# Statistics of Statistical Anisotropy Measures

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Confronting particle-cosmology with Planck and LHC
August 10-12, 2011

# Bipolar Spherical Harmonics

- Correlation is a two point function & can be expanded in bipolar spherical harmonics basis.

$$C(\hat{n}_1, \hat{n}_2) = \sum_{l_1, l_2, L, M} A^{LM}_{l_1 l_2} \{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{LM}$$

$A^{LM}_{l_1 l_2} \rightarrow$ Bipolar spherical harmonic(BipoSH) coefficients

$\{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{JM} \rightarrow$ Bipolar spherical harmonics

Convenient basis of expansion for functions depending on two vector directions

$$\{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{LM} = \sum_{m_1 m_2} C^{LM}_{l_1 m_1 l_2 m_2} Y_{l_1 m_1}(\hat{n}_1)\, Y_{l_2 m_2}(\hat{n}_2)$$

$|l_1 - l_2| \leq L \leq l_1 + l_2$ Triangularity conditions $M = m_1 + m_2$

Orthonormal functions for different set of $l_1, l_2, L, M$

# Statistical Isotropy in bipolar space

$$C(\hat{n}_1, \hat{n}_2) = C(\hat{n}_1 . \hat{n}_2) = C(\Theta)$$

Correlation function is invariant under the rotations

$$L = 0, M = 0, l_1 = l_2$$

**Statistical Isotropy**

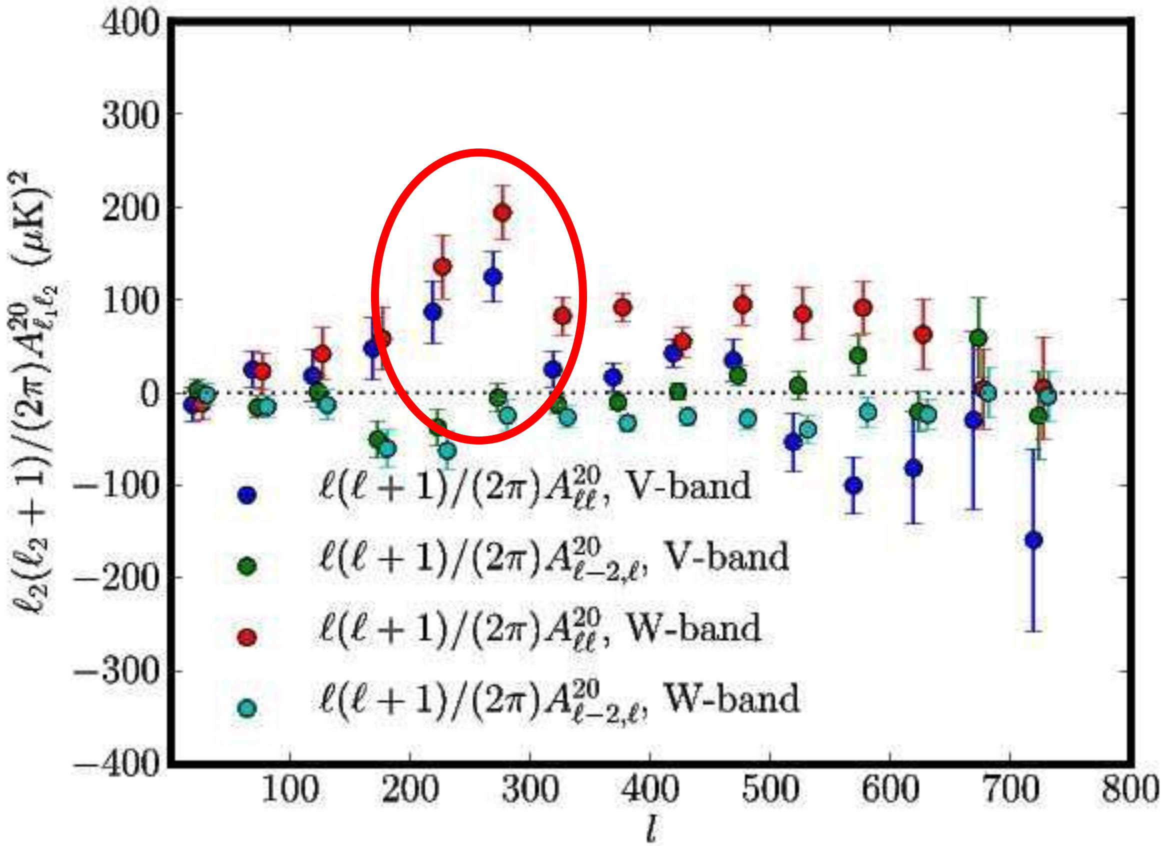
Any Statistical isotropy violation signal can be searched for in BipoSH coefficients!!

