

On the Relation of the Infinitesimal Particle Propagator to the Nature of Mass.

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(ricevuto il 22 Giugno 1971)

Summary. — The finite particle propagator can be constructed by a path integral method provided the infinitesimal propagator is known. Hitherto, however, it has not been possible to specify the relativistic infinitesimal propagator except in an *ad hoc* way. From consideration of the nature of mass, in a Machian cosmological sense, it is shown in the present paper that the infinitesimal propagator can be derived in relativistic quantum mechanics by a method similar to that used in the nonrelativistic path integral.

1. — Introduction.

Most physicists prefer the wave equation picture of quantum mechanics to the use of path integrals. Yet the widespread appeal to Feynman diagrams is itself an admission of the importance of the concepts underlying the path integral method. Undoubtedly, the difficulty of extending the nonrelativistic path integral description to Dirac particles has dissuaded many physicists from exploring this approach. The present paper seeks to clarify the awkward features encountered in the relativistic generalization. With the exception of one place where we have been obliged to use the mathematician's trick of extending the definition of a function (instead of giving a physical argument) we think we have been able to understand why the difficulties have arisen and how they can be removed.

(*) On a visit to the Department of Physics and Astronomy, University of Maryland, College Park, Md., under contract No. NGL 21-002-033.

The relevant issues arise already in the discussion of free particles, so for our purpose it is sufficient to discuss free particles. It was shown in two recent papers ^(1,2), hereafter referred to as I, II, that inclusion of the electromagnetic interaction does not raise further difficulties. Neither does the weak interaction, as we shall show in the following paper.

In the nonrelativistic case the action for a free particle is given by

$$(1) \quad S = \int_{t_1}^{t_2} \frac{1}{2} m \dot{\underline{a}}^2 dt, \quad t_2 > t_1,$$

where $\underline{a}(t)$ is a typical path Γ starting at point 1: (\underline{a}_1, t_1) and ending at point 2: (\underline{a}_2, t_2) , and m is the mass of the particle. In the path integral approach the amplitude for the particle to go from 1 to 2 along Γ is given by

$$(2) \quad P(\Gamma) = (\text{const}) \exp [iS],$$

and the quantum-mechanical propagator from 1 to 2 is given by summing $P(\Gamma)$ for all Γ :

$$(3) \quad K(2; 1) = \sum P(\Gamma) = \int \exp [iS] D^3 \underline{a}.$$

Here and throughout the paper we take $\hbar = 1$, $c = 1$.

The notation used in the path integral (3) implies that the summation over all Γ is carried out in the following way. Divide Γ into a large number of small segments $[\underline{a}^{(i)}, t^{(i)}]$, $i = 1, \dots, N-1$ denoting the points of division, and $[\underline{a}^{(0)}, t^{(0)}]$, $[\underline{a}^{(N)}, t^{(N)}]$ corresponding to 1, 2, respectively. The exponential dependence of $P(\Gamma)$ on S evidently permits us to write

$$(4) \quad P(\Gamma) = \prod_{i=1}^N P[\underline{a}^{(i)}, t^{(i)}; \underline{a}^{(i-1)}, t^{(i-1)}],$$

where $P[\underline{a}^{(i)}, t^{(i)}; \underline{a}^{(i-1)}, t^{(i-1)}]$ is the infinitesimal amplitude to go from point $i-1$ to point i . In the nonrelativistic case all paths are timelike and it is possible to choose $t^{(i)}$, $i = 1, \dots, N-1$, the same for every path. In particular, it can be arranged that

$$(5) \quad t^{(i)} - t^{(i-1)} = \varepsilon, \quad i = 1, \dots, N.$$

The path integral is now evaluated by first multiplying each $P[\underline{a}^{(i)}, t^{(i)}; \underline{a}^{(i-1)}, t^{(i-1)}]$ by $(m/2\pi i \varepsilon)^{\frac{3}{2}}$ and by integrating over all intermediate space points $\underline{a}^{(i)}$, $i = 1, \dots, N-1$. The factor $(m/2\pi i \varepsilon)^{\frac{3}{2}}$ is interpreted as the measure of the paths going from $\underline{a}^{(i-1)}, t^{(i-1)}$ to the element of 3-surface $d^3 \underline{a}^{(i)}$ at $\underline{a}^{(i)}, t^{(i)}$,

(1) F. HOYLE and J. V. NARLIKAR: *Ann. of Phys.*, **54**, 207 (1969).

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$i = 1, \dots, N - 1$. The result is

$$(6) \quad K(2; 1) = \left[\frac{m}{2\pi i(t_2 - t_1)} \right]^{\frac{1}{2}} \exp \left[\frac{im(a_2 - a_1)^2}{2(t_2 - t_1)} \right], \quad t_2 > t_1,$$

in which we have restored the notation 1, 2 in place of the points $i = 0, i = N$. In the nonrelativistic treatment the propagator is taken to be zero if $t_2 < t_1$:

$$(7) \quad K(2; 1) = 0, \quad t_2 < t_1,$$

and $K(2; 1)$ then satisfies the inhomogeneous Schrödinger equation

$$(8) \quad \left[\frac{\partial}{\partial t_2} + \frac{1}{2im} \nabla_2^2 \right] K(2; 1) = \delta_4(2, 1).$$

This procedure, due to Feynman, is very elegant. It gives a finer scale picture of the quantum-mechanical amplitude than does an immediate choice of (8). A particle can go from 1 to 2 by any Γ . The amplitude is the sum of all these possibilities with each Γ weighted by the phase factor $\exp [iS]$.

The first step in seeking a relativistic generalization of the above procedure is to restrict one's attention (for the moment) to forward-going paths, denoted by Γ^+ . The aim is to obtain the propagator K_0^+ defined by

$$(9) \quad K_0^+(2; 1) = \theta(t_2 - t_1) \sum_n u_n(2) \bar{u}_n(1),$$

where θ is the Heaviside function and the summation is over a complete set of suitably normalized stationary states u_n with energies E_n of the free-particle Dirac equation. This is the propagator appropriate to the Dirac hole theory.

