Does a nonzero tunneling probability imply particle production in time-independent classical electromagnetic backgrounds?

L. Sriramkumar* and T. Padmanabhan†
IUCAA, Post Bag 4, Ganeshkhind, Pune 411 007, India
(Received 3 April 1996)

In this paper, we probe the validity of the tunneling interpretation that is usually called forth in the literature to explain the phenomenon of particle production by time-independent classical electromagnetic backgrounds. We show that the imaginary part of the effective Lagrangian is zero for a complex scalar field quantized in a time-independent, but otherwise arbitrary, magnetic field. This result implies that no pair creation takes place in such a background. But we find that when the quantum field is decomposed into its normal modes in the presence of a spatially confined and time-independent magnetic field, there exists a nonzero tunneling probability for the effective Schrödinger equation. According to the tunneling interpretation, this result would imply that spatially confined magnetic fields can produce particles, thereby contradicting the result obtained from the effective Lagrangian. This lack of consistency between these two approaches calls into question the validity of attributing a nonzero tunneling probability for the effective Schrödinger equation to the production of particles by the time-independent electromagnetic backgrounds. The implications of our analysis are discussed. [S0556-2821(96)01124-1]

PACS number(s): 12.20.Ds, 04.62.+v, 11.15.Kc

I. INTRODUCTION

The phenomenon of pair creation by classical electromagnetic backgrounds was first studied by Schwinger more than four decades ago. In his classic paper, Schwinger considered a quantized spinor field interacting with a constant external electromagnetic background [1]. Obtaining an effective Lagrangian by integrating out the degrees of freedom corresponding to the quantum field, he showed that the effective Lagrangian had an imaginary part only when $(\mathbf{E}^2-\mathbf{B}^2)>0$, where $\mathbf{E}$ and $\mathbf{B}$ are the constant electric and magnetic fields, respectively (also see [2]). The appearance of an imaginary part in the effective Lagrangian implies an instability of the vacuum and Schwinger attributed the cause of this instability to the production of pairs corresponding to the quantum field by the electromagnetic background. The imaginary part of the effective Lagrangian, Schwinger concluded, should be interpreted as the number of pairs that have been produced, per unit four-volume, by the external electromagnetic field.

Though attempts have been made in literature to obtain the effective Lagrangian for a fairly nontrivial electromagnetic field (see, for instance, Refs. [3–7]), its evaluation for an arbitrary vector potential proves to be an uphill task. Because of this reason the phenomenon of particle production in classical electromagnetic backgrounds has been repeatedly studied in the literature by the method of normal mode analysis. In this approach, the normal modes of the quantum field are obtained by solving the wave equation it satisfies for the given electromagnetic background in a particular gauge. The coefficients of the positive frequency normal modes of the quantum field are then identified to be the annihilation operators. The evolution of these operators, therefore, follows the evolution of the normal modes. Then, by relating these operators defined in the asymptotic regions (either in space or in time), the number of particles that have been produced by the electromagnetic background can be computed.

Consider an electromagnetic background that can be represented by a time-*dependent* gauge. If we choose to study the evolution of the quantum field in such a gauge, then a positive frequency normal mode of the quantum field at late times will, in general, prove to be a linear superposition of the positive and negative frequency modes defined at early times. The coefficients in such a superposition are the Bogoliubov coefficients α and β. A nonzero Bogoliubov coefficient β would then imply that the *in*-vacuum state is not the same as the *out*-vacuum state. This in turn implies that the *in-out* transition amplitude is less than unity which can be attributed to the excitation of the modes of the quantum field by the electromagnetic background [8–13]. These excitations manifest themselves as real particles corresponding to the quantum field.

On the other hand, consider an electromagnetic background that can be described by a space-dependent gauge (by which we mean a gauge that is completely independent of time). If the evolution of the quantum field is studied in such a gauge, then because of the lack of dependence on time, the Bogoliubov coefficient β proves to be trivially zero. This could then imply that the electromagnetic background which is being considered does not produce particles.

An interesting situation arises when the same electromagnetic field can be described by a (purely) space-dependent gauge as well as a (purely) time-dependent gauge. If we choose to study the evolution of the quantum field in the time-dependent gauge, in general, β will prove to be nonzero, thereby implying (as discussed above) that particles are being produced by the electromagnetic background. But, in the space-dependent gauge β is trivially zero, thereby dis-

*Electronic address: lsk@iucaa.ernet.in
†Electronic address: paddy@iucaa.ernet.in

agreeing with the result obtained in the time-dependent gauge. Therefore, to obtain results that are gauge invariant, the phenomenon of particle production has to be somehow ''explained'' in the space-dependent gauge. In the literature, a ''tunneling interpretation'' is usually invoked to explain the phenomenon of particle production in such a situation [14–18]. In this approach, an effective Schrödinger equation is obtained after the quantum field is decomposed into normal modes in the space-dependent gauge. The nonzero tunneling probability for this Schrödinger equation is then attributed to the production of particles by the electromagnetic background.

