

Initial state of matter fields and trans-Planckian physics: Can CMB observations disentangle the two?

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The standard, scale-invariant, inflationary perturbation spectrum will be modified if the effects of trans-Planckian physics are incorporated into the dynamics of the matter field in a phenomenological manner, say, by the modification of the dispersion relation. The spectrum also changes if we retain the standard dynamics but modify the initial quantum state of the matter field. We show that, given *any* spectrum of perturbations, it is possible to choose a class of initial quantum states which can exactly reproduce this spectrum with the standard dynamics. We provide an explicit construction of the quantum state which will produce the given spectrum. We find that the various modified spectra that have been recently obtained from “trans-Planckian considerations” can be constructed from suitable squeezed states above the Bunch-Davies vacuum in the standard theory. Hence, the CMB observations can, at most, be useful in determining the initial state of the matter field in the standard theory, but it can *not* help us to discriminate between the various Planck scale models of matter fields. We study the problem in the Schrodinger picture and determine the criterion for negligible back reaction due to modified initial conditions.

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I. THE TANGLED WEB

The inflationary scenario [1–3] is at present the most attractive paradigm for generating the initial small scale inhomogeneities [4,5]. These perturbations leave their imprints as anisotropies in the Cosmic Microwave Background (CMB) [6] and later evolve, due to gravitational instability, into the large-scale structures that we see around us today. While there is no natural particle physics candidate for generating the inflationary phase, a single (or a few) “inflaton” field(s), with fine-tuned designer couplings, is (are) often introduced in order to reproduce the observed magnitude and shape of the perturbation spectrum.

In many of the models of inflation [3], the period of acceleration lasts sufficiently long so that length scales that are of cosmological interest today would have emerged from sub-Planckian length scales at the beginning of inflation. This suggests that physics at the very high energy scales can, in principle, modify the primordial perturbation spectrum [7] and these modifications can—in turn—leave their signatures on the CMB [8]. This has led to a considerable interest in understanding the effects of Planck scale physics on the inflationary perturbation spectrum [7–20] and the CMB [21–23].

Metric fluctuations during the inflationary epoch can be modeled by a quantized, massless, and minimally coupled scalar field [4,5,24]. In the absence of a workable quantum theory of gravity, the Planck scale effects on the perturba-

tion spectrum have to be studied by phenomenologically modifying the dynamics of the scalar field to take into account the quantum gravitational effects (for an early attempt in this direction, see Ref. [7]). The high energy models of the quantum scalar field that have been popular in the literature either introduce new features in the dispersion relation [8–12] or modify the standard uncertainty principle [13,14] or assume that the spacetime coordinates are noncommutative [15]. (For other approaches, see Refs. [16–19] and, for a recent review of many of these approaches, see Ref. [20]). Some of these models have been utilized to evaluate not only the Planckian corrections to the standard, scale-invariant perturbation spectrum, but also the resulting signatures on the CMB [21–23]. This suggests the possibility that sufficiently accurate measurements of the CMB anisotropies can help us understand physics beyond the Planck scale.

There is, however, one serious difficulty with this approach, which we shall now briefly describe.

We begin by noticing that, in a Friedmann universe, each mode $q_{\mathbf{k}}$ of the scalar field, labeled by the wave vector \mathbf{k} , evolves as an independent oscillator with time-dependent parameters that are related to the expansion factor $a(t)$. Given the quantum state $\psi_{\mathbf{k}}[q_{\mathbf{k}}, t_i(\mathbf{k})]$ for the mode $q_{\mathbf{k}}$ at an initial time $t_i(\mathbf{k})$, one can obtain the state at a later time t by

$$\psi_{\mathbf{k}}(q_{\mathbf{k}}, t) = \int d\bar{q}_{\mathbf{k}} K[q_{\mathbf{k}}, t; \bar{q}_{\mathbf{k}}, t_i(\mathbf{k})] \psi_{\mathbf{k}}[\bar{q}_{\mathbf{k}}, t_i(\mathbf{k})], \quad (1)$$

where $K(q_{\mathbf{k}}, t; \bar{q}_{\mathbf{k}}, t_i)$ is the (path integral) kernel for an oscillator with time-dependent parameters, which can be written down in terms of the classical solution (see, for

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example, Ref. [25]). To keep our discussion general, we have allowed for the possibility that the initial quantum state of each of the oscillators is specified at different times so that t_i can depend on \mathbf{k} . For instance, one may choose to specify the quantum state of each oscillator when the proper wavelength of that mode is equal to the Planck length [16,26]. Of course, when the initial state of all the modes are specified at a given time, $t_i(\mathbf{k}) = t_i$ will be independent of \mathbf{k} . The *dynamics* of the system are completely specified by the kernel K .

