

## BOUNDS ON VACUUM ENERGY DENSITY IN A GENERAL COSMOLOGICAL SCENARIO

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General limits on the cosmological constant (or equivalently the vacuum energy density) are derived for an inflationary universe. This is accomplished under the general assumption of global hyperbolicity and without the use of any special properties like spherical symmetry or homogeneity of the underlying spacetime. A clear upper limit of  $1/3$  is obtained for the vacuum energy density parameter  $\Omega_v$ , while the lower limit is found to depend on the age of the oldest object in the universe.

The inflationary scenario [1] in its several available versions claims to resolve the cosmological conundrums such as the horizon problem, monopole problem and at the same time predicts that the density parameter  $\Omega$  of the universe must be close to unity. All these models require a fine tuning of the net cosmological constant, during and after the inflationary phase. This cosmological constant  $\Lambda$  is equivalent to the vacuum energy density  $\rho_v$ ,

$$\rho_v = \Lambda c^2 / 8\pi G. \quad (1)$$

One could appeal to observations on departures from the exact Hubble law for distant galaxies and related cosmological phenomena to get more information or constraints on this remnant vacuum energy density. It is the purpose of this paper to determine the range of values for  $\rho_v$  that may exist today without any assumptions of exact symmetries for the universe or crucial dependence on the present value of  $H_0$ . In view of the uncertainties prevailing in cosmological observations today as well as the restrictive nature of exact symmetries usually adopted, it would be highly desirable to have some model-independent conclusions about the present value of  $\rho_v$ .

In order to retain the general features of the cosmological models, we relax here the following (usually adopted) assumption: that the space-like hypersurfaces evolving in time have spherical symmetry or the exact homogeneity and isotropy. In other words, we permit departures and small perturbations from the maximally symmetric model. With

regard to the global geometry we merely assume that the spacetime describing the universe admits a foliation by space-like hypersurfaces. All the physically reasonable cosmological models (e.g. Friedman or steady state) obey this criterion of global hyperbolicity [2]. In particular, the flat ( $k=0$ ) Friedman models with small perturbations will satisfy this criterion.

Einstein's equations with the cosmological constant can be written as

$$R^i_k - \frac{1}{2}\delta^i_k R = 8\pi G(T^i_k)^{\text{eff}}, \quad (2)$$

where

$$(T^i_k)^{\text{eff}} = T^i_k - (\Lambda c^2 / 8\pi G)\delta^i_k. \quad (3)$$

In Friedman models the universe is supposed to "originate" from the big-bang singularity at  $t=0$ . The age of the universe today can be expressed as

$$t_u = H_0^{-1} f(q_0), \quad (4)$$

where  $H_0$  is the present value of the Hubble constant and  $q_0$  is the deceleration parameter. The question arises as to whether we can introduce the concept of age of the universe for the more general scenario described above. In fact, this turns out to be possible if we assume that there are no modes carrying negative energy. This energy condition may be formulated [3] as

$$(T_{ij} - \frac{1}{2}g_{ij}T)V^i V^j \geq 0, \quad (5)$$

where  $V^i$  is a unit time-like vector.

Consider now the present epoch, which is characterised by a space-like hypersurface  $S_0$  which crosses our world line. Let us examine the evolution of non-space-like trajectories from  $S_0$  into the past: such trajectories could represent, for example, the world lines of galaxies or other material particles. Two major factors enter into determining the evolution of these trajectories. In the first place, the stress-energy density encountered by non-space-like curves induce gravitational focusing effects [3]. These effects are characterized by the expansion "parameter"  $\theta$  of the congruence of time-like geodesics which are orthogonal to  $S_0$  and which obey the Raychaudhuri equation

$$d^2x/dt^2 + F(t)x = 0, \quad (6)$$

where  $x$  is defined by  $\theta = x^{-1} dx/dt$  and

$$F(t) = \frac{1}{3}(R_{ij}V^iV^j + 2\sigma^2), \quad (7)$$

the quantity  $\sigma$  represents the shear of the congruence.