The discussion in the above paragraph can be illustrated by the following well-known, and instructive, example. Consider a constant electric field given by $\mathbf{E}=E\mathbf{x}$, where E is a constant and $\hat{\mathbf{x}}$ is the unit vector along the positive x axis. This electric field can be described either by the time-dependent gauge, $A_1^\mu=(0,-Et,0,0)$, or by the space-dependent gauge, $A_2^\mu=(-Ex,0,0,0)$. In the gauge A_1^μ, because of the time dependence, the positive frequency normal modes of the quantum field at $t=+\infty$ are related by a nonzero Bogoliubov coefficient β to the positive frequency modes at $t=-\infty$. The quantity $|\beta|^2$ then yields the number of particles that have been produced in a single mode of the quantum field at late times in the *in* vacuum [19,20]. But, if the evolution of the quantum field is studied in the gauge A_2^μ, because of time independence, β proves to be zero, thereby disagreeing with the result obtained in the gauge A_1^μ. The tunneling interpretation can be invoked in such a situation to explain particle production in the gauge A_2^μ. In this gauge, after the normal mode decomposition of the quantum field, an effective Schrödinger equation is obtained along the x direction. The nonzero tunneling probability $|T|^2$ for this Schrödinger equation is then interpreted as the number of particles that have been produced in a single mode of the quantum field [19,20]. The tunneling probability $|T|^2$ evaluated in the gauge A_2^μ, in fact, exactly matches the quantity $|\beta|^2$ obtained in the gauge A_1^μ. Also, these two quantities agree with the pair creation rate obtained by Schwinger from the imaginary part of the effective Lagrangian.

The fact that the quantities $|\beta|^2$ and $|T|^2$ agree, not only with each other but also with the pair creation rate obtained from the effective Lagrangian, for the case of a constant electric field has given certain credibility to the tunneling interpretation. Our aim, in this paper, is to probe the validity of the tunneling interpretation.

Consider an arbitrary electromagnetic background that can be described by a space-dependent gauge. Also assume that when the evolution of the quantum field is studied in such a gauge, there exists a nonzero tunneling probability for the effective Schrödinger equation. Can such a nonzero tunneling probability be always interpreted as particle production? We attempt to answer this question in this paper by comparing the results obtained from the effective Lagrangian with those obtained from the tunneling approach. We carry out our analysis for a spatially varying, time-independent magnetic field when it is described by a space-dependent gauge. We find that there exists, in general, a lack of consistency between the results obtained from the tunneling approach and those obtained from the effective Lagrangian. This inconsistency clearly calls into question the validity of the tunneling interpretation as it is presently understood in the literature.

This paper is organized as follows. In Sec. II, we show that the imaginary part of the effective Lagrangian for an arbitrary time-independent background magnetic field is zero. In Sec. III, we calculate the tunneling probability, which happens to be nonzero, for a particular spatially confined and time-independent magnetic field when it is represented by a space-dependent gauge. Finally, in Sec. IV we discuss the implications of our analysis to the study of particle production in time-independent electromagnetic and gravitational backgrounds.

II. EFFECTIVE LAGRANGIAN FOR A TIME-INDEPENDENT MAGNETIC FIELD BACKGROUND

The system we consider in this paper consists of a complex scalar field Φ interacting with an electromagnetic field represented by the vector potential A^μ. It is described by the Lagrangian density

$$\mathcal{L}(\Phi,A_\mu)=(\partial_\mu\Phi+iqA_\mu\Phi)(\partial^\mu\Phi^*-iqA^\mu\Phi^*)-m^2\Phi\Phi^*-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{1}$$

where q and m are the charge and the mass associated with a single quantum of the complex scalar field, the asterisk denotes complex conjugation, and

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu. \tag{2}$$

The electromagnetic field is assumed to behave classically; hence, A_μ is just a c number while the complex scalar field is assumed to be a quantum field so that Φ is an operator-valued distribution. We will also assume that the electromagnetic field is given to us *a priori*, i.e., we will not take into account the back reaction of the quantum field on the classical background. (Kiefer *et al.* show in Ref. [21] that the semiclassical domain as envisaged here does exist; also see Ref. [22] in this context. The issue of back reaction on the electromagnetic background has been addressed in Refs. [23–25].) In such a situation, we can obtain an effective Lagrangian for the classical electromagnetic background by integrating out the degrees of freedom corresponding to the quantum field as

$$\exp\left(i\int d^4x\,\mathcal{L}_{\rm eff}(A_\mu)\right)\equiv\int\mathcal{D}\Phi\int\mathcal{D}\Phi^*\exp\left(i\int d^4x\,\mathcal{L}(\Phi,A_\mu)\right), \tag{3}$$