It is obvious that the mathematics of a free scalar field in a Friedmann universe is trivial and is no more complicated than that of an oscillator with time-dependent parameters. The perennial interest in this problem (allowing so many e-prints to be written) arises from two conceptual issues:

- (1) To make any predictions, we need to know $\psi_{\mathbf{k}}(q_{\mathbf{k}}, t)$, which, in turn, requires knowing $\psi_{\mathbf{k}}[\bar{q}_{\mathbf{k}}, t_i(\mathbf{k})]$. We have absolutely no idea what to use for $\psi_{\mathbf{k}}[\bar{q}_{\mathbf{k}}, t_i(\mathbf{k})]$ and so different choices (usually called “vacuum states,” which is only a manner of speaking) can be investigated.
- (2) There are infinite number of such oscillators, leading to the standard, unresolved, issues of regularization in quantum field theory. To make any progress, we need to *make an assumption* regarding $\psi_{\mathbf{k}}[\bar{q}_{\mathbf{k}}, t_i(\mathbf{k})]$ and our results are only as valid as this assumption.

If we now further modify the dynamics (due to a phenomenological input regarding trans-Planckian physics), we will be changing the form of the kernel K . But, since we can only observe the integrated effect of K and $\psi_{\mathbf{k}}[\bar{q}_{\mathbf{k}}, t_i(\mathbf{k})]$, the observations can tell us something about the K (and trans-Planckian physics) only if we assume something about $\psi_{\mathbf{k}}[\bar{q}_{\mathbf{k}}, t_i(\mathbf{k})]$. The usual assumption is to consider the initial quantum state to be the Bunch-Davies vacuum [27], but it is only an assumption. The crucial question is whether the effects of trans-Planckian physics can be mimicked by a different choice of the initial state other than the Bunch-Davies vacuum.

We shall show that *any* (modified) spectrum of fluctuations can be reproduced from a suitably chosen squeezed state above the Bunch-Davies vacuum in the standard theory. We shall provide an explicit construction of the state for any spectrum of the perturbations that is observed. (We need to emphasize here that this reconstruction is hardly unique. As we shall discuss later, even within the class of squeezed states, there is some degree of freedom in choosing the parameters for reproducing a given spectrum. Obviously, the amount of freedom will be enlarged if we also choose to consider other classes of states, such as, for instance, n -particle states.) So, if some specific deviation from the standard scale-invariant spectrum is seen in the CMB, a conservative interpretation will be to attribute it to a deviation from the standard initial state of the theory. Unless this possibility is ruled out, one cannot claim that the observation supports, say, a particular model of trans-

Planckian phenomenology. Motivated by this result, we argue that the CMB can at most help us identify the quantum state of the scalar field in the standard theory, but it can *not* aid us in discriminating between the various Planck scale models of the matter fields.

The remainder of this paper is organized as follows. In Section II, we set up the formalism to study the evolution of a Gaussian quantum state in a Friedmann universe and apply the formalism to evaluate the perturbation spectrum in power-law inflation. In Section III, we show that any modified spectrum can be reproduced from a suitable squeezed state above the Bunch-Davies vacuum in the standard theory. We explicitly discuss four modified spectra that have recently been considered in the literature. In Section IV, we evaluate the energy density in these excited states and examine whether these modified spectra also lead to a large backreaction on the inflating background. Finally, in Section V, we conclude with a discussion on the wider implications of our analysis.

Our conventions and notations are as follows. We shall set $\hbar = c = 1$ and the metric signature we shall adopt is $(+, -, -, -)$.