Search for $L \neq 0$ BipoSH coefficients

A.Hajian and T. Souradeep, ApJ 597 L5 (2003)

# *Detection of SI violation*

$A_{ll}^{20}$




WMAP–seven year
Bennett et al. 2010

The quadrupolar bipolar power spectra, binned with l = 50, using the KQ75y7 mask.

**Error Bars?? Is distribution symmetric??**

# Statistical significance of detection??

## Statistics of BipoSH coefficients

Understanding is extremely crucial to assess the significance of any statistical isotropy violation detection!!

# Characteristic Function Approach....

Characteristic function is fourier transform of the density function,

$$\varphi_Z(t) = E[e^{itZ}]$$

If $X_1, X_2, ....X_n$ are independent random variables and $a_1, a_2, ...a_n$ are some constants.
The characteristic function of linear combination of $X_i$'s($Z = \sum_i^n a_i X_i$) is,

$$\varphi_{Z_n}(t) = \varphi_{X_1}(a_1 t)\varphi_{X_2}(a_2 t)......\varphi_{X_n}(a_n t)$$

Cumulant generating function,

$$g_Z(t) = \log[\varphi_Z(t)]$$

Cumulants

$$K_n = i^n g_Z^n(t)|_{t=0}$$

# Moments

## Relationship between Cumulants & Moments

$$
\begin{aligned}
\mu_1 &= K_1, \\
\mu_2 &= K_2, \\
\mu_3 &= K_3, \\
\mu_4 &= K_4 + 3K_2^2, \\
\mu_5 &= K_5 + 10K_3K_2, \\
\mu_6 &= K_6 + 15K_4K_2 + 10K_3^2 + 15K_2^3.
\end{aligned}
$$

## Normalized Moments

$$
\mu_n^{norm} = \frac{\mu_n}{\sigma^n}
$$

BipoSH coefficients can also be expressed as linear combination of random variables,

$$A_{l_1 l_2}^{LM} = \sum_{m_1 m_2} a_{l_1 m_1} a_{l_2 m_2} C_{l_1 m_1 l_2 m_2}^{LM}$$

Real and Imaginary parts of BipoSH coefficients,

$$A_{l_1 l_2}^{LM^{(R)}} = \sum_{m_1 m_2} (x_{l_1 m_1} x_{l_2 m_2} - y_{l_1 m_1} y_{l_2 m_2}) C_{l_1 m_1 l_2 m_2}^{LM}$$

$$A_{l_1 l_2}^{LM^{(I)}} = \sum_{m_1 m_2} (y_{l_1 m_1} x_{l_2 m_2} + x_{l_1 m_1} y_{l_2 m_2}) C_{l_1 m_1 l_2 m_2}^{LM}$$

RECALL

$$\varphi_{Z_n}(t) = \varphi_{X_1}(a_1 t)\varphi_{X_2}(a_2 t)......\varphi_{X_n}(a_n t)$$

# Decomposition of CMB temperature fluctuations

$$\Delta T(\hat{n}) = \sum_{l=1}^{\infty} \sum_{m=-l}^{m=+l} a_{lm} Y_{lm}(\hat{n}) \, .$$

where $a_{lm}$ are the spherical harmonic coefficients.