where we have set $\hbar=c=1$ for convenience. The effective Lagrangian can be expressed as

$$\mathcal{L}_{\rm eff}=\mathcal{L}_{\rm em}+\mathcal{L}_{\rm corr}, \tag{4}$$

where $\mathcal{L}_{\rm em}$ is the Lagrangian density for the free electromagnetic field, the third term in the Lagrangian density (1), and $\mathcal{L}_{\rm corr}$ is given by

$$\exp\left(i\int d^4x\mathcal{L}_{\rm corr}(A_\mu)\right)=\int \mathcal{D}\Phi\int \mathcal{D}\Phi^*\times\exp\left(i\int d^4x\{(\partial_\mu\Phi+iqA_\mu\Phi)\times(\partial^\mu\Phi^*-iqA^\mu\Phi^*)-m^2\Phi\Phi^*\}\right). \tag{5}$$

Integrating the action for the scalar field in the above equation by parts and dropping the resulting surface terms, we obtain that

$$\exp\left(i\int d^4x\mathcal{L}_{\rm corr}(A_\mu)\right)=\int \mathcal{D}\Phi\int \mathcal{D}\Phi^*\exp\left(-i\int d^4x\Phi^*\hat{D}\Phi\right)=(\det\hat{D})^{-1}, \tag{6}$$

where the operator $\hat{D}$ is given by

$$\hat{D}\equiv D_\mu D^\mu+m^2 \quad \text{and} \quad D_\mu\equiv\partial_\mu+iqA_\mu. \tag{7}$$

The determinant in Eq. (6) can be expressed as follows

$$\exp\left(i\int d^4x\mathcal{L}_{\rm corr}\right)=(\det\hat{D})^{-1}=\exp[-{\rm Tr}(\ln\hat{D})]=\exp\left(-\int d^4x\langle t,\mathbf{x}|\ln\hat{D}|t,\mathbf{x}\rangle\right), \tag{8}$$

and in arriving at the last expression, following Schwinger, we have chosen the set of basis vectors $|t,\mathbf{x}\rangle$ to evaluate the trace of the operator $\ln\hat{D}$. From the above equation it is easy to identify that

$$\mathcal{L}_{\rm corr}=i\langle t,\mathbf{x}|\ln\hat{D}|t,\mathbf{x}\rangle. \tag{9}$$

Using the following integral representation for the operator $\ln\hat{D}$:

$$\ln\hat{D}\equiv-\int_0^\infty\frac{ds}{s}\exp[-i(\hat{D}-i\epsilon)s] \tag{10}$$

(where $\epsilon\to0^+$), the expression for $\mathcal{L}_{\rm corr}$ can be written as

$$\mathcal{L}_{\rm corr}=-i\int_0^\infty\frac{ds}{s}\exp[-i(m^2-i\epsilon)s]K(t,\mathbf{x},s|t,\mathbf{x},0), \tag{11}$$

where

$$K(t,\mathbf{x},s|t,\mathbf{x},0)=\langle t,\mathbf{x}|e^{-i\hat{H}s}|t,\mathbf{x}\rangle \quad \text{and} \quad \hat{H}\equiv D_\mu D^\mu. \tag{12}$$

That is, $K(t,\mathbf{x},s|t,\mathbf{x},0)$ is the kernel for a quantum-mechanical particle (in four dimensions) described by the Hamiltonian operator $\hat{H}$. The variable s, that was introduced in Eq. (10) when the operator $\ln\hat{D}$ was expressed in an integral form, acts as the time parameter for the quantum-mechanical system. [The integral representation for the operator $\ln\hat{D}$ we have used above is divergent in the lower limit of the integral, i.e., near $s=0$. This divergence is usually regularized in field theory by subtracting from it another divergent integral, viz., the integral representation of an operator $\ln\hat{D}_0$, where $\hat{D}_0=(\partial^\mu\partial_\mu+m^2)$, the operator corresponding to that of a free quantum field. That is, to avoid the divergence, the integral representation for $\ln\hat{D}$ is actually considered to be

$$\ln\hat{D}-\ln\hat{D}_0\equiv-\int_0^\infty\frac{ds}{s}\{\exp[-i(\hat{D}-i\epsilon)s]-\exp[-i(\hat{D}_0-i\epsilon)s]\}. \tag{13}$$

Therefore, in what follows, the operator $\ln\hat{D}$ should be considered as $\ln\hat{D}-\ln\hat{D}_0$ though it will not be written so explicitly.]

