

The classical essence of black hole radiation

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We show that the mathematics of Hawking process can be interpreted classically as the Fourier analysis of an exponentially redshifted wave mode which scatters off the black hole and travels to infinity at late times. We use this method to derive the Planckian power spectrum for Schwarzschild, Reissner-Nordstrom and Kerr black holes.

I. INTRODUCTION

The study of black holes from astrophysical point of view and by astronomers has blossomed in the last decade because of the dramatic increase in the number of black hole candidates from the sole candidate (Cygnus X-1) some 25 years ago. This in turn requires a deeper familiarity with black hole physics and especially with *black hole radiation*, for astronomers and classical relativists. Hawking in his original work on the black hole radiation (Ref.[4]) has used a quantum field theoretical approach to arrive at this radiation. In the next few sections we will describe the classical essence of this radiation in a language which is free from the usual quantum field theoretic tools and therefore more suitable for the astronomers and relativists. Our derivation of Hawking radiation will also establish the close connection between the black hole radiation and the existence of an event horizon.

II. SCHWARZSCHILD BLACK HOLE

We start with the simplest case of the one parameter family of blackholes, namely the Schwarzschild black hole, which was previously discussed in Ref.[6]. Consider a radial light ray in the Schwarzschild spacetime which propagates from $r_{in} = 2M + \epsilon$ at $t = t_{in}$ to the event $\mathcal{P}(t, r)$ where $r \gg 2M$ and $\epsilon \ll 2M$. The trajectory can be found using the fact that for light rays

$$ds^2 = 1 - \frac{2M}{r} dt^2 - \frac{1}{1 - \frac{2M}{r}} dr^2 - r^2 d\Omega^2 = 0,$$

for radial light rays, $d\theta = d\phi = 0$, we have

$$\frac{dr}{dt} = 1 - \frac{2M}{r}, \quad (1)$$

from which the trajectory with the required initial condition, is

$$r = r_{in} - 2M \ln\left(\frac{r - 2M}{2M}\right) + 2M \ln\left(\frac{r_{in} - 2M}{2M}\right) + t - t_{in} \cong t - t_{in} + 2M \ln\left(\frac{\epsilon}{2M}\right) \quad (2)$$

where the last equality uses $r \gg 2M$, $\epsilon \ll 2M$. The frequency of a wave will be redshifted as it propagates on this trajectory. This redshift is basically due to the fact that the frequency is measured in terms of the proper time τ , which flows differently at different points of a stationary spacetime according to the following relation:

$$\tau = \frac{1}{c} \sqrt{g_{00}} x^0, \quad (3)$$

The frequency Ω at $r \gg 2M$ will be related to the frequency Ω_{in} at $r = 2M + \epsilon$ by

$$\Omega \cong \Omega_{in} [g_{00}(r = 2M + \epsilon)]^{1/2} \cong \Omega_{in} \left(\frac{\epsilon}{2M}\right)^{1/2} = \Omega_{in} \exp\left(-\frac{t - t_{in} - r}{4M}\right) \quad (4)$$

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If the wave packet, $\Phi(r, t) \propto \exp(i\theta(t, r))$, centered on this null ray has a phase $\theta(r, t)$, then the instantaneous frequency is related to the phase by $(\partial\theta/\partial t) = \Omega$. Integrating (4) with respect to t , we find the relevant wave mode to be

$$\Phi(t, r) \propto \exp i \int \Omega dt \propto \exp \left[-4Mi\Omega_{in} \exp \left(-\frac{t - t_{in} - r}{4M} \right) \right] \quad (5)$$