Gaussianity and reality of these fluctuations implies that real and imaginary part of spherical harmonic coefficients are mutually independent and both Gaussian.

$$\begin{aligned} \langle x_{lm} \rangle &= \langle y_{lm} \rangle = 0, \\ \sigma^2(x_{lm}) &= \sigma^2(y_{lm}) = \frac{C_l}{2}. \end{aligned} \quad \Bigg\} \quad \text{For } m \neq 0,$$

$$\begin{aligned} \langle x_{lm} \rangle &= 0, \\ \sigma^2(x_{lm}) &= C_l \end{aligned} \quad \Bigg\} \quad \text{For } m = 0, \text{ imaginary part vanishes.}$$

# Combinations of random variables

$$X_1 \sim N(0, \sigma_1^2) \qquad X_2 \sim N(0, \sigma_2^2)$$

$$Z = X_1 X_2 \longrightarrow f_Z(z) = \frac{K_0(\frac{|z|}{\sigma_1\sigma_2})}{\pi\sigma_1\sigma_2}$$

$$(X_1, Y_1) \sim N(0, \sigma_1^2) \qquad (X_2, Y_2) \sim N(0, \sigma_2^2)$$

$$Z = X_1 X_2 + Y_1 Y_2 \longrightarrow f_Z(z) \sim Laplace(0, 2/\sigma_1^2\sigma_2^2)$$

$$X \sim N(0, \sigma^2) \qquad Y \sim N(0, \sigma^2)$$

$$Z = X^2 - Y^2 \qquad f_Z(z) \sim \frac{1}{2\pi\sigma^2} K_0(\frac{|z|}{2\sigma^2})$$

# Application of characteristic function approach

$$l_1 = l_2, M = 0$$

$$A^{L0^{(R)}}_{l_1 l_1} = \sum_{m_1 (m_1 > 0)} (-1)^{m_1} 2(x^2_{l_1 m_1} + y^2_{l_1 m_1}) C^{L0}_{l_1 m_1 l_1 - m_1} + x^2_{l_1 0} C^{L0}_{l_1 0 l_1 0}$$

$\chi^2$ with 2 d.o.f

$\chi^2$ with 1 d.o.f

$type1$

$type2$

$$\varphi_{A^{L0}_{ll}}(t) = \prod \varphi_{type1}(t) \varphi_{type2}(t)$$

# Recipe

- BipoSH coefficients are linear combination of some random variables.

$$A^{JM}_{l_1 l_2} = \sum_{m_1 m_2} a_{l_1 m_1} a_{l_2 m_2} C^{JM}_{l_1 m_1 l_2 m_2}$$

- Find the characteristic function of each term present in linear sum.

- Assume NO non-linear correlation among terms, find characteristic function of these coefficients.

$$\varphi_{Z_n}(t) = \varphi_{X_1}(a_1 t)\varphi_{X_2}(a_2 t)......\varphi_{X_n}(a_n t)$$

- Find Cumulant generating function from characteristic function.

$$g_Z(t) = \log[\varphi_Z(t)]$$

- Find Cumulants from cumulant generating function.

$$K_n = i^n g^n_Z(t)|_{t=0}$$

- Finally, find moments from Cumulants.

# RESULTS!!

- Developed Faster code to calculate Bipolar coefficients (30x)
  - Can go up to high multipoles.

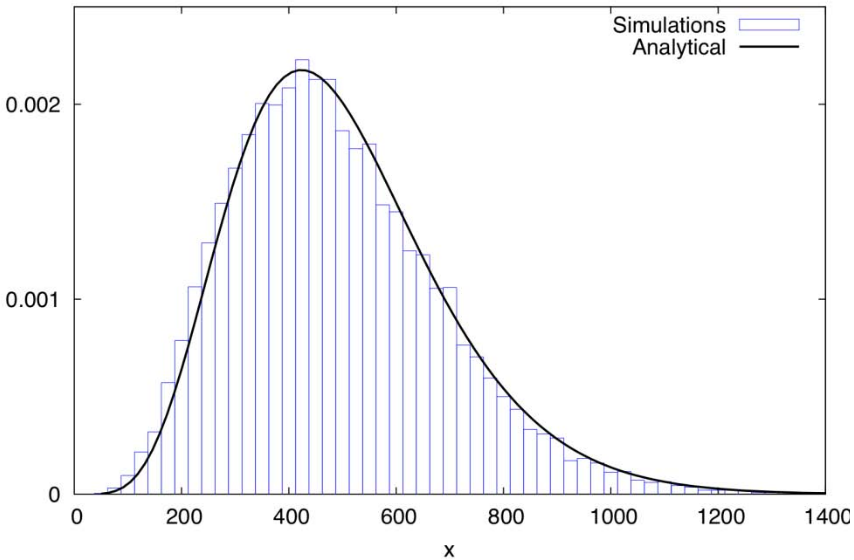
- Simulated Moments from 15000 Gaussian & isotropic realizations generated with best fit LCDM angular power spectrum.