Now, consider a background electromagnetic field described by the vector potential

$$A^\mu=(0,0,A(x),0), \tag{14}$$

where $A(x)$ is an arbitrary function of x. This vector potential does not produce an electric field but gives rise to a magnetic field $\mathbf{B}=(dA/dx)\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is the unit vector along the positive z axis. According to the Maxwell's equations, in the absence of an electric field, the magnetic field is related to the current $\mathbf{j}(x)$ as

$$\nabla\times\mathbf{B}=\mathbf{j}. \tag{15}$$

Then, the current that can give rise to the time-independent magnetic field we consider here is given by

$$\mathbf{j}=-\left(\frac{d^2A}{dx^2}\right)\hat{\mathbf{y}}, \tag{16}$$

where $\hat{\mathbf{y}}$ is the unit vector along the positive y axis. If we assume that $\mathbf{j}$ is finite and continuous everywhere and also vanishes as $|x|\to\infty$, then the magnetic field we consider here can be physically realized in the laboratory.

The operator $\hat{H}$ corresponding to the vector potential (14) is given by

$$\hat{H}\equiv\partial_t{}^2-\nabla^2+2iqA\partial_y+q^2A^2. \tag{17}$$

Then, the kernel for the quantum-mechanical particle described by the Hamiltonian above can be formally written as

$$K(t,\mathbf{x},s|t,\mathbf{x},0)=\langle t,\mathbf{x}|\exp[-i(\partial_t{}^2-\nabla^2+2iqA\partial_y+q^2A^2)s]\,|t,\mathbf{x}\rangle. \tag{18}$$

Using the translational invariance of the Hamiltonian operator $\hat{H}$ along the time coordinate t and the spatial coordinates y and z, we can express the above kernel as

$$K(t,\mathbf{x},s|t,\mathbf{x},0)=\int\frac{d\omega}{2\pi}\int\frac{dp_y}{2\pi}\int\frac{dp_z}{2\pi}\langle x|\exp\{-i[-\omega^2-d_x^2+(p_y-qA)^2+{p_z}^2]s\}|x\rangle. \tag{19}$$

Performing the ω and p_z integrations, we obtain that

$$K(t,\mathbf{x},s|t,\mathbf{x},0)=\frac{1}{4\pi s}\int\frac{dp_y}{2\pi}\langle x|e^{-i\hat{G}s}|x\rangle,$$

where

$$\hat{G}\equiv-d_x^2+(p_y-qA)^2. \tag{20}$$

The quantity $\langle x|e^{-i\hat{G}s}|x\rangle$ is then the kernel for the one-dimensional quantum-mechanical system described by the effective Hamiltonian operator $\hat{G}$. It can be expressed, using the Feynman-Kac formula, as

$$\langle x|\exp(-i\hat{G}s)|x\rangle=\sum_E|\Psi_E(x)|^2e^{-iEs}, \tag{21}$$

where Ψ_E is the eigenfunction of the operator $\hat{G}$ corresponding to an eigenvalue E, i.e.,

$$\hat{G}\Psi_E\equiv[-d_x^2+(p_y-qA)^2]\Psi_E=E\Psi_E, \tag{22}$$

so that $K(t,\mathbf{x},s|t,\mathbf{x},0)$ reduces to

$$K(t,\mathbf{x},s|t,\mathbf{x},0)=\frac{1}{4\pi s}\int\frac{dp_y}{2\pi}\sum_E|\Psi_E(x)|^2e^{-iEs}. \tag{23}$$

(It is assumed that the summation over E stands for integration over the relevant range when E varies continuously.) Since the potential term, $[p_y-qA(x)]^2$, in the Hamiltonian operator $\hat{G}$ is a positive definite quantity, the eigenvalue E can only lie in the range $(0,\infty)$. Substituting the expression for $K(t,\mathbf{x},s|t,\mathbf{x},0)$ in Eq. (11), we find that $\mathcal{L}_{\rm corr}$ is given by

$$\mathcal{L}_{\rm corr}=-\frac{i}{4\pi}\int\frac{dp_y}{2\pi}\sum_E|\Psi_E(x)|^2\int_0^\infty\frac{ds}{s^2}\times\exp[-i(m^2+E-i\epsilon)s]. \tag{24}$$

Differentiating the above expression for $\mathcal{L}_{\rm corr}$ twice with respect to m^2 (cf. [26]) and then carrying out the integration over the variable s, we obtain that

$$\mathcal{L}''_{\rm corr}=\frac{\partial^2\mathcal{L}_{\rm corr}}{\partial(m^2)^2}=\frac{1}{4\pi}\int\frac{dp_y}{2\pi}\sum_E\left(\frac{|\Psi_E(x)|^2}{m^2+E-i\epsilon}\right). \tag{25}$$