II. EVOLUTION OF THE QUANTUM STATE AND THE POWER SPECTRUM

A. General formalism

Consider a flat Friedmann universe described by the line-element

$$ds^2 = dt^2 - a^2(t)dx^2 = a^2(\eta)(d\eta^2 - dx^2), \quad (2)$$

where t is the cosmic time, $a(t)$ is the scale factor, and η denotes the conformal time with $d\eta = dt/a(t)$. The scalar as well as the tensor perturbations during the inflationary epoch can be modeled by a massless and minimally coupled, real scalar field, say, Φ , governed by the action [1,2,4,5,24,28]

$$S[\Phi] = \frac{1}{2} \int d^4x \sqrt{-g} \partial_\mu \Phi \partial^\mu \Phi. \quad (3)$$

The homogeneity and isotropy of the Friedmann metric (2) allows us to decompose the scalar field Φ as

$$\Phi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \tilde{q}_{\mathbf{k}}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (4)$$

As $\tilde{q}_{\mathbf{k}}$ is complex, we have 2 degrees of freedom corresponding to the real and imaginary parts of $\tilde{q}_{\mathbf{k}}$ for each \mathbf{k} . If we write $\tilde{q}_{\mathbf{k}} = (\tilde{A}_{\mathbf{k}} + i\tilde{B}_{\mathbf{k}})$, then, since Φ is a real scalar field, we can relate the variables for \mathbf{k} to that for $-\mathbf{k}$ as follows: $\tilde{A}_{\mathbf{k}}^* = \tilde{A}_{-\mathbf{k}}$ and $\tilde{B}_{\mathbf{k}}^* = -\tilde{B}_{-\mathbf{k}}$. Evidently, only half the modes are independent degrees of freedom. Therefore, we can work with a new set of real modes $q_{\mathbf{k}}$ for all values of \mathbf{k} with a suitable redefinition, say, by taking $q_{\mathbf{k}} = \tilde{A}_{\mathbf{k}}$ for one half of \mathbf{k} and $q_{-\mathbf{k}} = \tilde{B}_{\mathbf{k}}$ for the other half. Then, in terms of real the variables $q_{\mathbf{k}}$, the action (3) can be ex-

pressed as follows:

$$S[\Phi] = \frac{1}{2} \int d^3\mathbf{k} \int d\eta a^2 (q_{\mathbf{k}}'^2 - k^2 q_{\mathbf{k}}^2), \quad (5)$$

where the primes denote differentiation with respect to the conformal time η and $k = |\mathbf{k}|$. The action (5) describes a collection of independent oscillators with time-dependent mass a^2 and frequencies k .

A few clarifying remarks concerning our choice of the Fourier variable in Eq. (4) needs to be made at this stage of our discussion. It is sometimes useful to keep track of the real and imaginary parts of the $\tilde{q}_{\mathbf{k}}$ separately. In our case, this is unnecessary and we can define our system in terms of real $q_{\mathbf{k}}$. Also, in the literature, one usually finds the Fourier decomposition in Eq. (4) expressed in terms of $(\tilde{q}_{\mathbf{k}}/a)$ rather than with just $\tilde{q}_{\mathbf{k}}$ as we have done. A decomposition in terms of $(\tilde{q}_{\mathbf{k}}/a)$ leads to oscillators that have a unit mass, but a time-dependent frequency, say, $\omega_{\mathbf{k}}^2$. But, we find that the frequency $\omega_{\mathbf{k}}$ of this oscillator turns imaginary (i.e. the normal oscillator turns to an inverted oscillator) at super Hubble scales. In contrast, if we decompose the scalar field as in Eq. (4), then the mass of the oscillators varies as a^2 , but the frequency remains constant. In other words, in such a case, the oscillator remains rightside up and does not turn upside down during its evolution. But, such a decomposition has the drawback that, in an inflationary background, the parameters of the oscillator do not go to a constant value at the sub-Hubble scales, as would be ideally desired. We feel that, despite this drawback, one should work with an oscillator whose frequency remains real right through the course of its evolution. (This becomes particularly important, if, for instance, one is interested in comparing the wave function of the mode with, say, the instantaneous ground and excited states of an oscillator.) We would like to strongly emphasize here that, barring the choice of the Fourier variable for the scalar field, our results and interpretation are completely equivalent to the conventional one.