# BipoSH coefficients with $L = 0$ and $M = 0$

- **Equivalent to CMB angular power spectrum.**
- **Well known result chi-square distribution.**

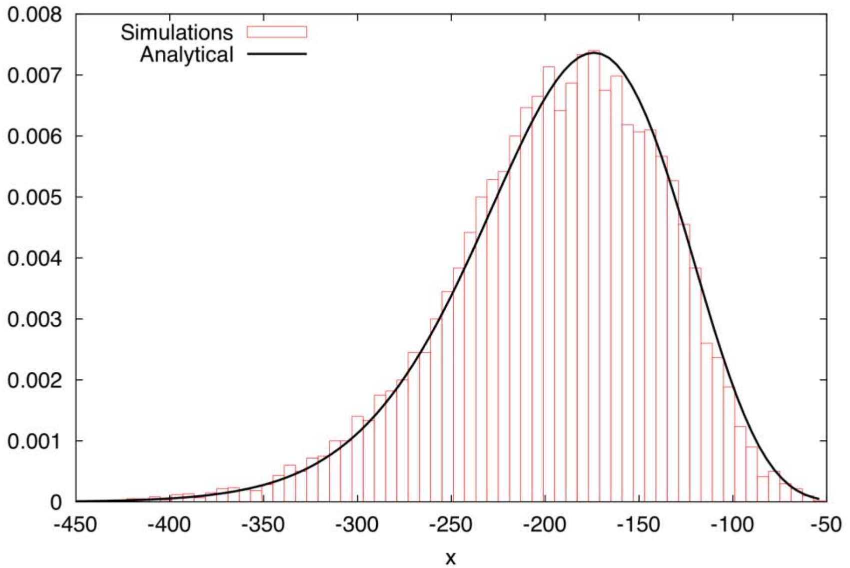
$$A_{ll}^{00} = (-1)^l C_l$$

PDF, l=6




Even multipoles- right skewed

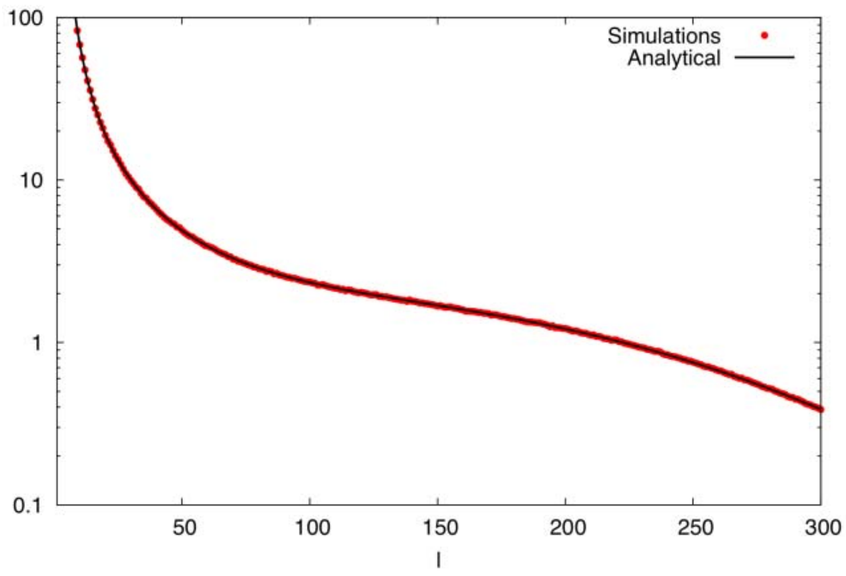
PDF, l=11



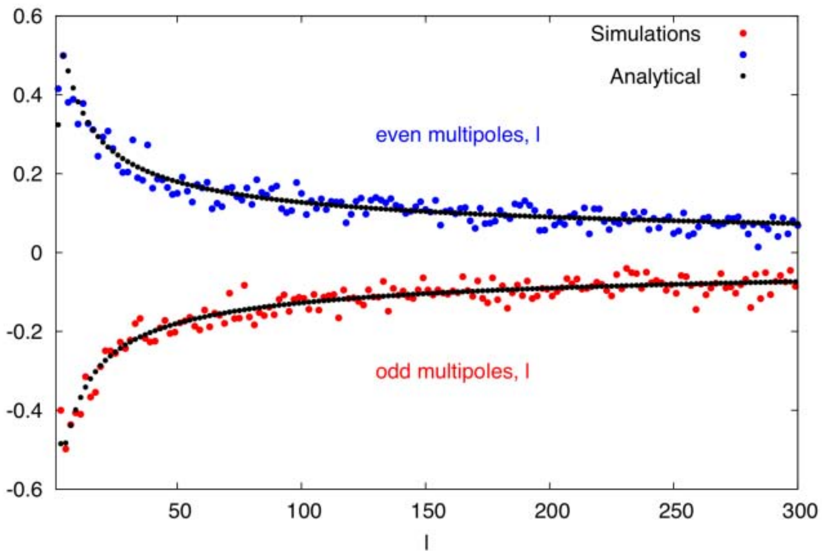

