM on a sinusoidal from the source is to generate a forest of sidebands spaced at the modulation frequency. The n^{th} sideband has an amplitude given by $J_n(f/f_o)$, where J_n is a cylindrical Bessel function of the first kind and the modulation index $n \approx \Delta f/f_o$. The Bessel functions have the important property that they cease to have an appreciable amplitude beyond n. At 1KHz, the daily Doppler modulation smears a signal into roughly 100 equally populated sidebands. If the frequency resolution of the analysis is sufficient to resolve the sidebands(spaced at 10μHz), then the smearing corresponds to a loss of 20dB($\approx \sqrt{100}$) in the S/N.

The Doppler shift due to rotation and orbital motion of the Earth in the SSB frame is calculated once the relative velocity between the source and the detector is known. Let A be the distance from the centre of the SSB frame to the centre of the Earth, R is the radius of the Earth and $\vec{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$ is the unit vector in the direction of the source. Then the time t_d registered by the detector for the same phase of the wave in terms of t_b is given by,

$$t_d = t_b - \int_0^{t_b} \frac{\dot{\vec{r}}_{tot}\cdot\vec{n}}{c}dt - \frac{\vec{r}_{tot}(0)\cdot\vec{n}}{c}, \qquad (7a)$$

$$= t_b - \frac{\Delta\vec{r}_{tot}\cdot\vec{n}}{c} - \frac{\vec{r}_{tot}(0)\cdot\vec{n}}{c}. \qquad (7b)$$

where $\vec{r}_{tot}$ is the position vector of the detector in the SSB frame. The phase $\phi(t)$ of the received signal for a single direction sky search (θ, ϕ) is given by,

$$\phi(t) = 2\pi f_\circ \Bigg[t - t_\circ + \Bigg\{ \frac{A}{c} \sin\theta \cos\phi' + \frac{R}{c} \sin\alpha \Big\{ \sin\theta \big(\sin\beta' \cos\varepsilon \sin\phi + \cos\phi \cos\beta' \big) + \sin\beta' \sin\varepsilon \cos\theta \Big\} \Bigg\} - \Bigg\{ \frac{A}{c} \sin\theta \cos\phi'_\circ + \frac{R}{c} \sin\alpha \Big\{ \sin\theta \big(\sin\beta'_\circ \cos\varepsilon \sin\phi + \cos\phi \cos\beta'_\circ \big) + \sin\beta'_\circ \sin\varepsilon \cos\theta \Big\} \Bigg\} \Bigg]. \quad (8)$$

where $\phi' = \omega_{orb} t - \phi$, $\beta' = \beta_\circ + \omega_{rot} t$, $\phi'_\circ = \omega_{orb} t_\circ - \phi$, $\beta'_\circ = \beta_\circ + \omega_{rot} t_\circ$. The figure 3 is a plot of the Doppler shift velocity variation $(\vec{v} \cdot \vec{n})/c$. The sinusoidal variation is a diurnal variation and down along the diagonal is the orbital variation which has a periodicity of one year.

3b. The FT of the frequency modulated GW signal : The detector is put on the Equator with the arms symmetrically placed about the North-South direction, *i.e.* $\alpha = \pi/2$, $\beta_0 = 0$, $\gamma = 0$, and the wave parameters $\theta = \pi/2, \phi = \psi = 0$. If we consider only the frequency modulated output of the signal following analysis ensues(for one day observation). We consider the signal $\cos(\phi(t))$ then its FT is given by,

$$\tilde{H}(\nu) = Re\left\{ e^{-ix} \left[\int_0^\pi \left(e^{i\nu\xi} e^{ix\cos\xi} + e^{i\nu\xi} e^{-ix\cos\xi} e^{-i\nu\pi} \right) d\xi \right] \right\}. \quad (9)$$

where $\omega_\circ = 2\pi f_\circ$, $\nu = \frac{\omega_0 - \omega}{\omega_r}$, $x = \frac{2\pi f_0 R}{c}$ and $f_\circ$ is the frequency of the pulsar signal. Using the identity,

$$e^{\pm ix\cos\xi} = J_0(\pm x) + 2 \sum_{k=1}^{\infty} i^k J_k(\pm x) \cos k\xi \quad (10)$$