In the Schrodinger picture, the scalar field Φ can be quantized by quantizing each independent oscillator $q_{\mathbf{k}}$. The Hamiltonian corresponding to the \mathbf{k} th oscillator is given by

$$H_{\mathbf{k}} = \frac{p_{\mathbf{k}}^2}{2a^2} + \frac{1}{2} a^2 k^2 q_{\mathbf{k}}^2, \quad (6)$$

where $p_{\mathbf{k}}$ is the momentum conjugate to the coordinate $q_{\mathbf{k}}$. Therefore, each of the oscillators satisfy the Schrodinger equation

$$i \frac{\partial \psi_{\mathbf{k}}}{\partial \eta} = -\frac{1}{2a^2} \frac{\partial^2 \psi_{\mathbf{k}}}{\partial q_{\mathbf{k}}^2} + \frac{1}{2} a^2 k^2 q_{\mathbf{k}}^2 \psi_{\mathbf{k}} \quad (7)$$

and the complete quantum state of the field is described by a wave function that is a product of $\psi_{\mathbf{k}}$ for all \mathbf{k} . Equivalently, the time evolution of the wave function can

be described by Eq. (1) in terms of the kernel $K[q_{\mathbf{k}}, t; \bar{q}_{\mathbf{k}}, t_i(\mathbf{k})]$.

As we do not expect a large-scale, spatially inhomogeneous classical scalar field to be present in the Universe, it is conventional to assume that the expectation value $\langle \psi_{\mathbf{k}} | \hat{q}_{\mathbf{k}} | \psi_{\mathbf{k}} \rangle$ vanishes in the quantum state of the field for $\mathbf{k} \neq 0$. (Since $\langle \psi_{\mathbf{k}} | \hat{q}_{\mathbf{k}} | \psi_{\mathbf{k}} \rangle$ satisfies the classical equations of motion, this condition can be satisfied at all times if suitable initial conditions are imposed at an early epoch.) As we shall discuss below, we will be assuming that the wave function $\psi_{\mathbf{k}}$ is a Gaussian in $q_{\mathbf{k}}$ and, in such a state, in free field theory, all the higher order correlation functions are completely determined by the two-point functions. Therefore, the power spectrum of the perturbations per logarithmic interval, viz. $[k^3 \mathcal{P}_{\Phi}(k)]$, is given by (see, for instance, Ref. [28])

$$k^3 \mathcal{P}_{\Phi}(k) = \frac{k^3}{2\pi^2} \int_{-\infty}^{\infty} dq_{\mathbf{k}} |\psi_{\mathbf{k}}(q_{\mathbf{k}}, \eta)|^2 q_{\mathbf{k}}^2. \quad (8)$$

The quantity on the right-hand side of this equation depends on time and the spectrum is to be evaluated at Hubble exit, i.e. when the physical wavelength $(k/a)^{-1}$ of the mode corresponding to the wavenumber \mathbf{k} is comparable to the Hubble radius H^{-1} , where $H = (a'/a^2)$.

Let us now consider the problem of determining the wave function $\psi_{\mathbf{k}}(q_{\mathbf{k}}, \eta)$. For a time-dependent oscillator, there is no concept of a unique ground (“vacuum”) state unless the parameters describing the oscillator go to a constant value asymptotically—which, in general, it does not, for the Friedmann universe. There is, however, a class of solutions to the time-dependent harmonic oscillator which are *form invariant* in the sense that the $q_{\mathbf{k}}$ dependence of the wave function $\psi_{\mathbf{k}}(q_{\mathbf{k}}, \eta)$ is the same at all η . A simple solution that has such a property is an exponential of a quadratic function of $q_{\mathbf{k}}$ and such states have been extensively investigated in the past in different contexts (see, for instance, Refs. [29,30]). When $\langle \psi_{\mathbf{k}} | \hat{q}_{\mathbf{k}} | \psi_{\mathbf{k}} \rangle = 0$, the mean value of the Gaussian vanishes and the quantum state of the mode can be described by the wave function [28–30]

$$\psi_{\mathbf{k}}(q_{\mathbf{k}}, \eta) = N_{\mathbf{k}}(\eta) \exp[-R_{\mathbf{k}}(\eta) q_{\mathbf{k}}^2], \quad (9)$$