odd multipoles- left skewed

# BipoSH coefficients with $M = 0$, $L \neq 0$

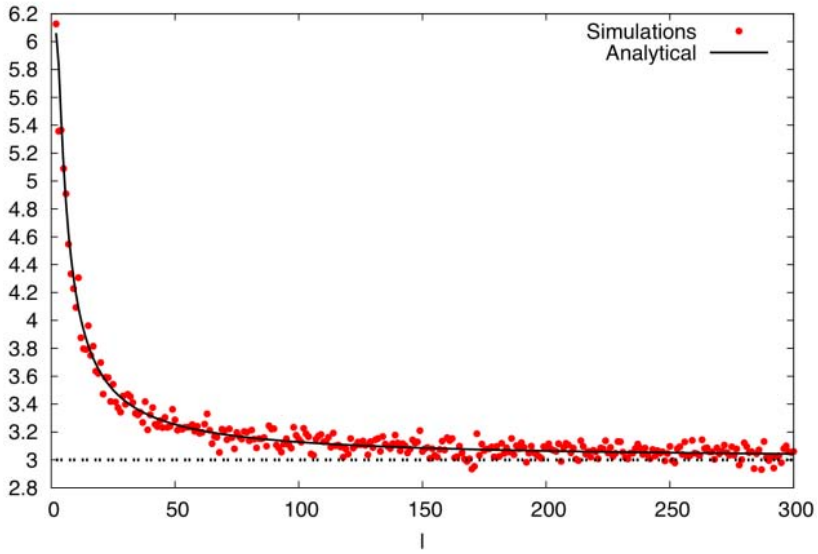
- All terms in linear combination are independent of each other.
- Characteristic function for these coefficients is product of the characteristic function of each term in linear combination.
- Only BipoSH coefficients with Asymmetric Distribution!!



