Why do naked singularities form in gravitational collapse?

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We investigate what are the key physical features that cause the development of a naked singularity, rather than a black hole, as the end-state of spherical gravitational collapse. We show that sufficiently strong shearing effects near the singularity delay the formation of the apparent horizon. This exposes the singularity to an external observer, in contrast to a black hole, which is hidden behind an event horizon due to the early formation of an apparent horizon.

I. INTRODUCTION

In the past decade or so, several scenarios have been discovered where the gravitational collapse of a massive matter cloud results in the development of a naked singularity [1]. The final outcome of gravitational collapse in general relativity is an issue of great importance and interest from the perspective of black hole physics as well as its astrophysical implications. When there is a continual collapse without any final equilibrium, either a black hole forms when the super-dense regions of matter are hidden from the outside observer within an event horizon of gravity, or a naked singularity results as the end product, depending on the nature of the initial data from which the collapse develops.

The theoretical and observational properties of a naked singularity would be quite different from those of a black hole (see [2] for further discussion of this). Thus it is of crucial importance to understand what are the key physical characteristics and dynamical features in collapse that give rise to a naked singularity, rather than a black hole. While many models of naked singularity formation within dynamically developing collapse scenarios have been found and analyzed [3], not much attention has been given to understanding this important aspect. We begin here an investigation of this question.

The main purpose of this paper is to identify the physical process which exposes the singularity. We find that it is shearing effects which, if sufficiently strong near the central worldline of the collapsing cloud, would delay the formation of the apparent horizon so that the singularity becomes visible and communication from the very strong gravity regions to outside observers becomes possible. When the shear is weak (and in the extreme case of no shear), the collapse necessarily ends in a black hole, because an early formation of the apparent horizon leads to the singularity being hidden behind an event horizon.

For spherical gravitational collapse of a massive matter cloud, the interior metric in comoving coordinates is

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\psi(t,r)}dr^2 + R^2(t,r)d\Omega^2\,. \qquad (1)$$

The matter shear is

$$\sigma_{ab} = e^{-\nu}\left(\frac{\dot R}{R} - \dot\psi\right)\left(\tfrac{1}{3}h_{ab} - n_a n_b\right), \qquad (2)$$

where $h_{ab} = g_{ab} + u_a u_b$ is the induced metric on 3-surfaces orthogonal to the fluid 4-velocity u^a, and n^a is a unit radial vector.

The initial data for collapse are the values on $t = t_{\rm i}$ of the three metric functions, the density, the pressures, and the mass function that arises from integrating the Einstein equations (for details see e.g. [4]),

$$F(t_{\rm i}, r) = \int \rho(t_{\rm i}, r) r^2 dr\,, \qquad (3)$$

where $4\pi F(t_{\rm i}, r_{\rm b}) = M$, the total mass of the collapsing cloud, and where $r > r_{\rm b}$ is a Schwarzschild spacetime. We use the rescaling freedom in r to set

$$R(t_{\rm i}, r) = r\,, \qquad (4)$$

so that the physical area radius R increases monotonically in r, and with $R'_{\rm i} = 1$ there are no shell-crossings on the initial surface. (We will be interested here only in the central shell-focusing singularity at $R = 0, r = 0$ which is a gravitationally strong singularity, as opposed to the shell-crossing ones which are weak, and through which the spacetime may sometimes be extended.) The evolution of the density and radial pressure are given by

$$\rho = \frac{F'}{R^2 R'}, \quad p_{\rm r} = \frac{\dot F}{R^2 \dot R}\,. \qquad (5)$$