where $N_{\mathbf{k}}(\eta)$ and $R_{\mathbf{k}}(\eta)$ are complex quantities. The normalization condition on the wave function then relates $N_{\mathbf{k}}$ and $R_{\mathbf{k}}$ as follows: $|N_{\mathbf{k}}|^2 = [(R_{\mathbf{k}} + R_{\mathbf{k}}^*)/\pi]^{1/2}$. Therefore, the only nontrivial aspect of the quantum state is encoded in the time dependence of the function $R_{\mathbf{k}}(\eta)$. It can be shown that (for details, see Ref. [31,32]), if we introduce a function $\mu_{\mathbf{k}}(\eta)$ through the relation $R_{\mathbf{k}} = -(ia^2/2) \times (\mu_{\mathbf{k}}'/\mu_{\mathbf{k}})$, then $\mu_{\mathbf{k}}$ satisfies the differential equation

$$\mu_{\mathbf{k}}'' + 2 \frac{a'}{a} \mu_{\mathbf{k}}' + k^2 \mu_{\mathbf{k}} = 0 \quad (10)$$

which is the same as the classical equation of motion satisfied by the oscillator variable $q_{\mathbf{k}}$. We find that the

perturbation spectrum (8) corresponding to the wave function (9) can be expressed as (see, for instance, Ref. [31,32])

$$k^3 \mathcal{P}_\Phi(k) = \frac{k^3}{2\pi^2} \left(\frac{|\mu_k|^2}{W(k)} \right), \quad (11)$$

where $W(k)$ is a k -dependent constant determined by the Wronskian condition for μ_k , viz.

$$(\mu_k \mu_k'^* - \mu_k' \mu_k^*) = -[iW(k)/a^2]. \quad (12)$$

Since the differential Eq. (10) has real coefficients, if s_k is a solution, so is s_k^* and the general solution is a linear superposition of the form, say, $\mu_k = [\mathcal{A}(k)s_k + \mathcal{B}(k)s_k^*]$. The quantum state $\psi_{\mathbf{k}}(q_{\mathbf{k}}, \eta)$, however, depends only on R_k which is independent of the overall scaling of μ_k . This feature translates into the power spectrum (11) as well; a global scaling of μ_k also changes the Wronskian $W(k)$ leaving $[|\mu_k|^2/W(k)]$ invariant. Hence, we can ignore the overall scaling in μ_k . We shall set $\mathcal{A}(k)$ to unity and choose the Wronskian $W(k)$ to be

$$W(k) = 1 - |\mathcal{B}(k)|^2. \quad (13)$$

Then, the power spectrum (11) reduces to

$$k^3 \mathcal{P}_\Phi(k) = \frac{k^3}{2\pi^2} \left(\frac{[1 + |\mathcal{B}(k)|^2]|s_k|^2 + 2\text{Re}[\mathcal{B}(k)s_k^{*2}]}{1 - |\mathcal{B}(k)|^2} \right). \quad (14)$$

(In the Heisenberg picture, one usually introduces the Bogoliubov coefficients $\alpha(k)$ and $\beta(k)$. These coefficients are related to $\mathcal{B}(k)$ through the relation: $\mathcal{B}(k) = [\beta(k)/\alpha(k)]$. In terms of $\alpha(k)$ and $\beta(k)$, the Wronskian condition (13) reduces to the standard normalization condition, viz. $(|\alpha(k)|^2 - |\beta(k)|^2) = 1$.) The dynamics impose no restrictions on the form of $\mathcal{B}(k)$, which is related to the choice of the quantum state at any specified time. The spectrum evidently depends on $\mathcal{B}(k)$ and with a suitable choice for $\mathcal{B}(k)$ we can generate any given, reasonable spectrum.

B. Perturbation spectrum in power-law and exponential inflation

1. Standard initial conditions

Let us now apply the above formalism to the case of power-law inflation. Power-law inflation corresponds to the situation wherein the scale factor $a(t)$ grows with t as

$$a(t) = a_0 t^p, \quad (15)$$

where $p > 1$. In terms of the conformal time η , this scale factor can be written as

$$a(\eta) = (-\mathcal{H}\eta)^{(\gamma+1)}, \quad (16)$$

where γ and \mathcal{H} are given by

$$\gamma = -\left(\frac{2p-1}{p-1}\right) \quad \text{and} \quad \mathcal{H} = (p-1)a_0^{1/p}. \quad (17)$$

Note that $\gamma \leq -2$ with $\gamma = -2$ corresponding to exponential inflation. Also, the quantity \mathcal{H} denotes the characteristic energy scale associated with inflation and, in the case of exponential inflation, it exactly matches the Hubble scale.