The quantity $(m^2+E-i\epsilon)^{-1}$ in the above expression can be written as

$$\left(\frac{1}{m^2+E-i\epsilon}\right)=\mathcal{P}\left(\frac{1}{m^2+E}\right)+i\pi\delta(m^2+E), \tag{26}$$

where $\mathcal{P}$ is the principal value of the corresponding argument. Since E is a positive semidefinite quantity, the argument of the δ function above never reduces to zero. Therefore, the second term in the above expression vanishes with the result that $\mathcal{L}''_{\rm corr}$ is a real quantity, thereby implying that $\mathcal{L}$ is also a real quantity. In fact, integrating $\mathcal{L}''_{\rm corr}$ twice with respect to m^2, we find that $\mathcal{L}_{\rm corr}$ can be expressed as

$$\mathcal{L}_{\rm corr}=\frac{1}{4\pi}\int\frac{dp_y}{2\pi}\sum_E|\Psi_E(x)|^2\alpha(\ln\alpha-1), \tag{27}$$

where $\alpha=(m^2+E)>0$ and ϵ has been set to zero. Then, clearly $\mathcal{L}_{\rm corr}$ is a real quantity. (To be rigorous, one has to take into account the two constants of integration that will appear on integrating $\mathcal{L}''_{\rm corr}$ with respect to m^2 {see [26]}, but these constants are irrelevant for our arguments here.)

Though we are unable to evaluate the effective Lagrangian for an arbitrary time-independent magnetic field in a closed form, we have been able to show that it certainly does not have any imaginary part. Therefore, we can unambiguously conclude that time-independent background magnetic fields do not produce particles. This, of course, agrees with Schwinger's result for a constant (time-independent) magnetic field background.

III. TUNNELING PROBABILITY IN A TIME-INDEPENDENT MAGNETIC FIELD BACKGROUND

We shall now calculate the tunneling probability for a specific time-independent background magnetic field in a space-dependent gauge. Consider the vector potential

$$A^\mu=(0,0,B_0L\tanh(x/L),0), \tag{28}$$

where B_0 and L are arbitrary constants. This vector potential does not produce any electric field but gives rise to the magnetic field

$$\mathbf{B}=B_0\,{\rm sech}^2(x/L)\hat{\mathbf{z}}, \tag{29}$$

where $\hat{\mathbf{z}}$ is the unit vector along the positive z axis. The magnetic field $\mathbf{B}$ goes to zero as $|x|\to\infty$, i.e., its strength is confined to an effective width L along the x axis. In the absence of an electric field, according to the Maxwell's equation (15), the magnetic field given by Eq. (29) can be produced by the current

$$\mathbf{j}=\left(\frac{2B_0}{L}\right){\rm sech}(x/L)\tanh(x/L)\hat{\mathbf{y}}, \tag{30}$$

where, as before, $\hat{\mathbf{y}}$ denotes the unit vector along the positive y axis. The current $\mathbf{j}$ is finite and continuous everywhere and also goes to zero as $|x|\to\infty$. Therefore, the magnetic field $\mathbf{B}$ given by Eq. (29) is physically realizable in the laboratory.

In an electromagnetic background, described by the vector potential A^μ, the complex scalar field satisfies the Klein-Gordon equation

$$(D_\mu D^\mu+m^2)\Phi=(\partial_\mu+iqA_\mu)(\partial_\mu+iqA_\mu)\Phi=0. \tag{31}$$

Substituting the vector potential (28) in the above equation, we obtain that

$$[\partial_t^2-\nabla^2+2iqB_0L\tanh(x/L)\partial_y+q^2B_0^2L^2\tanh^2(x/L)+m^2]\Phi=0. \tag{32}$$

Since the vector potential (28) is dependent only on the spatial coordinate x, the normal mode decomposition of the scalar field can be carried out as

$$\Phi_{\omega k_\perp}=N_{\omega k_\perp}e^{-i\omega t}e^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp}\psi_{\omega k_\perp}(x), \tag{33}$$

where $N_{\omega k_\perp}$ is the normalization constant, $k_\perp\equiv(k_y,k_z)$, and $x_\perp\equiv(y,z)$. The modes are normalized according to the scalar product

$$\begin{aligned}(\Phi_{\omega k_\perp},\Phi_{\omega' k'_\perp})=-i\int d\Sigma^\mu[\Phi_{\omega k_\perp}(D_\mu\Phi_{\omega' k'_\perp})^*\\ -\Phi^*_{\omega' k'_\perp}(D_\mu\Phi_{\omega k_\perp})]\\ =\delta(\omega-\omega')\,\delta(k_\perp-k'_\perp),\end{aligned} \tag{34}$$

where $d\Sigma^\mu$ is a timelike hypersurface. Substituting the normal mode $\Phi_{\omega k_\perp}$ in Eq. (32), we find that ψ satisfies the differential equation

$$\frac{d^2\psi}{d\rho^2}+[\omega^2-(k_y-qB_0L\tanh\rho)^2-k_z^2-m^2]L^2\psi=0, \tag{35}$$

where $\rho=(x/L)$ and we have dropped the subscripts on ψ. This differential equation can be rewritten as