Modifying (1) to the relativistic form

$$(10) \quad S = - \int_{\Gamma^+} m \, da$$

obviously is inadequate since (10) does not contain spin—here da is the element of proper time along the path Γ^+ . We evidently need to give $P(\Gamma^+)$ a 4×4 matrix structure. This can be done without undue difficulty however. We have

$$(11) \quad da^2 = \eta_{ik} da^i da^k,$$

where da^i is the co-ordinate displacement for the element da and η_{ik} is the Minkowski tensor. Now the Dirac matrices satisfy

$$(12) \quad \gamma_i \gamma_k + \gamma_k \gamma_i = 2I \eta_{ik},$$

where I is the unit 4×4 matrix, so (11) can be rewritten in the form

$$(13) \quad I \cdot da^2 = (\gamma_k da^k)^2.$$

Instead of using the square root of (11) in the action (10), we replace $-da$ by $\gamma_k da^k$:

$$(14) \quad S = + \int_{I^+} m \gamma_k da^k = + m((\gamma \cdot a)_2 - (\gamma \cdot a)_1),$$

where $(\gamma \cdot a)_1 = \gamma_k a_1^k$, $(\gamma \cdot a)_2 = \gamma_k a_2^k$; a_1^k , a_2^k being the co-ordinates of points 1 and 2.

Since there is no difficulty in interpreting $\exp [iS]$ as a von Neumann series, we might seek to repeat the nonrelativistic procedure, hoping eventually to arrive at the K_0^+ -propagator. A point of divergence from the nonrelativistic procedure arises already at (4) however. No such relation is generally valid when the amplitude is a matrix. That is to say, the definition (2) for the amplitude $P(I^+)$, S being given by (14), does not satisfy (4) and we cannot therefore proceed with the evaluation of (3) along the same lines as before. Quite apart from this difficulty, $P(I^+)$ is the same for every path from 1 to 2 and an amplitude that is thus path independent cannot lead to a satisfactory quantum theory.

These difficulties are resolved if we adopt (4) as the definition of path amplitude and restrict (2) to infinitesimal segments. Since the exponential law $\exp [A] \cdot \exp [B] = \exp [A + B]$ does not hold for general matrices A and B , $P(I^+)$ is now path dependent, as in the nonrelativistic case.

We proceed by taking (4) as the definition of $P(I^+)$, subject to a modification described in the next paragraph. $P(I^+)$ is then a chain of matrices representing the amplitudes for the individual segments of I^+ . Going from 1 to 2 we set the chain out from right to left.

We can now attempt an evaluation of the path integral along the same lines as before. We multiply each $P[\underline{a}^{(i)}, t^i; \underline{a}^{(i-1)}, t^{i-1}]$ by a suitable measure for the weight of paths from $i-1$ to the element of 3-surface $d^3\underline{a}^{(i)}$ at i and integrate over $t = t^{(i)}$, $i = 1, \dots, N-1$. To make the integrals invariant, however, we must also multiply by $\gamma \cdot n = \gamma_4$ at each $d^3\underline{a}^{(i)}$. This has the effect of linking one segment amplitude to another through the matrix γ_4 .

It turns out that proceeding in this way does lead to the finite propagator K_0^+ provided the product of a segment amplitude with the measure is chosen in the following way. Let the displacement between the end points of a segment be q^i , and write $\gamma \cdot q = \gamma_i q^i$, $q^2 = (\gamma \cdot q)^2 = q_i q^i$. The required product is

$$(15) \quad \frac{1}{\pi} \left[(\gamma \cdot q) \delta'(q^2) - \frac{1}{2} im \delta(q^2) \right].$$

We shall refer to (15) as the infinitesimal propagator.

It is easy to prove the statement of the previous paragraph. Let S^i be the displacement from 1 to 2. Writing $S^2 = S_i S^i$, we can show, by summing the series in (9), that

$$(16) \quad K_0^+(2; 1) = \frac{1}{2\pi} \theta(t_2 - t_1) (\nabla_2 - im) \left[\delta(S^2) - \frac{m}{2S} \theta(S^2) J_1(mS) \right], \quad \nabla = \gamma \cdot \nabla,$$

which is infinitesimally the same as (15). Hence if the path integral yields K_0^+ going from $t = t^{(0)}$ to $t = t^{(i)}$ it must yield K_0^+ going from $t = t^{(0)}$ to $t = t^{(i+1)}$. This induction follows immediately because K_0^+ satisfies

$$(17) \quad K_0^+(3; 1) = \int K_0^+(3; 2) (\gamma \cdot n)_2 K_0^+(2; 1) d^3x_2, \quad t_3 > t_2 > t_1.$$

In fact, (17) can easily be verified by using (9) for the propagators on the right-hand side.

This was the path integral method used in II to obtain K_0^+ . It is an improvement on the nonrelativistic case in that the awkward multiplication by $(m/2\pi i \epsilon)^{\frac{3}{2}}$ as a measure factor has been eliminated. On the other hand, the expectation that $\exp[iS]$, with S given by (14), would represent the amplitude has apparently not been fulfilled, even for the amplitude of an individual path segment. At first sight this attractive feature of the nonrelativistic case has been lost. Yet (15) can be written in the interesting form

$$(18) \quad \frac{1}{\pi} \exp \left[\frac{1}{2} im(\gamma \cdot q) \right] \cdot (\gamma \cdot q) \delta'(q^2),$$

because it is sufficient, infinitesimally speaking, to take only the first two terms in the power series of the exponential:

$$(19) \quad \exp \left[\frac{1}{2} im(\gamma \cdot q) \right] \approx 1 + \frac{1}{2} im(\gamma \cdot q),$$

and because $q^2 \delta(q^2) = -\delta'(q^2)$. The highest-order singularity in the infinitesimal propagator comes from the unity term in (19), whereas the highest-order singularity involving the mass comes from the second term. These are sufficient to generate the finite propagator (16). Details of the way the finite propagator is built from the infinitesimal propagator are given later in Sect. 3. It is not necessary to consider these details here, since the above induction proof is already sufficient.

The connection of the infinitesimal amplitude with $\exp[im(\gamma \cdot q)]$ has not therefore been wholly lost, although interestingly enough a factor $\frac{1}{2}$ has appeared in the exponent. This odd circumstance turns out to have a significant interpretation, as will be seen in the next Section.

Evidently the key problem in understanding the path integral for Dirac particles lies in the factor $(q \cdot \gamma) \delta'(q^2)$ in (18). It is this factor in the infinitesimal propagator which has hitherto impeded progress in developing the path integral approach. We shall attempt to resolve this problem in the following Section.