On writing $\mu_k = (f_k/a)$ in the differential Eq. (10), we find that f_k satisfies the following equation:

$$f_k'' + \left[k^2 - \left(\frac{a''}{a} \right) \right] f_k = 0. \quad (18)$$

In a Friedmann universe described by the scale factor (16), the general solution to this differential equation can be written as (see, for e.g., Ref. [33], p. 362)

$$f_k(\eta) = \frac{\sqrt{\pi\eta}}{2} \left(e^{-i\pi\gamma/2} H_{-(\gamma+\frac{1}{2})}^{(1)}(k\eta) + \mathcal{B}(k) e^{i\pi\gamma/2} H_{-(\gamma+\frac{1}{2})}^{(2)}(k\eta) \right), \quad (19)$$

where $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are the Hankel functions of the first and the second kind, respectively. The k -dependent constant $\mathcal{B}(k)$ is to be fixed by choosing suitable initial conditions for each of the modes. For the above solution, it can be easily shown that the Wronskian condition (12) leads to the relation (13) between $\mathcal{B}(k)$ and $W(k)$.

Let us first briefly review the standard theory (see, for e.g., Ref. [28]) in which the initial conditions are imposed on sub-Hubble scales, i.e. when the physical wavelengths $(k/a)^{-1}$ of the modes are much smaller than the Hubble radius H^{-1} . A natural choice for the initial condition will be the one in which each of the oscillators $q_{\mathbf{k}}$ is in its ground state at sub-Hubble scales. This condition implies that the wave function (9) has the following asymptotic form:

$$\lim_{(k/aH) \rightarrow \infty} \psi_{\mathbf{k}}(q_{\mathbf{k}}, \eta) \rightarrow \left(\frac{ka^2}{\pi} \right)^{1/4} \exp\left(-\frac{k}{2} a^2 q_{\mathbf{k}}^2 + i \frac{k}{2} \eta \right) \quad (20)$$

which, in turn, requires that, as $(k/aH) \rightarrow \infty$, we need to have $R_k \rightarrow (ka^2/2)$ and $N_k \rightarrow (ka^2/\pi)^{1/4} e^{-ik\eta/2}$. These conditions can be satisfied provided $f_k \rightarrow (e^{ik\eta}/\sqrt{2k})$ as $(k/aH) \rightarrow \infty$ (cf. Refs. [31,32]) This can be achieved by setting $\mathcal{B}(k) = 0$ in Eq. (19), so that we have

$$f_k(\eta) = \left(\frac{\sqrt{\pi\eta}}{2} \right) e^{-i\pi\gamma/2} H_{-(\gamma+\frac{1}{2})}^{(1)}(k\eta) \quad (21)$$

and this choice corresponds to what is known as the Bunch-Davies vacuum [27]. Note that, according to the Eq. (13), $\mathcal{B}(k) = 0$ implies $W(k) = 1$. Therefore, on substituting the above f_k in Eq. (11) and imposing the condition of Hubble exit, viz. that $(k/a) = H$, we obtain the spectrum of perturbations to be (for a recent discussion, see, for e.g., Ref. [34])

$$k^3 \mathcal{P}_\Phi(k) = C \left(\frac{\mathcal{H}^2}{2\pi^2} \right) \left(\frac{k}{\mathcal{H}} \right)^{2(\gamma+2)}, \quad (22)$$

where C is given by

$$C = \left(\frac{\pi}{4}\right) \left| H_{-(\gamma+\frac{1}{2})}^{(1)}[(\gamma+1)] \right|^2 |(\gamma+1)|^{-(2\gamma+1)}. \quad (23)$$

In the case of exponential inflation, corresponding to $\gamma = -2$, the perturbation spectrum in Eq. (22) reduces to

$$k^3 \mathcal{P}_\Phi(k) = \frac{\mathcal{H}^2}{2\pi^2} \quad (24)$$

which is a spectrum that is exactly scale-invariant. (Note that, for exponential inflation, $\mathcal{H} = H$.)