$$-\frac{d^2\psi}{d\rho^2}+(k_yL-qB_0L^2\tanh\rho)^2\psi=(\omega^2-k_z^2-m^2)L^2\psi, \tag{36}$$

which then resembles a time-independent Schrödinger equation corresponding to a potential $(k_yL-qB_0L^2\tanh\rho)^2/2$ and energy eigenvalue $(\omega^2-k_z^2-m^2)L^2/2$. The potential term in the effective Schrödinger equation above reduces to a finite constant as $|x|\to\infty$. Therefore, there exist solutions for ψ which reduce to $e^{\pm ik_Lx}$ as $x\to-\infty$ and $e^{\pm ik_Rx}$ as $x\to+\infty$, where k_L and k_R are given by

$$\begin{aligned}k_L&=[\omega^2-(k_y+qB_0L)^2-k_z^2-m^2]^{1/2},\\ k_R&=[\omega^2-(k_y-qB_0L)^2-k_z^2-m^2]^{1/2}.\end{aligned} \tag{37}$$

We will confine to values of ω and $k_\perp$ such that k_L and k_R are real.

The differential equation (35) can be solved by the ansatz (cf. [27])

$$\psi=e^{-a\rho}\mathrm{sech}^b\rho f(\rho), \tag{38}$$

where

$$a=ik_-L,\quad b=ik_+L,\quad \text{and}\quad k_\pm=(k_R\pm k_L)/2. \tag{39}$$

Substituting the above ansatz in Eq. (35), we find that f satisfies the differential equation

$$\begin{aligned}u(u-1)\frac{d^2f}{du^2}+[1+a+b-2(b+1)u]\frac{df}{du}\\ +[q^2B_0{}^2L^4-b(b+1)]f=0,\end{aligned} \tag{40}$$

where the variable u is related to ρ by the equation: $u=(1-\tanh\rho)/2$. The above equation is a hypergeometric differential equation and its general solution is a linear combination of two hypergeometric functions (cf. [28], pp. 562 and 563), i.e.,

$$\begin{aligned}f(u)=AF(b+\tfrac{1}{2}+c,b+\tfrac{1}{2}-c,1+a+b,u)\\ +Bu^{-a-b}F(\tfrac{1}{2}-a+c,\tfrac{1}{2}-a-c,1-a-b,u),\end{aligned} \tag{41}$$

where A and B are arbitrary constants and

$$c=(\tfrac{1}{4}+q^2B_0{}^2L^4)^{1/2}. \tag{42}$$

To calculate the tunneling probability for the effective Schrödinger equation (36), we have to choose the constants A and B such that $\psi\sim e^{ik_Rx}$ as $x\to+\infty$ (i.e., when $u\to0$). This can be achieved by setting $A=0$ and $B=2^{-b}$, so that

$$f(u)=2^{-b}u^{-a-b}F(\tfrac{1}{2}-a+c,\,\tfrac{1}{2}-a-c,1-a-b,u). \tag{43}$$

Substituting the above solution in Eq. (38) and using the relation (cf. [28], p. 559)

$$\begin{aligned}&F(\tfrac{1}{2}-a+c,\,\tfrac{1}{2}-a-c,1-a-b,u)\\ &\quad=PF(\tfrac{1}{2}-a+c,\tfrac{1}{2}-a-c,1-a+b,1-u)\\ &\qquad+Q(1-u)^{a-b}\\ &\qquad\times F(\tfrac{1}{2}-b-c,\,\tfrac{1}{2}-b+c,1+a-b,1-u),\end{aligned} \tag{44}$$

where

$$P=\left(\frac{\Gamma(1-a-b)\Gamma(a-b)}{\Gamma(\tfrac{1}{2}-b-c)\Gamma(\tfrac{1}{2}-b+c)}\right)$$

and

$$Q=\left(\frac{\Gamma(1-a-b)\Gamma(b-a)}{\Gamma(\tfrac{1}{2}-a+c)\Gamma(\tfrac{1}{2}-a-c)}\right), \tag{45}$$

we find that, as $x\to-\infty$, i.e., when $(1-u)\to0$,

$$\psi\to Pe^{ik_Lx}+Qe^{-ik_Lx}. \tag{46}$$

Consider a solution of the effective Schrödinger equation (36) which goes as $(Re^{ik_Lx}+e^{-ik_Lx})$ as $x\to-\infty$ and goes over to (Te^{ik_Rx}) as $x\to+\infty$. Then, it is easy to identify the expressions for R and T from equation (46). They are given by

$$R=\left(\frac{P}{Q}\right)=\left(\frac{\Gamma(\tfrac{1}{2}-a+c)\Gamma(\tfrac{1}{2}-a-c)\Gamma(a-b)}{\Gamma(\tfrac{1}{2}-b-c)\Gamma(\tfrac{1}{2}-b+c)\Gamma(b-a)}\right),$$

$$T=\left(\frac{1}{Q}\right)=\left(\frac{\Gamma(\tfrac{1}{2}-a+c)\Gamma(\tfrac{1}{2}-a-c)}{\Gamma(1-a-b)\Gamma(b-a)}\right), \tag{47}$$

so that

$$|R|^2=\left(\frac{\cosh 2\pi k_+L+\cos 2\pi c}{\cosh 2\pi k_-L+\cos 2\pi c}\right)$$

and

$$|T|^2=\left(\frac{k_L}{k_R}\right)\left(\frac{\cosh 2\pi k_+L-\cosh 2\pi k_-L}{\cosh 2\pi k_-L+\cos 2\pi c}\right). \tag{48}$$