We have the expression,

$$\tilde{H}(\nu) \simeq 2\pi \frac{\sin(\nu\pi)}{\nu\pi} \left[J_0(x) \cos x + 2 \sum_{k=1}^{\infty} J_k(x) \cos\left(x - \frac{3}{2} k\pi \right) \frac{\nu^2}{\nu^2 - k^2} \right]. \quad (11)$$

where we have neglected the fast oscillating term. For the particular values of $\nu = k, k = 0, 1, 2, ...$

$$\tilde{H}(\nu) = 2\pi J_k(x) \cos\left(x - \frac{k\pi}{2} \right) \quad (12)$$

An important question is whether a closed form expression for the sum in the second term exists ? If so it would be extremely useful. Figure 4 shows the Fourier transform of the pulsar signal including the orbital correction besides that due to the Earth's rotation about its axis. The distinct bands of the spectrum are clearly seen. Without FM, the peak at 48Hz has the amplitude $T/2 = 43200.0$ for one day observation time. The FT becomes more complex with the inclusion of the orbital correction and one obtains a double series of Bessel functions.

4. Conclusion

Signal analysis of the gravitational waveform of pulsars has been done by considering both the rotation and orbital motion of the Earth. Both FM and AM smear out the monochromatic wave signal into a small bandwidth about the signal frequency f_0. Both the reduction of the peak height and the spread into bands are shown. Other motions such as the Earth-Moon motion etc. which have bearing on the signal detection should also be accounted for in the computation of the response for the data analysis(Schutz, Kanti and Dhurandhar). These calculation help in understanding the response function for a monochromatic wave such as a pulsar and will be useful for the data analysis of such sources.

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Figure Captions

Figure I : The figure shows the orientation of the wave frame (X, Y, Z) with respect to the Solar System Barycentre(SSB) frame (x_b, y_b, z_b). The orientation of the wave frame is specified by the three Euler angles θ, ϕ, ψ, which are needed to rotate the wave frame to SSB frame. The angles (θ, ϕ) give the direction of incoming wave in the SSB frame, while ψ represents the polarization angle.

Figure II : The figure shows orientation of the detector on the Earth. The detector orientation is specified by the angles (α, β, γ) with respect to the Earth frame (x_E, y_E, z_E) with z_E as spin axis of the Earth. Here α is the colatitude, β is the current longitude($\beta = \beta_o + \omega_{rot}t$) and γ is the angle the x axis of the detector makes with the local meridian. The Earth's spin axis has eclipticity ε (the equatorial plane of the Earth is inclined by an angle ε with respect to orbital plane of the Earth). These angles specify the orientation of detector in SSB frame. The SSB frame and the Earth frame have a common x axis. The x axis of the SSB frame is taken to lie along the line joining the Sun to the Earth on 21^{st} March(Vernal equinox). The z axis of the SSB frame is orthogonal to the orbital plane of the Earth.

Figure III : The figure shows Doppler shift variation in SSB frame due to rotation and orbital motion of the Earth.

Figure IV : The figure shows Fourier transform of the monochromatic pulsar signal due to the orbital and rotation motion of the Earth at 48Hz for orientation of the detector $\alpha = \pi/2, \beta = 0.0$ and incoming direction of the wave $\theta = \pi/2, \phi = 0.0$ considering only h_+ polarization of the wave.