Suppose we go backward in time from t_2 to t_1 ($t_2 > t_1$). The propagator $K_0^-(1, 2)$ was defined in II by

$$(20) \quad K_0^-(1; 2) = -\theta(t_2 - t_1) \sum u_n(1) \bar{u}_n(2).$$

The series can be summed and gives

$$(21) \quad K_0^-(1; 2) = -\frac{1}{2\pi} \theta(t_2 - t_1) (\not{\nabla}_1 - im) \left[\delta(S^2) - \frac{m}{2S} \theta(S^2) J_1(mS) \right] = \\ = \frac{1}{2\pi} \theta(t_2 - t_1) (\not{\nabla}_2 + im) \left[\delta(S^2) - \frac{m}{2S} \theta(S^2) J_1(mS) \right].$$

Apart from a sign this is just what would be obtained by applying the path integral method backward from 2 to 1. Three changes arise in this calculation—each $\gamma \cdot q$ in the infinitesimal propagators is changed to $-(\gamma \cdot q)$, the linking γ_4 matrices are changed to $-\gamma_4$ (because the sign of the unit normal is changed at each spatial integration), and the matrices of the infinitesimal propagators are arranged in opposite order (left to right instead of right to left). Absorbing a minus sign from $-\gamma_4$ into an infinitesimal propagator restores the original $\gamma \cdot q$ but changes the sign of the mass term. This is the same as taking the complex conjugate of the original infinitesimal propagator. Since moreover the matrices are linked in opposite order, the path integral calculation evidently gives just the complex conjugate of the previous result, except that there is an additional sign change because the number of γ_4 matrices in our chains is one less than the number of infinitesimal propagators. Since $K_0^-(1; 2)$, as defined above, is the complex conjugate of $K_0^+(2; 1)$, the path integral calculation is seen to give the result stated above. Paths treated in this way were denoted in II as I^- paths.

The sign difference between the path integral calculation and K_0^- is merely a matter of definition. Using K_0^+ to propagate a wave function ψ forward in time we have

$$(22) \quad \psi(2) = \int K_0^+(2; 1) (\gamma \cdot n)_1 \psi(1) d^3 \underline{x}_1, \quad t_2 > t_1.$$

When we propagate backward, it is desirable to define K_0^- such that

$$(23) \quad \psi(1) = \int K_0^-(1; 2) (\gamma \cdot n)_2 \psi(2) d^3 \underline{x}_2, \quad t_2 > t_1.$$

This requires a minus sign to be included in (20), because $(\gamma \cdot n)_1 = \gamma_4$ in (22), whereas $(\gamma \cdot n)_2 = -\gamma_4$ in (23). The extra $\gamma \cdot n$ that arises when a wave function is propagated makes the number of γ_4 matrices equal to the number of segment amplitudes along the paths.

$K_0^+(2; 1)$ propagates to points 2 in and on the forward light-cone of point 1, and $K_0^-(1; 2)$ propagates to points 1 in and on the backward light-cone of point 2. Both are timelike in character.

According to (22) and (23), positive- and negative-energy states are propagated equally forwards and backwards. As pointed out above, this is the situation in the Dirac hole theory. In Feynman's picture, on the other hand, positive-energy states are propagated forward and negative-energy states backward. We were able to introduce the Feynman picture in II by distinguishing between I^+ and I^- paths, and by introducing the hypothesis that the positive-energy part of the wave function, ψ_+ say, gave the weighting of I^+ paths, while the negative-energy part of the wave function, ψ_- say, gave the weighting of the I^- paths. From this point of view, forward propagation satisfies

$$(24) \quad \psi_+(2) = \int K_0^+(2; 1)(\gamma \cdot n)_1 \psi_+(1) d^3x_1,$$

and backward propagation satisfies

$$(25) \quad \psi_-(1) = \int K_0^-(1; 2)(\gamma \cdot n)_2 \psi_-(2) d^3x_2.$$

This hypothesis was sufficient for the main purpose of II, which was to discuss the electromagnetic interaction. Such a separation of the wave function is not very satisfactory however. A different procedure was therefore suggested at the end of II. We take the free-particle propagator from 1 to 2 to be

$$(26) \quad \frac{1}{2} [K_0^+(2; 1) + K_0^-(2; 1)],$$

irrespective of the time order of the two points. The Heaviside functions appearing in the definitions (9), (20) require the propagator to be $\frac{1}{2}K_0^+(2; 1)$ if $t_2 > t_1$, but to be $\frac{1}{2}K_0^-(2; 1)$ if $t_2 < t_1$. The propagator (26) is the amplitude for the particle to go directly from 1 to 2. A particle can also arrive at 2 indirectly. First, let there be propagation from 1 and 2 to other points 3, in particular to points distant in the universe from 1 and 2. As a consequence, physical processes occur at these other points, causing amplitudes to be propagated back to 1 and 2. The sum of all such amplitudes evaluated in a suitable way constitutes the response of the universe. It was pointed out in II that if the response takes the form

$$(27) \quad \frac{1}{2} \left[\sum_{x_n > 0} u_n(2) \bar{u}_n(1) - \sum_{x_n < 0} u_n(2) \bar{u}_n(1) \right],$$

in analogy to the situation in electrodynamics, the sum of (26) and (27) is the Feynman propagator $K_+(2; 1)$. This question will be further discussed in the fourth Section of the present paper.

2. - The mass interaction.

The idea which we shall develop in this Section can be understood by returning to the apparently unpromising attempt to generalize the nonrelativistic path integral in terms of

$$(28) \quad K(2; 1) = \sum_{\Gamma} P(\Gamma) = \int P(\Gamma) D^3\Gamma,$$

$$(29) \quad P(\Gamma) = \exp [iS],$$

$$(30) \quad S = -\int N da,$$

where we have written N for the mass. This formulation was dismissed on the grounds that it does not contain spin. However, before dealing with spin it is instructive to proceed with this scheme and to consider the situation that would arise if N were to tend to infinity. Then the only paths that would contribute effectively to the integral would be those made up of null segments. Any path Γ from 1 to 2 containing an element da that was nonzero would have large action, and an infinitesimal variation of Γ would produce a large variation in the phase angle of $P(\Gamma)$, causing cancellation in the path integral. Now our relativistic formulation of the path integral was indeed in terms of paths built from null segments. The null property arose from the factor $(\gamma \cdot q) \delta'(q^2)$ in (18), which is just the factor we are seeking to understand. These considerations therefore suggest that there are two contributions to the mass of a particle, a finite part m and a very large part N . We have to understand how this can be, and how the particle is able to distinguish m and N separately.