The central singularity at $r = 0$, where density and curvature are infinite, is naked if there are outgoing non-spacelike geodesics which reach outside observers in the future and terminate at the singularity in the past. Outgoing radial null geodesics of Eq. (1) are given by

$$\frac{dt}{dr} = e^{\psi - \nu}\,. \tag{6}$$

Consider first the case of homogeneous-density collapse, $\rho = \rho(t)$. Writing $f = e^{-2\psi}R'^2 - 1$, the Einstein equations give $f - e^{-2\nu}\dot{R}^2 = -F/R$. Then Eq. (6) can be written as [5]

$$\frac{dR}{du} = \left(1 - \sqrt{\frac{f + F/R}{1+f}}\right)\frac{R'}{\alpha r^{\alpha-1}}\,, \tag{7}$$

where $u = r^\alpha$ ($\alpha > 1$). If there are outgoing radial null geodesics terminating in the past at the singularity with a definite tangent, then at the singularity we have $dR/du > 0$. For homogeneous density, the entire mass of the cloud collapses to the singularity simultaneously at the event $(t = t_\mathrm{s}, r = 0)$, so that $F/R \to \infty$. By Eq. (7), $dR/du \to -\infty$, so that no radial null geodesics can emerge from the central singularity. It can be similarly shown that all the later epochs $t > t_\mathrm{s}$ are similarly covered.

We have thus shown that *for spherical gravitational collapse with homogeneous density (and arbitrary pressures), the final outcome is necessarily a black hole.* We note that this conclusion does not require homogeneity of the pressures p_r and $p_\perp$, and is independent of their behavior. The result generalizes the well-known Oppenheimer-Snyder result for the special case of dust, where the homogeneous cloud collapses to form a black hole always.

An immediate consequence is that *if the final outcome of spherical gravitational collapse is not a black hole, then the density must be inhomogeneous.* In any physically realistic scenario, the density will be typically higher at the center, so that generically collapse is inhomogeneous.

II. INHOMOGENEOUS DUST

Consider now a collapsing inhomogeneous dust cloud ($p = 0$), with density higher at the center. The metric is Tolman-Bondi-Lemaître, given by Eq. (1) with $\nu = 0$ and $e^{2\psi} = R'^2/(1+f)$, and

$$\dot{R}^2 = f(r) + \frac{F(r)}{R}\,. \tag{8}$$

These models are fully characterized by the initial data, specified on an initial surface $t = t_\mathrm{i}$ from which the collapse develops, which consist of two free functions: the initial density $\rho_\mathrm{i}(r) = \rho(t_\mathrm{i}, r)$ (or equivalently, the mass function $F(r)$), and $f(r)$, which describes the initial velocities of collapsing matter shells. At the onset of collapse the spacetime is singularity-free, so that by Eq. (5),

$$F(r) = r^3\bar{F}(r)\,, \quad 0 < \bar{F}(0) < \infty\,. \tag{9}$$

The initial density $\rho_\mathrm{i}(r)$ is

$$\rho_\mathrm{i}(r) = r^{-2}F'(r)\,. \tag{10}$$

The shell-focusing singularity appears along the curve $t = t_\mathrm{s}(r)$ defined by

$$R(t_\mathrm{s}(r), r) = 0\,. \tag{11}$$

As the density grows without bound, trapped surfaces develop within the collapsing cloud. These can be traced explicitly via the outgoing null geodesics, and the equation of the apparent horizon, $t = t_\mathrm{ah}(r)$, which marks the boundary of the trapped region, is given by

$$R(t_\mathrm{ah}(r), r) = F(r)\,. \tag{12}$$

If the apparent horizon starts developing earlier than the epoch of singularity formation, then the event horizon can fully cover the strong gravity regions including the final singularity, which will thus be hidden within a black hole. On the other hand, if trapped surfaces form sufficiently later during the evolution of collapse, then it is possible for the singularity to communicate with outside observers.