(This form of the wave can also be obtained by directly integrating the wave equation in Schwarzschild space with appropriate boundary conditions). Equation (4) shows that, despite being in a static spacetime, the frequency of the wave (measured by an observer at fixed $r \gg 2M$) depends nontrivially on t , for a fixed t_{in} and ϵ . Such an observer will not see a monochromatic radiation. Therefore an observer using the time coordinate t will Fourier decompose these modes with respect to the frequency ω defined using t as:

$$\Phi(t, r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) e^{-i\omega t} d\omega \quad (6)$$

where

$$f(\omega) = \int_{-\infty}^{\infty} \Phi(t, r) e^{i\omega t} dt \propto \int_0^{\infty} x^{-4iM\omega-1} \exp(-4Mix\Omega_{in}) dx \quad (7)$$

and $x = \exp([-t + t_{in} + r]/4M)$. To evaluate the above integral we rotate the contour to the imaginary axis, i.e. $x \rightarrow y = ix$,

$$f(\omega) \propto e^{-2\pi M\omega} \int_0^{i\infty} Y^{z-1} e^{-Y} dY \quad (8)$$

where $z = -4iM\omega$ and $Y = -4M\Omega_{in}y$. Using the fact that the integral in the right hand side of the above relation is one of the representations of Gamma function we get the corresponding power spectrum to be

$$|f(\omega)|^2 \propto (\exp(8\pi M\omega) - 1)^{-1} \quad (9)$$

where we have used the fact that $|\Gamma(ix)|^2 = (\pi/x \sinh \pi x)$. In terms of the conventional units the above relation becomes

$$|f(\omega)|^2 \propto \left(\exp\left(\frac{8\pi GM\omega}{c^3}\right) - 1 \right)^{-1} \equiv \left(\exp\left(\frac{\omega}{\omega_0}\right) - 1 \right)^{-1} \quad (10)$$

where

$$\omega_0 = \frac{c^3}{8\pi GM} \quad (11)$$

As one can see no \hbar appears in the above analysis and ω_0 can be thought of as the characteristic frequency of the problem by a radio astronomer who thinks in terms of frequency. On the other hand an X-ray or a γ -ray astronomer -who thinks in terms of photons- will introduce the energy $E = \hbar\omega$ into the above relation in the following form:

$$|f(\omega)|^2 \propto \left(\exp\left(\frac{\hbar\omega}{\hbar\omega_0}\right) - 1 \right)^{-1} \equiv \left(\exp\left(\frac{E}{k_B T}\right) - 1 \right)^{-1} \quad (12)$$

which shows that the corresponding power spectrum is Planckian at temperature

$$T = \frac{\hbar c^3}{8\pi GM k_B} \quad (13)$$

III. REISSNER-NORDSTROM BALCK HOLE

The same approach can be used to study the radiation in the space of static charged black holes which are characterized by two parameters M and Q . The equation governing the outgoing null radial geodesics in R-N spacetime has the following form

$$\frac{dr}{dt} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \quad (14)$$

In terms of the conventional units the above equation will take the following form

$$\frac{dr}{dt} = c - \frac{2M}{r} \left(\frac{G}{c}\right) + \frac{Q^2}{r^2} \left(\frac{G\hbar}{c^2}\right). \quad (15)$$

The event horizon of the Reissner-Nordstrom black hole is at $r_+ = M + (M^2 - Q^2)^{1/2}$. Considering a light ray propagating from $r_{in} = r_+ + \epsilon$ at $t = t_{in}$ to the event $\mathcal{P}(t, r)$ where $r \gg r_+$ and $\epsilon \ll r_+$ we will find the trajectory in the following form

$$r \cong t - t_{in} + \frac{r_+^2}{2(M^2 - Q^2)^{1/2}} \ln \epsilon \quad (16)$$

The redshifted frequency ω will be related to the frequency at $r = r_+ + \epsilon$ by

$$\Omega \cong \Omega_{in} [g_{00}(r = r_+ + \epsilon)]^{1/2} \cong \omega_{in} \left(\frac{\epsilon}{2M}\right)^{1/2} = \omega_{in} \exp\left(-\frac{t - t_{in} - r}{\frac{(M + (M^2 - Q^2)^{1/2})^2}{(M^2 - Q^2)^{1/2}}}\right) \quad (17)$$