In previous work^(3,4) we have developed a classical Machian approach to the nature of mass which we shall now review briefly. The aim of the work was to obtain a gravitational theory based on the action

$$(31) \quad \sum_{a < b} \iint \tilde{G}(A, B) da db.$$

⁽³⁾ F. HOYLE and J. V. NARLIKAR: *Proc. Roy. Soc.*, A **282**, 191 (1964).

⁽⁴⁾ F. HOYLE and J. V. NARLIKAR: *Proc. Roy. Soc.*, A **294**, 138 (1966).

Here $\tilde{G}(A, B)$ is a symmetrical biscalar describing the inertial interaction between points A and B on the world-lines of particles a and b . $\tilde{G}(X, A)$ can be regarded as a Green's function generated at point A and evaluated at the general field point X . It is taken to satisfy the conformally invariant scalar wave equation

$$(32) \quad \square_x \tilde{G}(X, A) + \frac{1}{6}R(X)\tilde{G}(X, A) = +(-g)^{-\frac{1}{2}}\delta_4(X, A),$$

R being the scalar Riemannian curvature. The gravitational equations are obtained by requiring the action (31) to be stationary with respect to small variations of the metric tensor g_{ik} . They can be expressed in the form

$$(33) \quad F(R_{ik} - \frac{1}{2}g_{ik}R) = -3(T_{ik} + \Phi_{ik}) + (g_{ik}\square F - F_{;ik}).$$

The Ricci tensor appears here as it does in Einstein's equations. The other quantities in (33) are defined in terms of mass fields $m^{(a)}(x), \dots$ at point X generated by the particles a, b, \dots :

$$(34) \quad m^{(a)}(x) = \int \tilde{G}(X, A) da, \dots$$

Thus

$$(35) \quad F = \frac{1}{2} \sum_{a \neq b} m^{(a)} m^{(b)},$$

$$(36) \quad \Phi_{ik} = -\frac{1}{2} \sum_{a \neq b} [m_i^{(a)} m_k^{(b)} + m_k^{(a)} m_i^{(b)} - g_{ik} m_i^{(a)} m^{(b)l}],$$

$$(37) \quad T^{ik} = \sum_a \int \delta_4(X, A) (-g)^{-\frac{1}{2}} m_a(A) \frac{da^i}{da} \frac{da^k}{da} da,$$

$$(38) \quad m_a(A) = \sum_{b \neq a} m^{(b)}(A).$$

When the gravitational field is weak

$$(39) \quad \tilde{G}(X, A) \approx \frac{1}{4\pi} \delta(S_{XA}^2)$$

and all mass fields are positive. In a many-particle universe we then have

$$(40) \quad \Phi_{ik} \approx - \left[\sum_a m^{(a)} \right]_i \left[\sum_b m^{(b)} \right]_k + \frac{1}{2} g_{ik} \left[\sum_a m^{(a)} \right]_i \left[\sum_b m^{(b)} \right]^l,$$

$$(41) \quad F \approx \frac{1}{2} \left[\sum_a m^{(a)} \right]^2 > 0.$$

Now because the theory is conformally invariant we can choose a particular

conformal frame in which F is constant. This is because the conformal transformation

$$(42) \quad g^*_{ik} = \Omega^2 g_{ik},$$

with Ω a well-defined nonzero function of X , changes the mass fields

$$(43) \quad m^{*(a)} = \Omega^{-1} m^{(a)},$$

so that F is changed in accordance with

$$(44) \quad F^* = \Omega^{-2} F.$$

If F is not constant to begin with, we can evidently make F^* constant by taking $\Omega \propto F^{\frac{1}{2}}$.

We see from (41) that

$$(45) \quad m = \sum_a m^{(a)}$$

must be constant in the particular conformal frame in which F is constant. All terms on the right-hand side of (33) except the T_{ik} term now disappear. Defining the gravitational constant G by

$$(46) \quad 8\pi G = \frac{3}{F} > 0,$$

we obtain for the field equations

$$(47) \quad R_{ik} - \frac{1}{2} g_{ik} R = -8\pi G T_{ik},$$

the same as Einstein's equations.

We may ask under what circumstances do eqs. (33) *not* reduce to (47). This occurs when the approximations (40) and (41) cannot be made. This could happen if there were only a small number of particles (but in practical cases the number of particles is always very large), or if the mass fields were of variable sign (in which case the sum of all cross products $\sum_{a \neq b} m^{(a)} m^{(b)}$ may not overwhelm the sum of squares $\sum_a [m^{(a)}]^2$). The masses are certainly all positive when the fields are weak, but this is a sufficient condition for (40) and (41) to hold, not a necessary condition. There can be strong-field cases in which (40) and (41) still hold good. On the other hand, we must contemplate that (40) and (41) may not hold good in some strong-field problems. Equations (33) do not then reduce to the Einstein equations.

Two points may be noted before we proceed. In weak fields G must be > 0 . We have therefore *deduced* that weak fields must be attractive. Second, the transformation (43) makes the Dirac equation conformally invariant.

From the present point of view, mass is cosmological in origin. The observed large-scale homogeneity and isotropy of the universe requires a metric of the well-known Robertson-Walker form:

$$(48) \quad ds^2 = dt^2 - Q^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \right],$$

in which k may be zero or ± 1 . It will be sufficient for our present purpose to consider the case $k = 0$. In this co-ordinate system matter has constant values of r, θ, φ . We define

$$(49) \quad \tau = \int_{t_0}^t \frac{dt}{Q}$$

for some constant t_0 . Then

$$(50) \quad ds^2 = Q^2 [d\tau^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)],$$

which is conformal to flat space. Since our physical theory is conformally invariant we can make the transformation $\Omega = Q^{-1}$ and proceed to work in flat space.

A point of some subtlety now arises. In the Friedmann cosmologies Q has a zero at a time $t = 0$. In the Einstein-de Sitter model, for example, $Q \propto t^{\frac{2}{3}}$. The conformal transformation $\Omega = Q^{-1}$ therefore fails at $t = 0$, since Q must not be zero. This means that we cannot pass from (48) through

$$(51) \quad g^*_{ik} = Q^{-2} g_{ik}$$

to the whole of

$$(52) \quad ds^{*2} = d\tau^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

but only to the half-space $0 < \tau$. But how if we elect to invert the situation by starting with (52), particularly how if we elect to start with the whole of the space (52), $-\infty < \tau < \infty$? An attempt to return to (48) through the inverse transformation $\Omega = Q$ now encounters the situation that Q is zero at $\tau = 0$, and our point of view is that the Friedmann singularity has arisen not through any difficulty in the physics but through a singularity in Ω . The Friedmann singularity has become a mathematical construct only.