The second factor which we have to consider concerns the focal or conjugate points that arise due to the gravitational focusing mentioned above. The conjugate points arise when infinitesimally separated time-like geodesics of the congruence representing the world lines of material particles start intersecting each other. Specifically, a point  $q$  along a time-like geodesic  $\gamma$  is said to be conjugate to  $S_0$  if one of the Jacobi vector fields describing the separation of  $\gamma$  from nearby trajectories of the congruence, vanish at the point  $q$ . Such a situation arises when the expansion  $\theta$  of the congruence becomes infinite at  $q$ . Alternatively, a zero of the differential equation (6) would represent a conjugate point, since such a zero indicates the divergence of  $\theta$  at that point for the congruence of time-like geodesics under consideration.

With this background we appeal to the following important property of globally hyperbolic spacetimes which describe our *general* cosmological universe [3]. For any event  $q$  of the spacetime, there is a time-like geodesic from  $q$  orthogonal to  $S_0$  along which the proper lengths of all time-like curves from  $q$  to  $S_0$  are maximised, and this geodesic does not contain any conjugate points between  $S_0$  and  $q$ . This constraint must be satisfied by all material trajectories from  $S_0$  when we study their evolution into the past.

We are, therefore, led to the conclusion that the

material trajectories must end up in a singularity before a conjugate point is encountered [4]. It can be shown [4] that if  $F(t) \geq k^2 \geq 0$ , then the maximal possible extension into the past of any non-space-like curve from  $S_0$  is given by  $\frac{1}{2}\pi\sqrt{3/k}$ . If we compute  $F(t)$  with the help of eq. (2) neglecting the contribution due to material pressure and shear (both of which would only add to the gravitational focusing), we get

$$R_{ij}V^iV^j \geq 4\pi G\rho_m - \Lambda c^2. \quad (8)$$

Using the above result we obtain the maximum possible age  $t_{\max}$ :

$$\begin{aligned} t_{\max} &= \frac{1}{2}\pi \sqrt{\frac{3}{4\pi G\rho_m - \Lambda c^2}} \\ &= \frac{1}{4}\sqrt{\frac{\pi}{G}} \sqrt{\frac{3}{\rho_m - 2\rho_v}}. \end{aligned} \quad (9)$$

We emphasize that only global hyperbolicity was assumed in obtaining the above result.

We further assume that the cosmological constant is a residue from an inflationary phase. Since inflation implies  $\rho = \rho_c$ , we can write

$$\rho_c = \rho_m + \rho_v = \rho. \quad (10)$$

For a dust-filled universe, the pressure term arises purely from vacuum energy density:

$$p = -\rho_v. \quad (11)$$

Defining

$$\Omega_v = \rho_v/\rho_c, \quad \Omega_m = \rho_m/\rho_c, \quad (12)$$

we get

$$t_{\max}^2 = \frac{3\pi}{16G\rho_c} \frac{1}{\rho_c - 3\rho_v}, \quad (13)$$

or, in other words,

$$1 - 3\Omega_v = \frac{3\pi}{16G\rho_c t_{\max}^2} = \frac{1}{2} \left( \frac{\pi}{H_0 t_{\max}} \right)^2, \quad (14)$$

Since the r.h.s.  $> 0$ , a clear upper limit on  $\Omega_v$  is given by

$$\Omega_v < \frac{1}{3}. \quad (15)$$

Clearly  $t_{\max} > t_u$  where  $t_u$  is the age of the oldest known objects in the universe (e.g. globular clus-

ters). Combining this constraint with (15) we get the bounds

$$\frac{1}{3} [1 - \frac{1}{2} (\pi/H_0 t_u)^2] < \Omega_v < \frac{1}{3} . \quad (16)$$