Now if we repeat the analysis of the Schwarzschild case for R-N spacetime, in exactly the same way, we find that the corresponding power spectrum for a wave packet which has scattered off the R-N black hole and travelled to infinity at late times has the following Planckian form

$$|f(\omega)|^2 \propto \left(\exp\left[\frac{2\pi[M + (M^2 - Q^2)^{1/2}]^2}{(M^2 - Q^2)^{1/2}}\right] \omega - 1\right)^{-1} \quad (18)$$

at temperature $T = \frac{(M^2 - Q^2)^{1/2}}{2\pi[M + (M^2 - Q^2)^{1/2}]^2}$ which is the standard result and reduces to that of the Schwarzschild case when $Q = 0$.

IV. HAWKING RADIATION OF A KERR BLACK HOLE

In applying the approach of the last two sections to the radiation of Kerr black holes we should be more careful because- unlike the Schwarzschild and R-N black holes- the event horizon and infinite redshift surface do not coincide. We will see that in this case the infinite redshift surface acts as a boundary for the outgoing null geodesics originating from inside the ergosphere, on which we should be concerned about the continuity problem. In Kerr spacetime the principal null congruences play the same role as the radial null geodesics in Schwarzschild and R-N spacetimes, so we consider them in our derivation of the Hawking radiation by Kerr black holes. The equation governing the principal null congruences ($\theta = \text{const.}$) is given by

$$\frac{dr}{dt} = 1 - \frac{2M}{r} + \frac{a^2}{r^2} \quad (19)$$

If we restrict our attention to the case $a^2 < M^2$, the above equation can be integrated to give

$$t = r + \left(M + \frac{M^2}{(M^2 - a^2)^{1/2}}\right) \ln|r - r_+| + \left(M - \frac{M^2}{(M^2 - a^2)^{1/2}}\right) \ln|r - r_-| \quad (20)$$

where

$$r_{\pm} = M \pm (M^2 - a^2)^{1/2} \quad (21)$$

are the event horizons of the Kerr metric. Now as in the previous sections we consider a light ray propagating from point $r_+ + \epsilon$ at $t = t_{in}$ to the event $\mathcal{P}(r, t)$ where $r, t \gg M$ and $\epsilon \ll M$. Starting from a point very close to the outer event horizon ($r_+ + \epsilon$) the trajectory would have the following form

$$r \cong t - t_{in} + \left(M + \frac{M^2}{(M^2 - a^2)^{1/2}}\right) \ln \epsilon \quad (22)$$

The frequency Ω at r will be related to the frequency Ω_{in} of a light ray emitted by a *locally nonrotating observer* (Ref.[1]) at $r = r_+ + \epsilon$ (inside the ergosphere) by (see appendix A)

$$\Omega = \Omega_{in} \frac{(g_{00} - g_{03}^2/g_{33})^{1/2}}{(1 + (g_{03}/g_{00})a\sin^2\theta)} \propto \Omega_{in}\epsilon^{1/2} = \Omega_{in}\exp\left(-\frac{(t - t_{in} - r)(M^2 - a^2)^{1/2}}{2(M^2 + M(M^2 - a^2)^{1/2})}\right) \quad (23)$$

repeating the procedure of the last two sections to the above redshifted frequency we find the following power spectrum for a wave packet scattered off the Kerr black hole at late times

$$|f(\omega)|^2 \propto \left(\exp\left[\frac{4\pi[M^2 + M(M^2 - a^2)^{1/2}]}{(M^2 - a^2)^{1/2}}\right]\omega - 1\right)^{-1} \quad (24)$$

which is Planckian at temperature

$$T = \frac{(M^2 - a^2)^{1/2}}{4\pi[M^2 + M(M^2 - a^2)^{1/2}]} \quad (25)$$

which is again the standard result (Ref.[2]) and reduces to (13) for $a = 0$.