In Minkowski space (32) is accurately satisfied:

$$(53) \quad \tilde{G}(X, A) = \frac{1}{4\pi} \delta(S^*_{xA}).$$

Since the r, θ, φ co-ordinates have not been changed, and since matter was homogeneously distributed with respect to the Robertson-Walker line element, matter is also homogeneously distributed with respect to (52). Because of this, and because of the simple form (53) for $\tilde{G}(X, A)$, it is easy to evaluate the mass integrals (34). Indeed, the problem is exactly the same mathematically as the evaluation of the electrostatic potential in a homogeneous universe in flat space, for the case in which all particles have the same « charge ». To prevent an infinite result we limit the calculation to the slab $-T \leq \tau \leq T$, where T is large and will later be permitted to tend to infinity.

The biscalar $\tilde{G}(X, A)$ is symmetric, $\tilde{G}(X, A) = \tilde{G}(A, X)$. We have therefore to deal with advanced potentials as well as retarded potentials. At a field point X , we write $N + m$ for the total retarded potential and $N - m$ for the total advanced potential. The difference $2m$ arises because in general X is not at the central plane $\tau = 0$. The mass m is finite and depends on the time co-ordinate of the point X . It is this time dependence which leads to the red-shift in the light of distant galaxies⁽⁵⁾. The mass N tends to infinity with T . A somewhat similar result was obtained by HAWKING⁽⁶⁾, working in the conventional Robertson-Walker models. In this case, because the extension beyond $\tau = 0$ was not made, the retarded mass was finite and the advanced mass infinite.

In the classical electromagnetic theory the universe contributes (cf. II, eq. (71))

$$(54) \quad \frac{1}{2} \sum_a [A_i^{(a)}(X)_{\text{ret}} - A_i^{(a)}(X)_{\text{adv}}],$$

where $A_i^{(a)}$ is the 4-potential due to particle a . In the quantum theory the response of the universe appears as a contribution to the phase angle of an exponential factor in the amplitude. We shall adopt nearly the same procedure for the mass, by taking the amplitude for an elementary path segment directed in the forward time sense (τ increasing if $\tau > 0$) to be

$$(55) \quad \exp [\frac{1}{2} i(N + m)(\gamma \cdot q)] - \exp [-\frac{1}{2} i(N - m)(\gamma \cdot q)],$$

which can be written in the form

$$(56) \quad \exp [\frac{1}{2} im(\gamma \cdot q)] \{ \exp [\frac{1}{2} iN(\gamma \cdot q)] - \exp [-\frac{1}{2} iN(\gamma \cdot q)] \}.$$

⁽⁵⁾ F. HOYLE and J. V. NARLIKAR: *Gamow Memorial Volume*.

⁽⁶⁾ S. W. HAWKING: *Proc. Roy. Soc., A* **285**, 313 (1965).

By placing the retarded and advanced masses in separate exponentials—a departure from the electromagnetic case—we give a time sense to the propagator. The first exponential of (55) is the amplitude to go forwards (τ increasing if $\tau > 0$), the second exponential is the amplitude to go backwards. The difference of the two exponentials gives the net flow in the positive sense. It is seen that the retarded mass acts in the forward-going part of the amplitude—*i.e.* in the sense away from the plane $\tau = 0$ —and the advanced mass in the backward-going part of the amplitude. What would happen if we reversed the roles of the masses? Then we should have

$$(57) \quad \exp \left[\frac{1}{2} i(N - m)(\gamma \cdot q) \right] - \exp \left[-\frac{1}{2} i(N + m)(\gamma \cdot q) \right],$$

again for the net flow away from $\tau = 0$. The net flow backwards, towards $\tau = 0$, is just minus the flow forwards:

$$(58) \quad \exp \left[-\frac{1}{2} i(N + m)(\gamma \cdot q) \right] - \exp \left[\frac{1}{2} i(N - m)(\gamma \cdot q) \right] = \\ = \exp \left[-\frac{1}{2} im(\gamma \cdot q) \right] \{ \exp \left[-\frac{1}{2} iN(\gamma \cdot q) \right] - \exp \left[\frac{1}{2} iN(\gamma \cdot q) \right] \},$$

which is like (56) except that the sign of $\gamma \cdot q$ is reversed. This is just the same as the switch from K_0^+ to K_0^- . Hence we see that the sense in which a particle « goes » is determined by the senses in which the advanced and retarded masses act. If the advanced mass acts away from $\tau = 0$, giving an amplitude $\exp \left[\frac{1}{2} i(N - m)(\gamma \cdot q) \right]$, and the retarded mass acts towards $\tau = 0$, giving an amplitude $\exp \left[-\frac{1}{2} i(N + m)(\gamma \cdot q) \right]$, we regard the particle as going backwards. This is the case for a positron. But if the advanced mass acts towards $\tau = 0$, giving an amplitude $\exp \left[-\frac{1}{2} i(N - m)(\gamma \cdot q) \right]$, and the retarded mass away from $\tau = 0$, giving an amplitude $\exp \left[\frac{1}{2} i(N + m)(\gamma \cdot q) \right]$, we regard the particle as going forwards. Pair creation and annihilation imply a switch in the senses in which the masses act.

Returning to (56) we have now obtained $\exp \left[\frac{1}{2} im(\gamma \cdot q) \right]$ as a factor, the same as in (18). The $\frac{1}{2}$ has appeared for the same reason it does in the electromagnetic theory. The $\delta(S^{*2}_{x_A})$ in the propagator (53) can be written in an obvious notation in the form

$$(59) \quad \delta(S^{*2}_{x_A}) = \delta[(\tau_x - \tau_A)^2 - (\underline{x} - \underline{a})^2] = \\ = \frac{1}{2|\underline{x} - \underline{a}|} \left[\delta(\tau_x - \tau_A - |\underline{x} - \underline{a}|) + \delta(\tau_x - \tau_A + |\underline{x} - \underline{a}|) \right].$$

The factor $\frac{1}{2}$ multiplies the usually calculated forms of the retarded and advanced fields. This detail encourages us to think the present considerations are along the right lines.