The Wronskian condition for the effective Schrödinger equation (36) then leads us to the relation

$$|R|^2-\left(\frac{k_R}{k_L}\right)|T|^2=1. \tag{49}$$

So, the tunneling probability *is* nonzero for the time-independent magnetic field we have considered here. It is, in fact, given by $|T|^2$ in Eq. (48).

The implications of our analysis are discussed in the following section.

IV. CONCLUSIONS

A time-independent magnetic field does not give rise to any electric field and a pure magnetic field cannot do any work on charged particles. Therefore, it seems plausible that such a magnetic field does not produce particles. This expectation is, in fact, corroborated by the result we have obtained in Sec. II, viz., that the imaginary part of the effective Lagrangian for a time-independent, but otherwise arbitrary, magnetic field is zero. Our analysis in Secs. II and III has been carried out assuming that the time-independent magnetic field is described by a space-dependent gauge. In such a gauge, β is trivially zero and if we had considered only a nonzero β to imply particle production, then this result would have proved to be consistent with the result we have obtained in Sec. II.

But this is not the whole story. According to the tunneling interpretation, in a time-independent gauge it is the tunneling probability for the effective Schrödinger equation that has to be interpreted as particle production. In Sec. III, we find that there exists a nonzero tunneling probability for a spatially confined, time-independent magnetic field. If the tunneling interpretation is true, this result would then imply that such a background can produce particles thereby contradicting the result we have obtained in Sec. II.

The tunneling probability can, in fact, prove to be nonzero in a more general case and is certainly not an artifact of our specific example. This can be seen as follows: Consider an arbitrary electromagnetic field described by the vector potential

$$A^\mu=(\phi(x),0,A(x),0), \tag{50}$$

where $\phi(x)$ and $A(x)$ are arbitrary functions of x. If the decomposition of the normal modes is carried out as was done in Eq. (33), then the effective Schrödinger equation for the x coordinate corresponding to the above vector potential turns out to be

$$-\frac{d^2\psi}{dx^2}+[(k_y-qA)^2-(\omega-q\phi)^2]\psi=(-k_z^2-m^2)\psi. \tag{51}$$

If we also assume that $\phi(x)$ and $A(x)$ vanish at the spatial infinities, then it is clear that the solutions for ψ will reduce to plane waves as $|x|\to\infty$. When such solutions are possible, in general, there is bound to exist a nonzero tunneling probability for the effective Schrödinger equation. Thus, quite generally, the tunneling interpretation will force us to conclude that the electromagnetic field described by the above potential produces particles. In particular, the tunneling probability will prove to be be nonzero even when $\phi=0$, the case which corresponds to a pure time-independent magnetic field. But for such a case, we have shown in Sec. II that the effective Lagrangian is real and hence there can be no particle production. Thus, we again reach a contradiction between the results obtained from the tunneling interpretation and those obtained from the effective Lagrangian.

On the other hand, consider the following situation. If we choose $A(x)$ to be zero and $\phi(x)$ to be nonzero in the above vector potential, then such a vector potential will give rise to a time-independent electric field. Such an electric field is always expected to produce particles. But in the space-dependent gauge we have chosen here β is trivially zero and if we consider only a nonzero β to imply particle production, then we will be forced to conclude that time-independent electric fields will not produce particles. It is to salvage such a situation that the tunneling interpretation has been repeatedly invoked in the literature. But then our analyses in the last two sections show that tunneling probability can be nonzero even if effective Lagrangian has no imaginary part.

There appears to be three possible ways of reacting to this contradiction. We shall examine each of them below.

(i) We may begin by noticing that in quantum field theory, there is always a tacit assumption that not only the fields but also the potentials should vanish at spatial infinities. If we take this requirement seriously, we may disregard the results for constant electromagnetic fields (the only case for which explicit results are known by more than one method) as unphysical. Then, we only need to provide a *gauge-invariant criterion* for particle production in electromagnetic fields described by potentials which vanish at infinity.