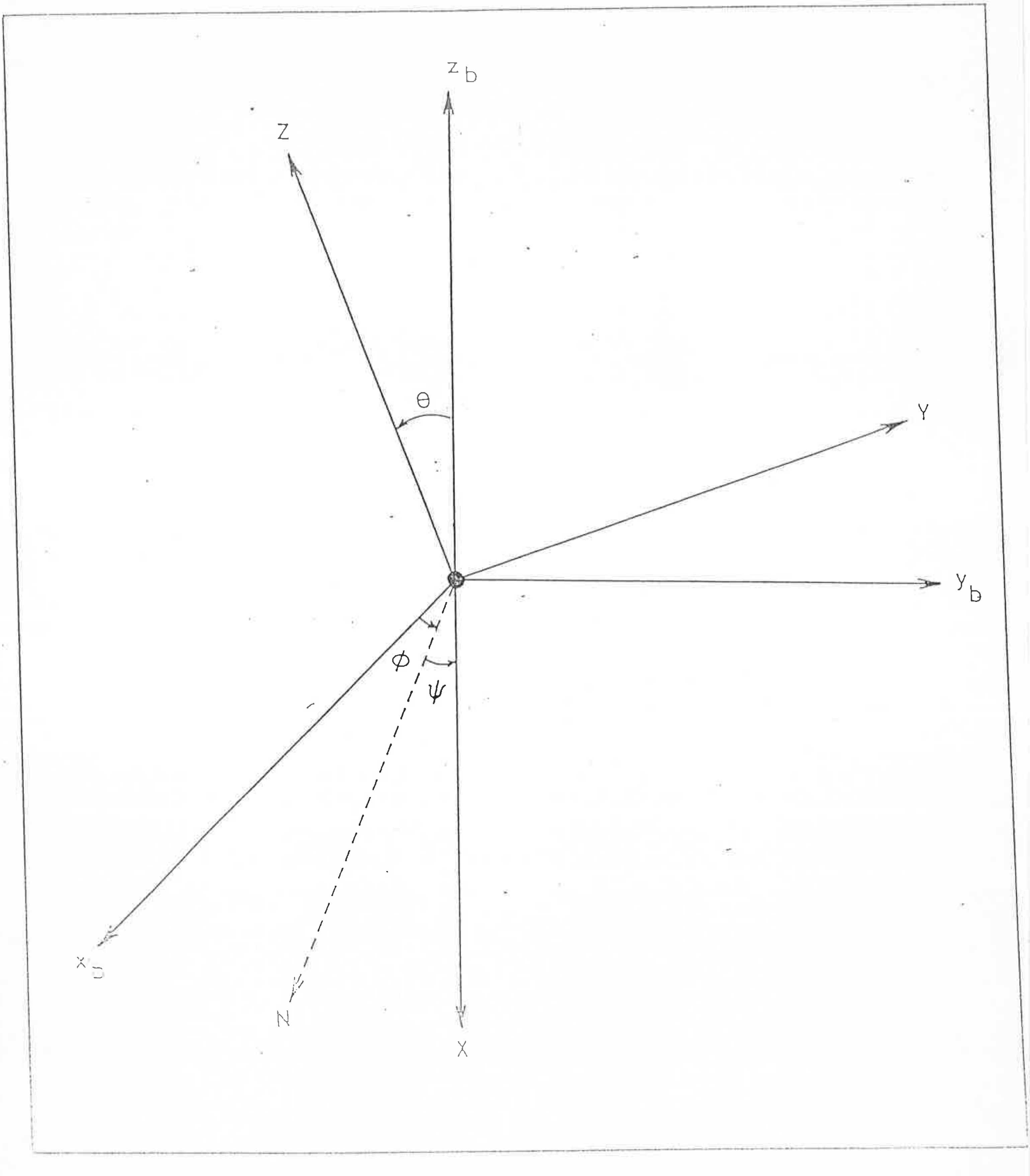


Figure I