It remains to analyse the second factor in (56). We do this for the case

in which the displacement q^i is timelike. Expanding the exponentials in power series gives

$$(60) \quad \exp \left[\frac{1}{2} iN(\gamma \cdot q) \right] - \exp \left[-\frac{1}{2} iN(\gamma \cdot q) \right] = \frac{2i(\gamma \cdot q)}{q} \sin \left(\frac{1}{2} Nq \right).$$

Next we note that over $0 \leq q < \infty$ the function (60) is the same in the limit $N \rightarrow \infty$ as the function

$$(61) \quad \frac{4i(\gamma \cdot q)}{q} \sin \left(\frac{1}{2} Nq^2 \right) = P \text{ say.}$$

By this we mean that (60) and (61) give the same result in the limit $N \rightarrow \infty$ when integrated over $0 \leq q < \infty$ against any well-behaved test function. From here on we shall work with (61) rather than (60), because (61), equivalent to the left-hand side of (60) when q^i is timelike, can be extended without difficulty to the case where q^i is spacelike. This is the mathematical trick mentioned in the Introduction.

To obtain the infinitesimal propagator it is necessary to sum all the infinitesimal segment amplitudes that go to a surface element, which we denote by dS . The contribution $(\gamma \cdot n) dS WP$ appears in the path integral, W being the measure factor and $\gamma \cdot n$ the unit normal to dS . Since $\gamma \cdot n$ and P are dimensionless and dS is the cube of a length, it is clear that W must be the inverse cube of a length. The only invariant possibility is q^{-3} , $q \neq 0$. Hence we write

$$(62) \quad W = \frac{\text{constant}}{q^3}, \quad q \neq 0,$$

expecting that a principal part will be needed in order to exclude $q = 0$.

A sphere of radius q in a Euclidean + + + + space has surface area $2\pi^2 q^3$, so in this space a source of unit strength would yield the constant $1/2\pi^2$ in (62). To give equal weight to paths inside and outside the light-cone in Minkowski space we consider the effective surface to be doubled, to $4\pi^2 q^3$, so that $W = 1/4\pi^2 q^3$, $q \neq 0$. The infinitesimal propagator is given by the limit $N \rightarrow \infty$ of the principal part at $q^2 = 0$ of the product WP :

$$(63) \quad \text{P.P.} \left[\frac{1}{4\pi^2 q^3} \exp \left[\frac{1}{2} im(\gamma \cdot q) \right] \frac{4i(\gamma \cdot q)}{q} \sin \left(\frac{1}{2} Nq^2 \right) \right].$$

Apart from a trivial factor $-i$, the limiting value of (63) is the same as (18). By this we mean that (63) integrated over the range $-\infty < q^2 < \infty$ against any well-behaved function of q^2 gives the same result in the limit $N \rightarrow \infty$ as does (18), the factor $-i$ excepted. Since all paths from 1 to 2 have the same number of segments the effect of a $-i$ factor in the infinitesimal prop-

agator is to multiply the finite propagator by either $\pm i$ or ± 1 . This is simply a phase factor and has no physical relevance. We could have removed the $-i$ by including $+i$ in W .

It follows that we have been able to formulate the idea set out at the beginning of this Section in a quantitative way. In doing so we have gained insight into the concept of mass, particularly in relation to the cosmological structure, and we have also gained insight into the difference between particles and antiparticles. On the other hand, the step of replacing (60) by (61), while strictly valid for timelike displacements, loses the connection with the left-hand side of (60) for spacelike displacements. This appears to us to be physically unsatisfactory although our feeling in this respect is subjective since for $q^2 > 0$ (60) and (61) are the *same function* in the limit $N \rightarrow \infty$. Nevertheless we have looked for alternative ways of arriving at (18) from (56).

One possibility is to set the amplitude zero for spacelike displacements. Removing paths outside the light-cone would be expected to change the constant in W to $2\pi^2$. We then get

$$(64) \quad \begin{cases} q^2 > 0, & \exp\left[\frac{1}{2}im(\gamma \cdot q)\right] \frac{4i(\gamma \cdot q)}{q} \sin\left(\frac{1}{2}Nq^2\right) \frac{1}{2\pi^2q^3}, \\ q^2 < 0, & 0. \end{cases}$$

For the test functions $\sum_{n=0}^{\infty} a_n q^{2n}$ with $a_0 = 0$, (64) is the same as (18) in the limit $N \rightarrow \infty$, again apart from a trivial $-i$. But, for test functions with $a_0 \neq 0$, (64) and (18) are not the same. On the other hand, if W were an operator

$$(65) \quad W \equiv \frac{1}{2\pi^2q} \frac{d}{dq^2},$$

with the property that the derivative acted on $\exp[\frac{1}{2}im(\gamma \cdot q)] \cdot (\gamma \cdot q)$ as well as on the test function, then (64) in the limit $N \rightarrow \infty$ would indeed be the same as (18). We have not been able to interpret (65) in a satisfactory physical way, however. The difficulty lies in the fact that two limiting processes, $q^2 \rightarrow 0$ as well as $N \rightarrow \infty$, are involved. These processes were handled unambiguously with the aid of the principal part in (63) but they seem difficult to separate with respect to (64) and (65).

3. - The neutrino and mass scattering.

No coupling constant was included in the action (31) because so far we have considered particles of the same kind. When particles of different masses are considered, then coupling constants are needed to determine the relative values of the mass. We regard the neutrino as a particle with a relatively

small but nonzero coupling constant. We then have $m \simeq 0$ relative to other particles but N for the neutrino still tends to infinity. The infinitesimal propagator (18) becomes

$$(66) \quad \frac{1}{\pi} (\gamma \cdot q) \delta'(q^2).$$

Putting $m = 0$ in the finite propagator (16) also leads to (66), with a factor $\theta(t_2 - t_1)$. The finite propagator $K_0^-(2; 1)$ again gives (66) when $m = 0$, but with a factor $-\theta(t_1 - t_2)$. The sign difference between (16) and (21) for K_0^+ and K_0^- is a matter of convention. It was introduced so that $(\gamma \cdot n)_1$ and $(\gamma \cdot n)_2$ in (22) and (23) could be of opposite signs. If we choose to take $\gamma \cdot n$ always in the positive time sense then the sign difference disappears and (66) is the neutrino propagator forward or backward.