For the sake of clarity, we consider marginally bound collapse, $f = 0$, although the conclusions can be generalized to hold for the general case. Then Eq. (8) can be integrated to give

$$R^{3/2}(t, r) = r^{3/2} - \tfrac{3}{2}(t - t_\mathrm{i})F^{1/2}(r)\,, \tag{13}$$

and Eqs. (11) and (12) lead to

$$t_\mathrm{s}(r) = t_\mathrm{i} + \frac{2}{3}\left[\frac{r^3}{F(r)}\right]^{1/2}\,, \tag{14}$$

$$t_\mathrm{ah}(r) = t_\mathrm{s}(r) - \frac{2}{3}F(r)\,. \tag{15}$$

The central singularity at $r = 0$ appears at the time

$$t_0 = t_\mathrm{s}(0) = t_\mathrm{i} + \frac{2}{\sqrt{3\rho_\mathrm{c}}}\,, \tag{16}$$

where where $\rho_\mathrm{c} = \rho_\mathrm{i}(0)$. Unlike the homogeneous dust case (Oppenheimer-Snyder), the collapse is not simultaneous in comoving coordinates, and the singularity is described by a curve, the first point being $(t = t_0, r = 0)$.

For inhomogeneous dust, Eqs. (2) and (13) give

$$\sigma^2 \equiv \frac{1}{2}\sigma_{ab}\sigma^{ab} = \frac{r}{6R^4R'^2F}\left(3F - rF'\right)^2\,. \tag{17}$$

A generic (inhomogeneous) mass profile has the form

$$F(r) = F_0r^3 + F_1r^4 + F_2r^5 + \cdots\,, \tag{18}$$

near $r = 0$, where $F_0 = \rho_\mathrm{c}/3$. Homogeneous dust (Oppenheimer-Snyder) collapse has $F_n = 0$ for $n > 0$, and Eq. (17) implies $\sigma = 0$. The converse is also true in this case: if we impose vanishing shear $\sigma = 0$, we get $F_n = 0$. Whenever there is a negative density gradient, e.g., when there is higher density at the center, then $F_n \neq 0$ for some $n > 0$, and it follows from Eq. (17)

that the shear is then necessarily nonzero. Note that if we want the density profile to be analytic, we can set all odd terms F_{2n-1} to zero; however, we note that this is not as such required by our own analysis, which is independent of any assumptions on F_n.

The important question is: what is the effect of such a shear on the evolution and development of the trapped surfaces? In other words, we want to determine the behavior of the apparent horizon in the vicinity of the central singularity at $R = 0, r = 0$. To this end, let the first non-vanishing derivative of the density at $r = 0$ be the n-th one ($n > 0$), i.e.,

$$F(r) = F_0 r^3 + F_n r^{n+3} + \cdots \,, \quad F_n < 0 \,, \tag{19}$$

near the center. By Eqs. (17) and (15),

$$\sigma^2(t,r) = \frac{n^2 {F_n}^2}{6F_0}\left[1 - 3F_0^{1/2}(t-t_\mathrm{i}) + \frac{9}{4}F_0(t-t_\mathrm{i})^2\right] r^{2n} + O(r^{2n+1}) \,, \tag{20}$$

$$t_\mathrm{ah}(r) = t_0 - \frac{2}{3}F_0 r^3 - \frac{F_n}{3{F_0}^{3/2}} r^n + O(r^{n+1}) \,. \tag{21}$$

The time-dependent factor in square brackets on the right of Eq. (20) decreases monotonically from 1 at $t = t_\mathrm{i}$ to 0 at $t = t_0$. Thus the qualitative role of the shear in singularity formation can be seen by looking at the initial shear. The initial shear $\sigma_\mathrm{i} = \sigma(t_\mathrm{i}, r)$ on the surface $t = t_\mathrm{i}$ grows as r^n, $n \geq 1$, near $r = 0$. A dimensionless and covariant measure of the shear is the relative shear, $|\sigma/\Theta|$, where

$$\Theta = 2\frac{\dot{R}}{R} + \frac{\dot{R}'}{R'} \,, \tag{22}$$

is the volume expansion. It follows that

$$\left|\frac{\sigma}{\Theta}\right|_\mathrm{i} = \frac{-nF_n}{3\sqrt{6}F_0}\, r^n \left[1 + O(r)\right] \,. \tag{23}$$