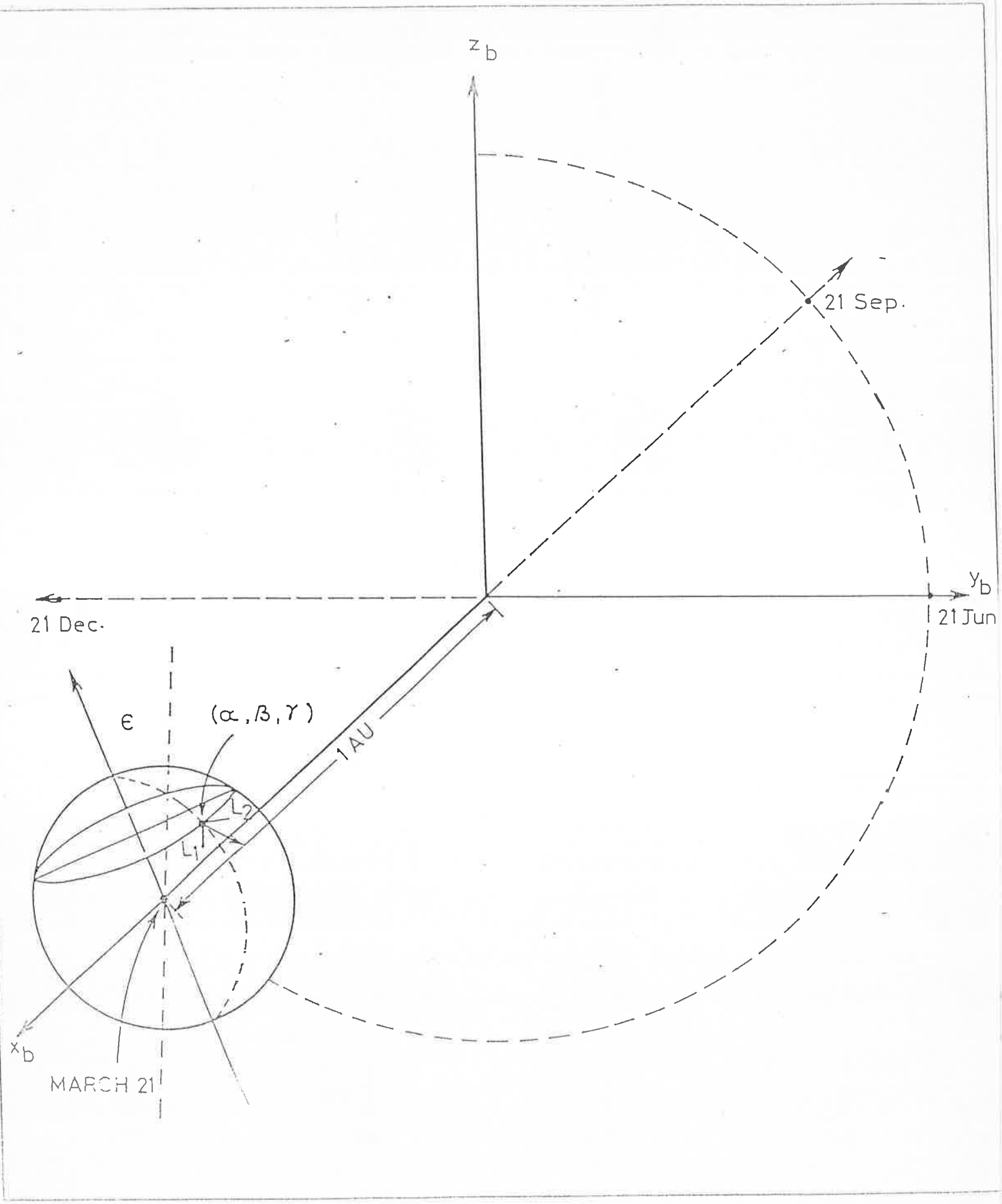


Figure II

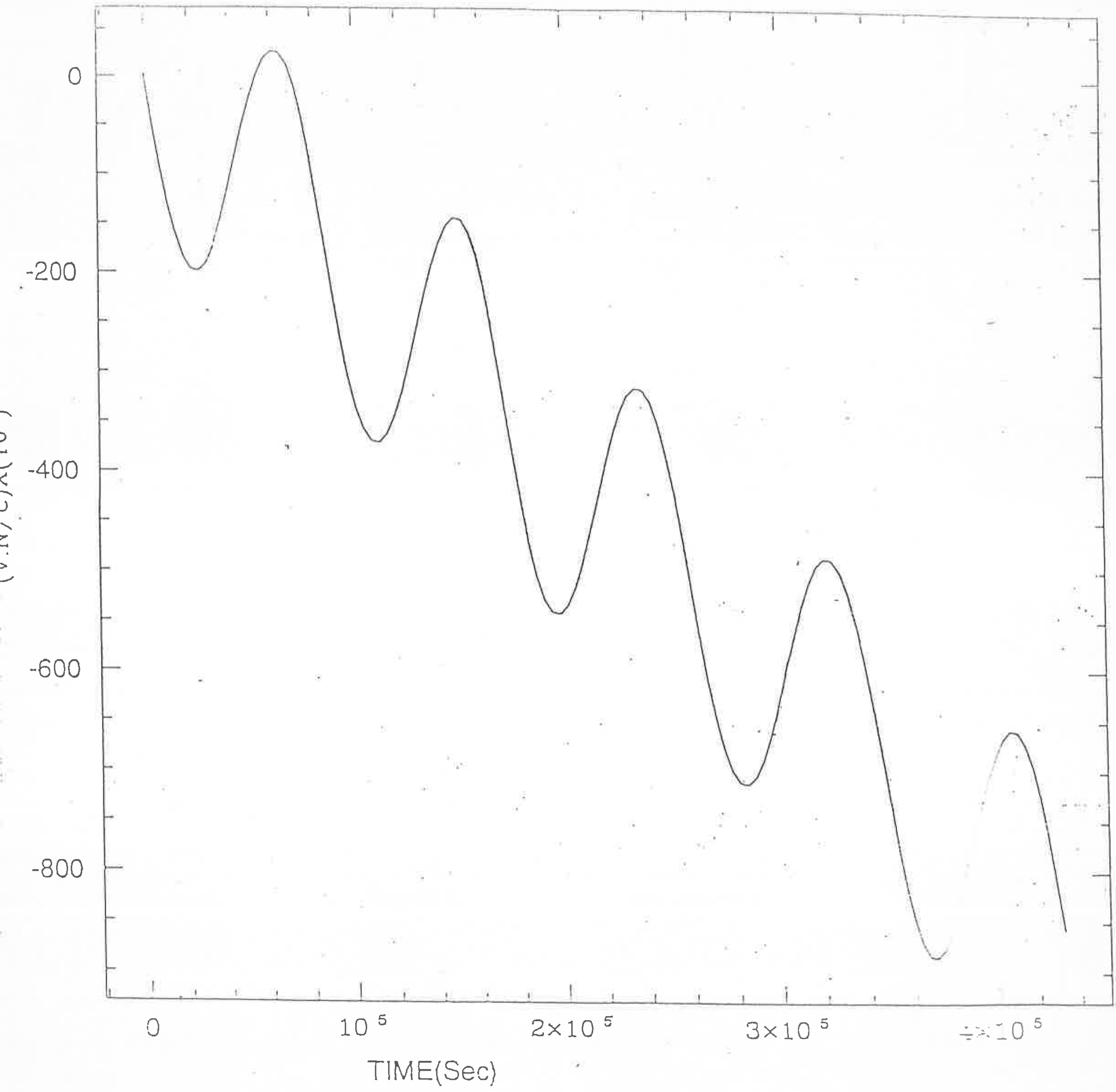