When the perturbation method is used to solve a problem in electrodynamics we have a picture of the particle being scattered by the electromagnetic field. The order in e^2 determines the number of scatterings in that particular order. Between scatterings the particle is free. In a similar way a particle with $m \neq 0$ can be regarded as being scattered by the mass field generated by the universe. A perturbation expansion can be made. The order in m determines the number of scatterings in that order. Between scatterings the particle is free, not in the usual sense of a free particle, but free of the mass interaction—the particle is a neutrino between scatterings. In principle there is no difference between the mass expansion and the charge expansion. In practice there is the important difference that the expansion in m can actually be summed, whereas the expansion in e^2 can only be summed in certain cases. The result of summing the mass expansion is the finite propagator.

All forms of the finite propagator, K_0^+ , K_0^- , $\frac{1}{2}(K_0^+ + K_0^-)$ and K_+ , satisfy the inhomogeneous Dirac equation. To avoid specifying which propagator we are considering, we write

$$(67) \quad (\nabla_2 + im)K(2; 1) = \delta_4(2, 1).$$

Equation (67) can be solved in successive approximations. We define $K^{(0)}(2; 1)$ such that

$$(68a) \quad \nabla_2 K^{(0)}(2; 1) = \delta_4(2, 1),$$

and let $K^{(n)}(2; 1)$, $n = 1, \dots$, satisfy

$$(68b) \quad \nabla_2 K^{(n)}(2; 1) = -imK^{(n-1)}(2; 1).$$

Then

$$(69) \quad K(2; 1) = \sum_{n=0}^{\infty} K^{(n)}(2; 1)$$

satisfies (67).

We define

$$(70) \quad I(2; 1) = \frac{\delta(S_{21}^2)}{4\pi} \quad \text{or} \quad \frac{\delta_{\pm}(S_{21}^2)}{4\pi},$$

where $\delta_{\pm}(S_{21}^2)$ are the positive- and negative-frequency parts in the Fourier expansion of $2\delta(S_{21}^2)$. Then

$$(71) \quad K^{(0)}(2; 1) = \nabla_2 I(2; 1)$$

satisfies (68a), and

$$(72) \quad K^{(1)}(2; 1) = -imI(2; 1)$$

satisfies (68b) for $n = 1$. For higher values of n we use the reduction

$$(73) \quad \begin{aligned} K^{(n)}(2; 1) &= -im \int K^{(0)}(2; 3) K^{(n-1)}(3; 1) d\tau_3 = \\ &= -im \int \nabla_2 I(2; 3) K^{(n-1)}(3; 1) d\tau_3 = (-im)^2 \int I(2; 3) K^{(n-2)}(3; 1) d\tau_3, \end{aligned}$$

to obtain the following solution of (68b) for general n :

$$(74) \quad \left\{ \begin{aligned} K^{(2n)}(P'; P) &= \\ &= -(-im)^{2n} \nabla_P \int \dots \int I(P'; n) I(n; n-1) \dots I(2; 1) I(1; P) d\tau_1 \dots d\tau_n, \\ K^{(2n+1)}(P'; P) &= \\ &= (-im)^{2n+1} \int \dots \int I(P'; n) I(n; n-1) \dots I(2; 1) I(1; P) d\tau_1 \dots d\tau_n. \end{aligned} \right.$$

To avoid confusion in the notation we have introduced the points P, P' in (74). The integrations $d\tau_1 \dots d\tau_n$ are over all space-time. A surface integral at infinity has been dropped in the last step of (73).

If $\delta(S_{21}^2)/4\pi$ is chosen for $I(2; 1)$ then the summation (69) gives

$$(75) \quad \frac{1}{2} [K_0^+(2; 1) + K_0^-(2; 1)].$$

If $\delta_+(S_{21}^2)/4\pi$ is chosen then (69) gives the Feynman propagator $K_+(2; 1)$, while $\delta_-(S_{21}^2)/4\pi$ gives $K_-(2; 1)$, a propagator like Feynman's but with reversed time sense.

The time-symmetric direct-particle theory requires us to choose $\delta(S^2)/4\pi$ for I , giving the time-symmetric propagator (75). The integrals (74) can be regarded as a summation over paths. The particle goes from P to the element $d\tau_1$ where it is scattered by the mass field. It then goes to $d\tau_2$, where it is

again scattered, then to $d\tau_3$, and so on, until after n scatterings the particle arrives at P' . Between scatterings the particle is massless like a neutrino —that is to say it goes in a null direction (*).

At the end of the Introduction we argued in favour of (75). To obtain the usual form of electrodynamics it was then necessary to introduce a response from the universe given by (27). We shall consider this question further in the next Section.

4. - The response of the universe.

In II we remarked on the similarity of (27) to the electrodynamic response of the universe. The latter is shown in Fig. 1. Segments at points A, \tilde{A} on

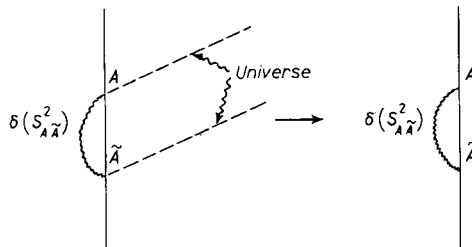


Fig. 1.

the path of a charged particle interact directly through the symmetric $\delta(S_{A\tilde{A}}^2)$ function. There are also interactions from A and from \tilde{A} to all other charged particles in the universe. Summation of these interactions contributes just the principal part of $\delta_+(S_{A\tilde{A}}^2)$, so the symmetric $\delta(S_{A\tilde{A}}^2)$, taken together with the response of the universe, gives $\delta_+(S_{A\tilde{A}}^2)$.

In the previous Section we saw that time-symmetric propagation of a particle from \tilde{A} to A is determined by the choice $\delta(S_{A\tilde{A}}^2)/4\pi$ for the function $I(A; \tilde{A})$. This choice leads to the time-symmetric $\frac{1}{2}[K_0^+(A; \tilde{A}) + K_0^-(A; \tilde{A})]$ for direct propagation from \tilde{A} to A . However, the particle can go out into the universe from A and from \tilde{A} . If the resulting interaction with the universe again provides the principal part of $\delta_+(S_{A\tilde{A}}^2)$, so that the combined effect of $\delta(S_{A\tilde{A}}^2)$ and of the response of the universe is to give $\delta_+(S_{A\tilde{A}}^2)$ for $4\pi I(A; \tilde{A})$, then the effective propagator is $K_+(A; \tilde{A})$.