Scaling  $t_u$  and  $H_0$  as

$$t_u = 1.5 \times 10^{10} p \text{ years},$$

$$H_0 = 100 h_0 \text{ km s}^{-1} \text{ Mpc}^{-1} ,$$

eq. (16) can be written as

$$\frac{1}{3} (1 - 2/h_0^2 p^2) < \Omega_v < \frac{1}{3} . \quad (17)$$

It is seen that a positive lower bound on  $\Omega_v$  is obtained for

$$ph_0 > \sqrt{2} \approx 1.4 . \quad (18)$$

Thus, for example, for  $h_0 = 1$ , observations of ages as high as 21 billion years would imply a positive lower bound on  $\Omega_v$ . As the consistent age limit for the universe is usually proposed to be in the range [5]  $13.8 \text{ byr} \leq t_u \leq 24 \text{ byr}$ , it is conceivable that  $\Omega_v$  is bounded from below. On the other hand, if quasars are taken to be old objects in the universe having ages of the order of 30 byr then eq. (16) would provide definitive lower limits for  $\Omega_v$ .

The scenario assumes a new dimension if dynamical considerations are introduced. It is well known that  $\rho = \rho_c$  cannot be achieved by luminous matter alone. The "dark matter" which has to be invoked can come in many forms, conveniently classified as "cold" or "hot". If either one of these choices proves to be successful, still our conclusions would be valid, however, investigations show that neither "cold" nor "dark" matter by itself can account for observations at all scales. This has motivated the study of scenarios in which a cold dark matter candidate decays in the recent past. In such models the present day universe will be radiation dominated because of the existence of relativistic decay products [6].

In such a situation, we must have

$$\rho = \frac{1}{3} \rho_m - \rho_v , \quad (19)$$

giving (compare with eq. (13))

$$t_{\text{max}}^2 = \frac{3\pi}{16G} \frac{1}{2\rho_m - 5\rho_v} . \quad (20)$$

This, in turn, leads to the following bounds on  $\Omega_v$ :

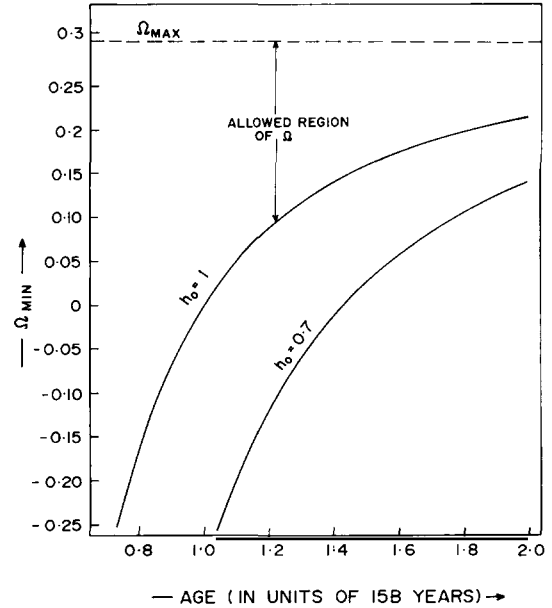


Fig.1. The minimum allowed value for the density parameter as a function of the age of the universe.

$$\frac{2}{3} [1 - (\pi/2H_0 t_u)^2] < \Omega_v < \frac{2}{3} . \quad (21)$$

Again following the earlier convention, it is seen that eq. (21) gives a positive lower bound to  $\Omega_v$  when  $ph_0 \gtrsim 1$ . It is clear that for a range of values of  $h_0$  and  $p$  this inequality will be satisfied and hence  $\Omega_v$  would be bounded from below (see fig. 1),

For comparison, we point out that  $\Omega_v$  and the cosmological constant are related by

$$\Lambda = (3.2 \times 10^{-56} \text{ cm}^{-2}) h_0^2 \Omega_v . \quad (22)$$

We have derived bounds on the cosmological constant or equivalently the vacuum energy density in an inflationary scenario without making any explicit assumptions about the exact symmetries of the underlying cosmological model. In particular, we do not demand the space-like hypersurfaces evolving in time to have any special properties like spherical symmetry or homogeneity and isotropy, except to satisfy the criterion of global hyperbolicity. It is then possible to set an upper bound to the (dimensionless) vacuum energy density  $\Omega_v$  and to derive conditions for obtaining a positive lower bound for  $\Omega_v$  from plausible estimates of the ages of the oldest objects in the universe.

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