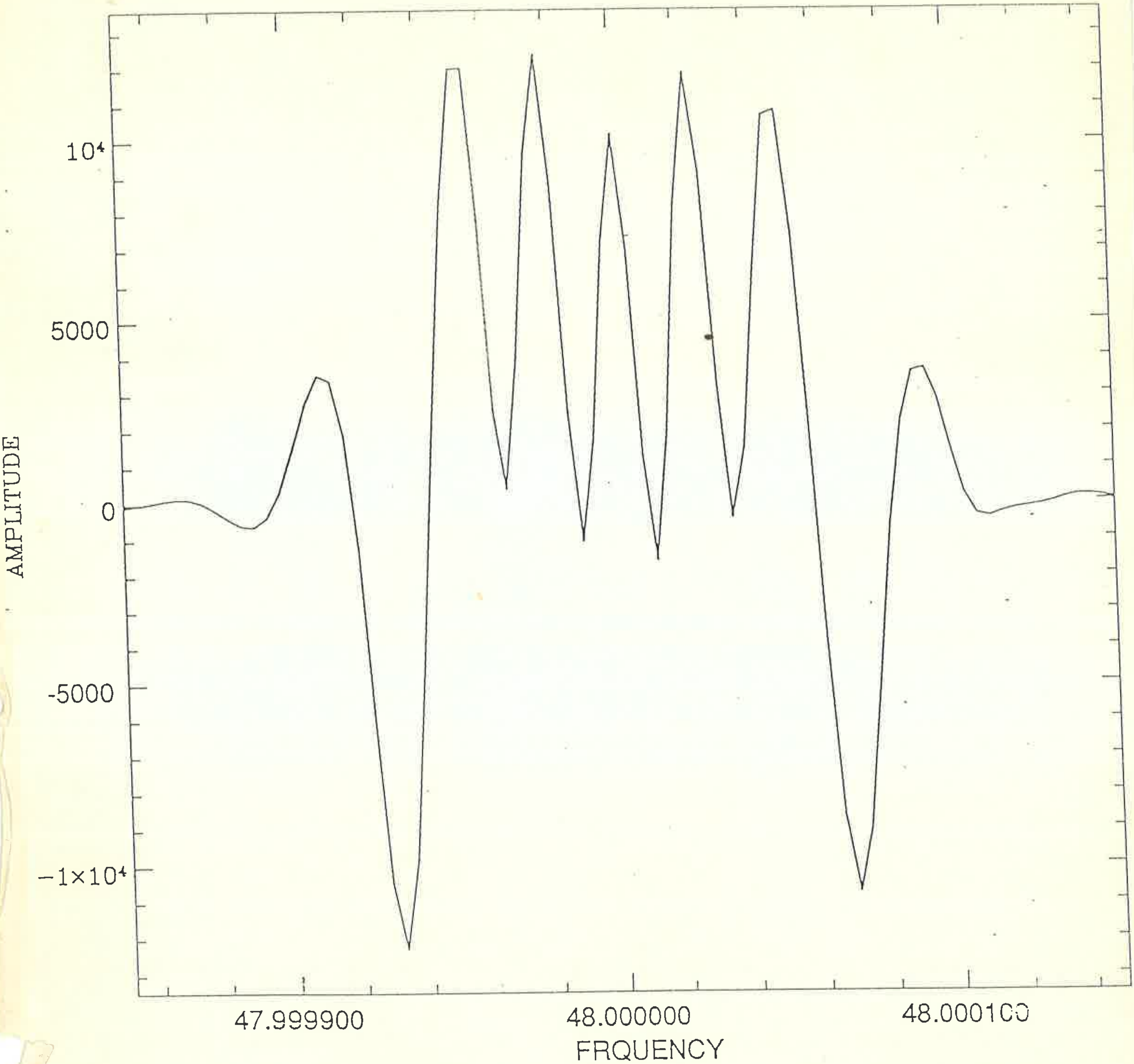


Figure III



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FRQUENCY

Figure IV