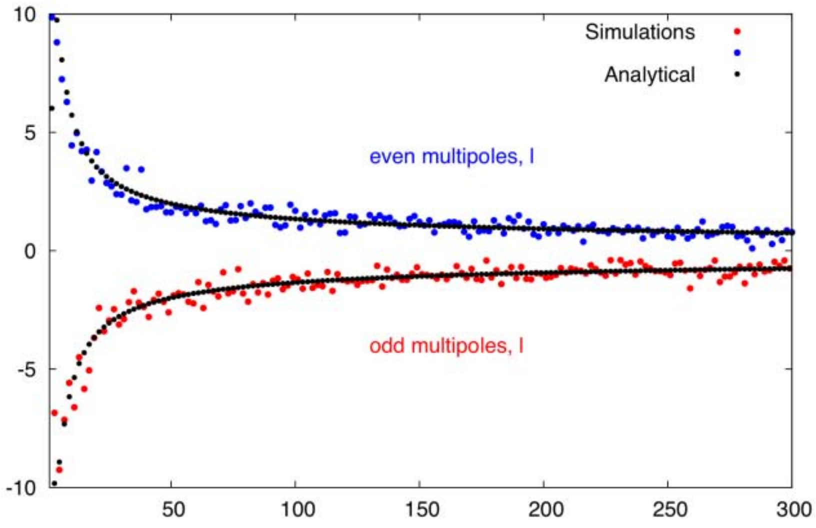

Standard Deviation




Skewness




Kurtosis




$5^{th}$ Moment

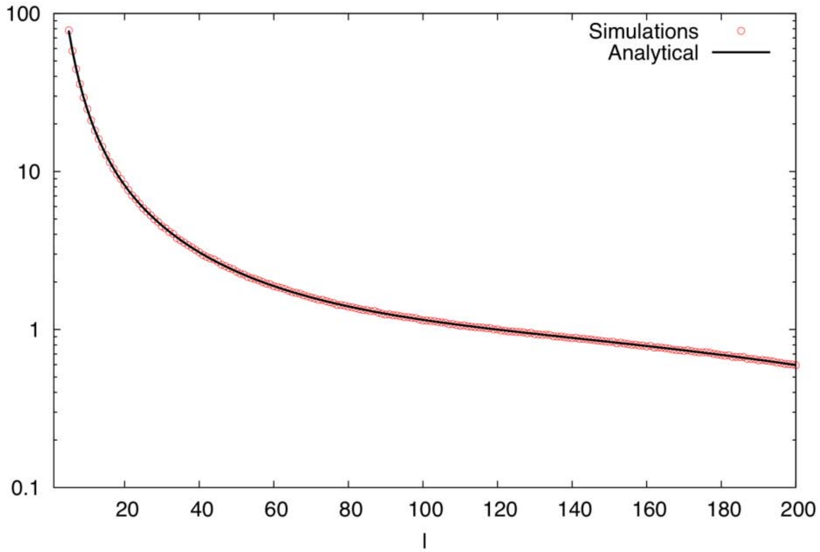
- These coefficients are always REAL.

- Odd moments for these coefficients oscillate between positive and negative values for even and odd multipoles respectively.

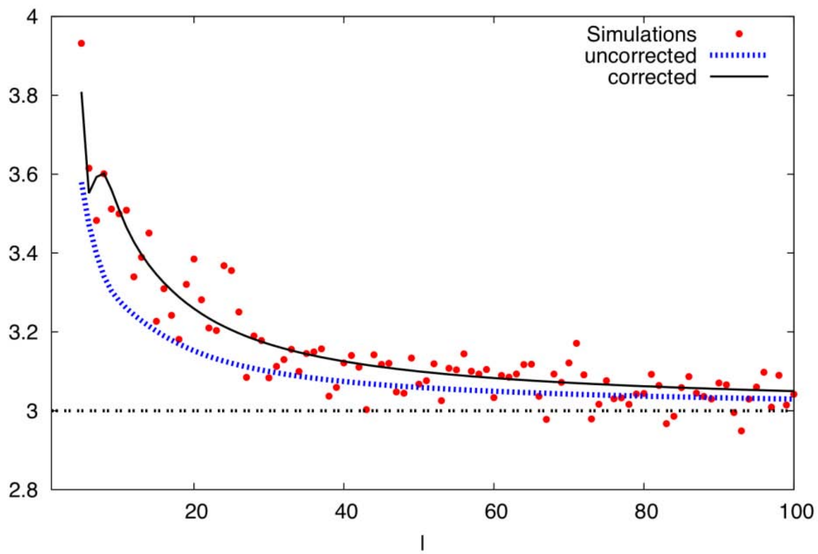
# BipoSH coefficients with $M \neq 0$

- Assumed independence among terms leads to mismatch between analytically derived moments and simulations.
- Terms in linear combination are linearly uncorrelated.
- Distribution is Symmetric, all odd moments vanishes.
- Account for non-linear correlations and simulation matches analytical moments.

$$\bar{\mu}_n = \tilde{\mu}_n + \text{correction}$$




Standard Deviation




Kurtosis

# Conclusions

- Complete statistical information available for BipoSH coefficients with M=0.

- BipoSH coefficients with M=0 have Asymmetric distribution with even and odd multipoles being left and right skewed.

- Remaining coefficients have symmetric distribution.

- For coefficients with M not equal to zero, it turns out that terms in expansion are non-linearly correlated.

- To account these non-linear correlations, we supply a correction term to moments(up to kurtosis).

- These details need to be taken in to account to have better assessment of any statistical isotropy violation detections in future data!!

# THANK YOU