A detailed calculation for the electrodynamic case was given in Sect. 4 of I. A similar calculation for particle absorption could be given, but since

(*) In their book *Quantum Mechanics and Path Integrals* FEYNMANN and HIBBS give a problem (p. 34) in which the propagator in one space-one time dimension is built up from paths of this kind. The resulting propagator satisfies the two-dimensional Dirac equation.

it would be rather lengthy we shall give the response condition in the form used in II.

Consider the interaction of an element $d\tilde{a}_i$ of an electron line at \tilde{A} with the element da^i at A . The electromagnetic interaction of $d\tilde{a}_i$ with the universe produces a response field at A equal to $\frac{1}{2}\delta_+(S_{A\tilde{A}}^2)d\tilde{a}_i$, which interacts with da^i to give

$$(76) \quad \frac{1}{2}\delta_+(S_{A\tilde{A}}^2)d\tilde{a}_i da^i.$$

Similarly the electromagnetic interaction of da^i with the universe produces a response field at \tilde{A} equal to $-\frac{1}{2}\delta_-(S_{A\tilde{A}}^2)da^i$, which interacts with $d\tilde{a}_i$ to give

$$(77) \quad -\frac{1}{2}\delta_-(S_{A\tilde{A}}^2)d\tilde{a}_i da^i.$$

The contributions (76) and (77) together give a response contribution equal to the principal part of $\delta_+(S_{A\tilde{A}}^2)$ multiplying $d\tilde{a}_i \cdot da^i$. The sign difference between (76) and (77) arises because $d\tilde{a}_i$ is a vector directed from \tilde{A} towards A , whereas da^i is directed from A away from \tilde{A} . In fact, (77) can trivially be written as $\frac{1}{2}\delta_-(S_{A\tilde{A}}^2)d\tilde{a}_i(-da^i)$, which is the time-reversed form of (76).

Again, for $t_A > t_{\tilde{A}}$ in the propagator case, the particle going from \tilde{A} interacts with the universe which sends paths to A contributing $\frac{1}{2}\delta_+(S_{A\tilde{A}}^2)$. Similarly the particle going from A interacts with the universe which responds at \tilde{A} with the time-reversed function $\frac{1}{2}\delta_-(S_{A\tilde{A}}^2)$. In this case we take the difference

$$(78) \quad \frac{1}{2}[\delta_+(S_{A\tilde{A}}^2) - \delta_-(S_{A\tilde{A}}^2)]$$

because we are seeking the net flow of the particle from \tilde{A} to A . The difference is the same as in (55). It gives the principal part of $\delta_+(S_{A\tilde{A}}^2)$, changing the symmetric $I = \delta(S_{A\tilde{A}}^2)/4\pi$ to $\delta_+(S_{A\tilde{A}}^2)/4\pi$, yielding $K_+(A; \tilde{A})$ in accordance with the analysis of Sect. 3.

The Feynman propagator $K_+(2; 1)$ is nonzero outside the light-cone of point 1. Since $K_+(2; 1)$ falls off as $\exp[-mS_{21}]$ it is usually thought that this effect is small, confined as it is to the order of the Compton wavelength. But in the path integral formulation we are concerned with time steps ε in (5) very much smaller than m^{-1} . Indeed, we showed in II that the renormalization procedures of QED involve cut-off distances that may well be less than m^{-1} by 10^{-40} . From the path integral point of view we have to think of $S_{21} \approx m^{-1}$ as a very large distance. The Feynman propagator $K_+(2; 1)$ permits a particle to arrive at point 2 even though the displacement from 1 to 2 is grossly space-like. Unless we abandon our physical concepts in a very major respect the particle cannot have come directly from point 1. It must have arrived via the universe.

It is interesting to compare the four propagators K_0^\pm , $\frac{1}{2}[K_0^+ + K_0^-]$ and K_+ .

All four determine the wave function at point 2 inside a closed surface S in accordance with

$$(79) \quad \psi(2) = \int_S K(2; 1)(\gamma \cdot n)_1 \psi(1) dS_1,$$

where the element of 3-surface dS_1 is at point 1 which ranges over S . $(\gamma \cdot n)_1$ is the unit inward normal at point 1. For simplicity, we take S to be the two time sections $t = t_2 \pm \tau$, $\tau > 0$. When $K = K_0^+$ the wave function $\psi(2)$ is determined wholly by the plane $t = t_2 - \tau$, while $K = K_0^-$ determines $\psi(2)$ wholly from the plane $t = t_2 + \tau$. The propagator $\frac{1}{2}[K_0^+ + K_0^-]$ gives $\frac{1}{2}\psi(2)$ from $t = t_2 - \tau$ and $\frac{1}{2}\psi(2)$ from $t = t_2 + \tau$. So far as these three propagators are concerned it makes no difference whether ψ is a positive- or a negative-energy state. For K_+ , on the other hand, $\psi(2)$ comes wholly from $t = t_2 - \tau$ if ψ is a positive-energy state and from $t = t_2 + \tau$ if the energy is negative. According to the view developed above $\frac{1}{2}[K_0^+ + K_0^-]$ is the correct local propagator. However the response of the universe doubles the $\frac{1}{2}\psi(2)$ contribution from $t = t_2 - \tau$ for a positive-energy state and cancels the $\frac{1}{2}\psi(2)$ contribution from $t = t_2 + \tau$. The doubling and cancellation are reversed for a negative-energy state.

● RIASSUNTO (*)

Si può costruire il propagatore finito delle particelle per mezzo di un metodo di integrale di percorso purché sia noto il propagatore infinitesimo. Sinora però non è stato possibile specificare il propagatore infinitesimo relativistico eccetto che *ad hoc*. Da considerazioni sulla natura della massa, nel senso cosmologico machiano, si mostra in questo articolo che si può dedurre il propagatore infinitesimo nella meccanica quantistica relativistica con un metodo simile a quello usato nell'integrale di percorso non relativistico.

(*) *Traduzione a cura della Redazione.*

О связи бесконечно малого пропагатора частиц с природой массы.

Резюме (*). — Конечный пропагатор частиц может быть сконструирован с помощью метода интегрирования по путям, при условии, что известен бесконечно малый пропагатор. До сих пор однако не было возможно определить релятивистский бесконечно малый пропагатор, за исключением специального способа. Из рассмотрения природы массы, в космологическом смысле Макиана, в настоящей работе показывается, что бесконечно малый пропагатор может быть выведен в релятивистской квантовой механике с помощью метода, аналогичного методу в нерелятивистском случае, использующему интегрирование по путям.

(*) *Переведено редакцией.*