It is now possible to see how such an initial shear distribution determines the growth and evolution of the trapped surfaces, as prescribed by the apparent horizon curve $t_\mathrm{ah}(r)$, given by Eq. (21). If we assume the initial density profile is smooth at the center, then $\rho_\mathrm{i}(r) = \rho_\mathrm{c} + \rho_2 r^2 + \cdots$, with $\rho_2 \leq 0$, which corresponds to $F(r) = F_0 r^3 + F_2 r^5 + \cdots$, with $F_2 \leq 0$. Now suppose that ρ_2 (and hence F_2) is nonzero. Then Eq. (21) implies that the apparent horizon curve initiates at $r = 0$ at the epoch t_0, and increases near $r = 0$ with increasing r, moving to the future. Note that as soon as F_2 is nonzero, even with very small magnitude, the behavior of the apparent horizon changes qualitatively. Rather than going back into the past from the center, as would happen in the homogeneous case with $F_2 = 0$, it is future pointed. This is what leads to a locally naked singularity. The singularity may be globally naked, i.e. visible to faraway observers, depending on the nature of the density function at large r.

A naked singularity occurs when a comoving observer (at fixed r) does not encounter any trapped surfaces until the time of singularity formation, whereas for a black hole, trapped surfaces form before the singularity. Thus for a black hole, we require

$$t_\mathrm{ah}(r) \leq t_0 \text{ for } r > 0 \,, \text{ near } r = 0 \,. \tag{24}$$

In the general case (not necessarily smooth initial density), this condition is violated for $n = 1, 2$, as follows from Eq. (21). The apparent horizon curve initiates at the singularity $r = 0$ at the epoch t_0, and increases with increasing r, moving to the future, i.e. $t_\mathrm{ah} > t_0$ for $r > 0$ near the center. The behavior of the outgoing families of null geodesics has been analyzed in detail in these cases, and it is known that the geodesics terminate at the singularity in the past [4], which results in a naked singularity. In such cases the extreme strong gravity regions can communicate with outside observers. For the case $n = 3$, Eq. (24) shows that we can have a black hole if $F_3 \geq -2F_0^{5/2}$, or a naked singularity, if $F_3 < -2F_0^{5/2}$. This is illustrated in Fig. 1. For $n \geq 4$, Eq. (24) is always satisfied, and a black hole forms.

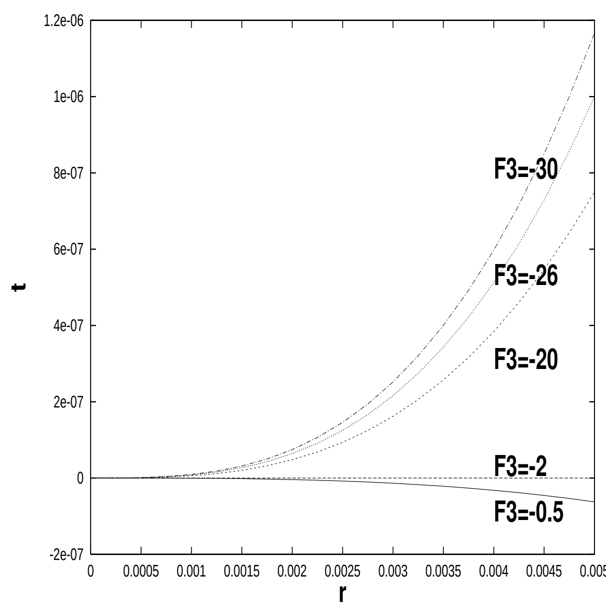

FIG. 1. Apparent horizon curves near $r = 0$ for the $n = 3$ case, with $F_0 = 1$. The labels on the curves give the values of F_3, the nonvanishing coefficient quantifying the shear. A black hole forms if $F_3 \geq -2$.