This turns out to be a difficult task, even conceptually. To begin with, we do not know how to generalize Schwinger's analysis and compute the effective Lagrangian for a spatially varying electromagnetic field. The only procedure available for us to study the evolution of the quantum field in such backgrounds is based on the method of normal mode analysis where we go on to obtain the tunneling probability $|T|^2$. But then the potential term in the effective Schrödinger equation is not gauge invariant, as can be easily seen from its form in Eq. (51). So the tunneling interpretation, even if it is adhered to, has the problem that it may not yield results that are gauge invariant. In fact, the situation is more serious; the entire tunneling approach can be used only *after* a particular gauge has been chosen. In some sense, the battle has already been lost.

Operationally also, it is doubtful whether the tunneling approach will yield results that are always consistent with the effective Lagrangian. As the analysis in this paper shows, there is at least one case, that of a spatially confined magnetic field, for which one can obtain a formal expression for effective Lagrangian and then compare it with the results obtained from the normal mode analysis. These results are clearly in contradiction with each other.

(ii) One may take the point of view that particle production in an electromagnetic field is a gauge-dependent phenomenon. It appears to be a remedy worse than disease and is possibly not acceptable. In addition to philosophical objections, one can also rule out this possibility by the following argument. We note that in the laboratory we can produce electromagnetic fields by choosing charges and current distributions but we have no operational way of implementing a gauge. So, given a particular electromagnetic field, in some region of the laboratory, we will either see particles being produced or not. It is difficult to see where the gauge can enter this result.

This point has some interesting similarities (and differences) with the question of particle definition in a gravitational field. If we assume that the choice of gauge in electromagnetic backgrounds is similar to the choice of a coordinate system in gravity, then one would like to ask whether the concept of particle is dependent on the coordinate choice. People seem to have no difficulty in accepting a coordinate dependence of particles (and particle production) in the case of gravity though the same people might not like the particle concept to be gauge dependent in the case of electromagnetism. To some extent, this arises from the idea that a coordinate choice is implementable by choosing a special class of observers, say, while a gauge choice in electromagnetism is not implementable in practice.

(iii) Finally, one may take the point of view that tunneling interpretation is completely invalid and one should rely entirely on the effective Lagrangian for interpreting the particle production. In this approach one would calculate the effective Lagrangian for a given electromagnetic field (possibly by numerical techniques, say) and will claim that particle production takes place only if the effective Lagrangian has an imaginary part. Further, one would confine oneself to those potentials which vanish at infinity, thereby ensuring proper asymptotic behavior.

This procedure is clearly gauge invariant in the sense that the effective Lagrangian is (at least formally) gauge invariant. Of course, one needs to provide a procedure for calculating the effective Lagrangian without having to choose a particular gauge. Given such a procedure, we have an unambiguous, gauge-invariant criterion for particle production for all potentials which vanish asymptotically. In fact, the effective Lagrangian for a spatially varying electromagnetic background can be formally expressed in terms of gauge-invariant quantities that involve the derivatives of the potentials and the fields.

This point could also have an interesting implication for gravitational backgrounds. The analogue of a constant electromagnetic background in gravity corresponds to spacetimes whose $R_{\mu\nu\rho\sigma}$'s are constants. The effective action in gravity can then possibly be expressed in terms of coordinate-invariant quantities constructed from $R_{\mu\nu\rho\sigma}$'s, just as it was possible to express the effective Lagrangian for a constant electromagnetic background in terms of gauge-invariant quantities involving $F_{\mu\nu}$'s.

We would like to stress here the following points. Equations (36) and (51) resemble a Schrödinger equation only in a formal sense. The actual time-dependent differential equation that we ought to deal with is the functional Schrödinger equation defined on the configuration space of all fields (see Refs. [22,29]). It is possible that such an approach would lead to an unambiguous way of dealing with particle creation. The results obtained from an analysis of the functional Schrödinger equation might not, in general, agree with the tunneling probability calculated for equations such as (36) or (51). It would be interesting to know the conditions under which the particle creation rate obtained from an analysis of the functional Schrödinger equation coincides with the tunneling probability evaluated, say, for Eq. (51). However, given the mathematical difficulties associated with solving functional differential equations, it is difficult to arrive at clear conclusions regarding the results for *arbitrary* electromagnetic backgrounds.

Comparing the three choices listed above, it seems that the third one is the most reasonable. Therefore, we conclude that the results obtained from the effective Lagrangian can be relied upon whereas the tunneling approach has to be treated with caution. It is likely, however, that the tunneling interpretation will prove to be consistent with the effective Lagrangian approach if we demand that an auxiliary gauge-invariant criterion has to be satisfied by the electromagnetic background before we can attribute a nonzero tunneling probability to particle production. But it is not obvious as to how such a condition can be obtained from the normal mode analysis. The wider implications of this result are under investigation.

ACKNOWLEDGMENTS

L.S.K. was supported by the Council of Scientific and Industrial Research, India. The authors would wish to thank Sukanya Sinha, Abhijit Kshirsagar, and Anil Gangal for discussions.

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