V. DISCUSSION

In this letter we gave a simple derivation of black hole radiation which strips the Hawking process to its bare bones and establishes the following two facts: (i) The key input which leads to the Planckian spectrum is the exponential redshift given by equations (4,17 & 23) of modes which scatter off the black hole and travel to infinity at late times, which in turn requires the existence of an event horizon. It is well known that frequencies of outgoing waves at late times in black hole evaporation correspond to super planckian energies of the ingoing modes near the horizon. One might ask where do the ingoing modes corresponding to the outgoing modes come from?. This where the quantum field theory plays its role in the black hole radiation. According to quantum field theory vacuum is a dynamical entity and space is nowhere free of vacuum fluctuations. The vacuum field fluctuations can be thought of as a superposition of ingoing and outgoing modes. A *collapsing star* will introduce a mismatch between these virtual modes causing the appearance of a real particle at infinity. The calculation shows that the energy carried by the radiation is extracted from the black hole (Ref.[4]). What we have done is to mimic the essence of this process by considering a classical mode propagating from near event horizon to infinity. (ii) The analysis given in the previous sections is entirely classical and no \hbar appears anywhere. The mathematics of Hawking evaporation is puerly classical and lies in the Fourier transform of an exponentially redshifted wave mode (for a more detailed discussion of classical versus quantum features see ref. [7]).

APPENDIX A : GRAVITATIONAL REDSHIFT BY A KERR BLACK HOLE

In this appendix we derive the gravitational redshift of a light ray emitted from inside the ergosphere and received by a Lorentzian observer at infinity (as given by equation (17) of the text). The general relation for the redshift between a source and an observer located at events \mathcal{P}_1 and \mathcal{P}_2 in a stationary spacetime, is given by (Ref[8])

$$\frac{\omega_{\mathcal{P}_1}}{\omega_{\mathcal{P}_2}} = \frac{(k_a u^a)_{\mathcal{P}_1}}{(k_a u^a)_{\mathcal{P}_2}} \quad (A1)$$

where k^a is the wave vector and u^a 's are the 4-velocities of the source and the observer. The numerator and denominator are evaluated at the events \mathcal{P}_1 and \mathcal{P}_2 respectively. One should note that the null geodesic (or equivalently its tangent vector) joining the source and the observer should be continuous over the boundary which in this case is the infinite redshift surface. The principal null congruences we are considering here, indeed satisfy this condition. Since there are no static observers inside the ergosphere we choose as our source the *locally nonrotating observer* (Ref.[1]) whose angular velocity in Boyer-Lindquist coordinates is given by (Ref[5])

$$\Omega = -\frac{g_{03}}{g_{33}} = \frac{2Mra}{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta} \quad (A2)$$

so the 4-velocities of the source and the static Lorentzian observer are given by (Ref[5])

$$u^a|_S = \frac{1}{(g_{00} + 2\Omega g_{03} + \Omega^2 g_{33})^{1/2}}(1, 0, 0, \Omega) \quad \& \quad u^a|_\infty = (1/\sqrt{g_{00}}, 0, 0, 0) \quad (A3)$$

Substituting (A2) and (A3) in (A1) we have

$$\omega|_\infty = \omega|_S \left(\frac{k_0|_\infty}{(k_0 u^0 + k_3 u^3)|_S} \right)$$

Using the fact that the frequency measured with respect to the coordinate time, k_0 , is constant and that $k_3/k_0 = -a\sin^2\theta$ for the principal null congruences (Ref[3]) we have

$$\omega|_\infty = \omega|_S \left(\frac{1}{(u^0 - a\sin^2\theta u^3)|_S} \right) \quad (A4)$$

Now Substituting from (A2) and (A3) in (A4) we obtain the following result

$$\omega|_\infty = \omega|_S \frac{(g_{00} - g_{03}^2/g_{33})^{1/2}}{(1 + (g_{03}/g_{00})a\sin^2\theta)} \quad (A5)$$

which is the relation used in the text.

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