When the dust density is homogeneous, the apparent horizon starts developing earlier than the epoch of singularity formation, which is then fully hidden within a black hole. There is no density gradient, and no shear. On the other hand, if a density gradient is present at the center, then the trapped surface development is delayed via shear, and, depending on the "strength" of the density gradient/shear at the center, this may expose the singularity. It is the rate of decrease of shear as we approach the center $r = 0$ on the initial surface $t = t_i$, given by Eq. (23), that determines the end-state of collapse. *When the shear falls rapidly to zero at the center, the result is necessarily a black hole; if shear falls more slowly, there is a naked singularity.* It is thus seen that naked singularities are caused by the sufficiently strong shearing forces near the singularity, as generated by the inhomogeneities in density distribution of the collapsing configuration. When shear decays rapidly near the singularity, the situation is effectively like the shear-free (and homogeneous density) case, with a black hole end-state.

It provides a useful insight to note that when a black hole forms, the apparent horizon typically springs into being as a finite-sized surface, at a finite r, then moving to the center $r = 0$. This is what happens, for example, in the Oppenheimer-Snyder black hole formation in homogeneous dust collapse. In such cases, the event horizon, which does typically start at a point, could have formed earlier than the apparent horizon. On the other hand, in the case of a naked singularity, it follows from Eqs. (15) and (21), that the apparent horizon starts at $r = 0$, and then is future directed in time, i.e. $t_{\rm ah}$ grows with increasing coordinate radius r along the apparent horizon curve $R = F$. These two behaviors of the apparent horizon curve are very different, and governed by shearing effects. A comoving observer will *not* encounter any trapped surfaces until the time of singularity formation in the naked singularity case, whereas in the black hole case, the apparent horizon typically develops *before* the epoch of singularity formation. This is what we mean by delayed formation of the apparent horizon, caused by shearing effects.

The relation between density gradients and shear may be understood via the nonlocal (or free) gravitational field. Density gradients act as a source for the electric Weyl tensor [6]

$$\mathrm{D}^b E_{ab} = \tfrac{1}{3}\mathrm{D}_a\rho\,, \tag{25}$$

where D_a is the covariant spatial derivative. (The magnetic Weyl tensor vanishes for spherical symmetry.) In turn, the gravito-electric field is a source for shear (equivalently, the shear is a gravito-electric potential [6]):

$$u^c\nabla_c\sigma_{ab} + \frac{2}{3}\Theta\sigma_{ab} + \sigma_{ac}\sigma^c{}_b - \frac{2}{3}\sigma^2 h_{ab} = -E_{ab}\,. \tag{26}$$

Thus density gradients may be directly related to shear:

$$\begin{aligned}\mathrm{D}_a\rho = -4\sigma\mathrm{D}_a\sigma - 2\Theta\mathrm{D}^b\sigma_{ab} - 3\mathrm{D}^b\left(u^c\nabla_c\sigma_{ab}\right)\\ - 3\sigma_a{}^b\mathrm{D}^c\sigma_{bc} - 3\mathrm{D}^b\left(\sigma_{ac}\sigma^c{}_b\right)\,,\end{aligned} \tag{27}$$

where we have used the shear constraint $\mathrm{D}^b\sigma_{ab} = \frac{2}{3}\mathrm{D}_a\Theta$. Equation (27) makes explicit the link between the behavior of density gradients and shear near the center, which was discussed above. The free gravitational field, which mediates this link, can also provide a covariant characterization of singularity formation. By Eqs. (23) and (26), the relative gravito-electric field E/Θ^2 (where $E^2 = \frac{1}{2}E^{ab}E_{ab}$) near $r = 0$ is given at $t = t_{\rm i}$ by

$$\left(\frac{E}{\Theta^2}\right)_{\rm i} = \frac{-7nF_n}{18\sqrt{6}F_0}\, r^n\left[1 + O(r)\right]\,. \tag{28}$$

Thus naked singularities in spherical dust collapse are signalled by a less rapid fall-off of the relative gravito-electric field as we approach the singularity. Equations (23) and (28) provide two equivalent ways of expressing the result. This specifies how much shear is sufficient to create a (locally) naked singularity.

For the case of dust collapse, the role of shear in deciding the end-state of collapse is fairly transparent. To understand how shear affects the formation of the apparent horizon for general matter fields with pressures included is much more complicated, in particular since $F = F(t, r)$, whereas $\dot{F} = 0$ for dust. In fact, even in some general classes of non-dust models (with nonzero pressure), it is possible to characterize collapse covariantly. Above we showed that *homogeneous density* implies a black hole end-state. The next logical step would be to consider models for which the *initial* density is homogeneous. For example, if the mass function is

$$F(t, r) = f(r) - R^3(t, r)\,,\ f(r) = 2r^3\,, \tag{29}$$

then Eq. (5) shows that $\rho_{\rm i}$ and $(p_{\rm r})_{\rm i}$ are constants. The density and pressure may however develop inhomogeneities as the collapse proceeds, depending on the choice of the remaining functions, including in particular the initial velocities of the collapsing shells, and the collapse may then end up in either a black hole or a naked singularity, depending on that (for a discussion on this for the case of dust collapse, we refer to [5]). In fact, we can show that *zero shear implies a black hole* for these models. By Eqs. (2), (5) and (29), the shear-free condition leads to $R'/R = 1/r$, and Eq. (5) then shows that $\rho = \rho(t)$, i.e. the density evolution is necessarily homogeneous. As shown above, the collapse thus necessarily ends in a black hole. For the class of models given by Eq. (29), whenever the collapse ends in a naked singularity, the shear must necessarily be nonvanishing. Although this class of models is somewhat special, the result indicates that the behavior of the shear remains a crucial factor even when pressures are nonvanishing.

III. CONCLUSIONS

Since black holes and naked singularities are of great interest in gravitation theory and astrophysics, it is important to understand *why* these objects develop. The physics of this needs to be probed carefully in order to make further progress towards cosmic censorship, or to understand the physical implications of naked singularities.

It would appear that the only way a singularity can be laid bare is by distorting the apparent horizon surface and so delay trapped surface formation suitably. As we have shown here, the shear provides a rather natural explanation for the occurrence of (locally) naked singularities. Our main result is that sufficiently strong shearing effects in spherical collapsing dust delay the formation of the apparent horizon, thereby exposing the strong gravity regions to the outside world and leading to a (locally) naked singularity. When shear decays rapidly near the singularity, the situation is effectively like the shear-free case, with a black hole end-state. An important point is that naked singularities can develop in quite a natural manner, very much within the standard framework of general relativity, governed by shearing effects.

In the case of spherical dust collapse, shear and density inhomogeneity are equivalent, i.e., the one implies the other. Although shear contributes positively to the focusing effect via the Raychaudhuri equation,

$$\dot{\Theta} + \frac{1}{3}\Theta^2 = -\frac{1}{2}\rho - 2\sigma^2\,, \tag{30}$$

its dynamical action can make the collapse incoherent and dispersive. (It is this feature which also plays the crucial role in avoidance of the big-bang singularity in singularity-free cosmological models [7].) *Depending on the rate of fall-off of shear near the singularity, its dispersive effect can play the critical role of delaying formation of the apparent horizon, without directly hampering the process of collapse.* The dispersive effect of shear always tends to delay formation of the apparent horizon, but is only able to expose the singularity when the shear is strong enough near the singularity.

We have considered here spherical collapse. Very little is known about nonspherical collapse, either analytically or numerically, towards determining the outcome in terms of black holes and naked singularities. However, phenomena such as trapped surface formation and apparent horizon are independent of any spacetime symmetries, and it is also clear that a naked singularity will not develop in general unless there is a suitable delay of the apparent horizon. This suggests that the shear will continue to be pivotal in determining the final fate of gravitational collapse, independently of any spacetime symmetries. In any case, our main purpose here has been to try to understand and find the physical mechanism which leads the collapse to the development of a naked singularity rather than a black hole in some of the well-known classes exhibiting such behavior. What we find is that the shear provides a covariant dynamical explanation of the phenomenon of naked singularity formation in spherical gravitational collapse.

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