Let us now consider a more general situation with $\mathcal{B}(k) \neq 0$. It is convenient to write

$$\mathcal{B}(k) = B(k) \exp[ib(k)], \quad (25)$$

so that the resulting power spectrum can be expressed as

$$k^3 \mathcal{P}_\Phi(k) = C \left(\frac{\mathcal{H}^2}{2\pi^2}\right) \left(\frac{k}{\mathcal{H}}\right)^{2(\gamma+2)} [1 - B^2(k)]^{-1} (1 + B^2(k) + 2B(k) \cos[b(k) + \pi\gamma - 2\theta]), \quad (26)$$

where θ is the phase of the Hankel function $H_\nu^{(1)}$ at Hubble exit, given by the relation

$$H_{-(\gamma+\frac{1}{2})}^{(1)}[(\gamma+1)] = \left| H_{-(\gamma+\frac{1}{2})}^{(1)}[(\gamma+1)] \right| e^{i\theta}. \quad (27)$$

If we write $\cos[b(k) + \pi\gamma - 2\theta] = d(k)$, then the power spectrum (26) reduces to

$$k^3 \mathcal{P}_\Phi(k) = C \left(\frac{\mathcal{H}^2}{2\pi^2}\right) \left(\frac{k}{\mathcal{H}}\right)^{2(\gamma+2)} \left(\frac{1 + B^2(k) + 2B(k)d(k)}{1 - B^2(k)}\right), \quad (28)$$

where $-1 \leq d(k) \leq 1$.

In the next section, we shall show that *any* modified spectrum obtained from a high energy model can be constructed in the standard theory by simply choosing suitable forms for the functions $B(k)$ and $b(k)$. But, before we do that, we shall briefly comment on an alternative procedure for imposing the initial conditions wherein the initial condition for different modes are imposed at different times.

2. Specifying initial conditions when $[k/a(\eta_k)] \simeq L_P^{-1}$

In the last section, we chose the quantum state by imposing a condition as $(k/aH) \rightarrow \infty$ and one may question whether we are sure of the short distance, sub-Planck scale physics well enough to make this choice. An alternative procedure, discussed extensively in the literature [16], attempts to address this question by choosing to impose the initial conditions *for the different oscillators at different times*. Specifically, one chooses the initial condition for the oscillator $q_{\mathbf{k}}$ at a time η_k such that $[k/a(\eta_k)] \simeq L_P^{-1}$, where L_P is the Planck length. The initial state of the oscillator uniquely determines the function $\mathcal{B}(k)$ and the corresponding power spectrum can then be

constructed using Eq. (26). However, it should be emphasized here that, apart from the fact that it has to be consistent with the observations, there exists no restrictions on the initial state to be chosen at η_k .

To illustrate the point, let us consider the case of exponential expansion for which $a(\eta) = (-\mathcal{H}\eta)^{-1}$. Let us assume that the oscillator $q_{\mathbf{k}}$ is in the ground state at a given time, say, η_i , which then requires that, at this instant, $R_k = -(ia^2/2)(\mu'_k/\mu_k) = (ka^2/2)$. For the case of exponential expansion, the general solution for f_k as given by Eq. (19) can be expressed in terms of simple functions as follows (see, for instance, Ref. [33], pp. 437–438):

$$f_k(\eta) = \frac{1}{\sqrt{2k}} \left[\left(1 + \frac{i}{k\eta}\right) e^{ik\eta} + \mathcal{B}(k) \left(1 - \frac{i}{k\eta}\right) e^{-ik\eta} \right]. \quad (29)$$

On imposing the condition $R_k = (ka^2/2)$ at η_i , we can determine the function $\mathcal{B}(k)$ to be

$$\mathcal{B}(k) = (1 + 2ik\eta_i)^{-1} e^{2ik\eta_i}. \quad (30)$$

Our original choice of the Bunch-Davies vacuum corresponds to assuming $\eta_i \rightarrow -\infty$ so that $\mathcal{B}(k)$ vanishes for all k . The modified procedure will be to choose η_i differently for different k by imposing the initial condition at, say, $[k/a(\eta_k)] = L_P^{-1}$. For exponential inflation, this translates to $(k\eta_k) = -(\mathcal{H}L_P)^{-1} \equiv -\xi^{-1}$, a constant. For $\eta_i = \eta_k$, the $\mathcal{B}(k)$ above reduces to [16,20]

$$\mathcal{B}(k) = [1 - (2i/\xi)]^{-1} e^{-(2i/\xi)} \quad (31)$$

which is a *constant independent of k* .

We would like to discuss here in some detail a few points concerning the power spectrum resulting from the above $\mathcal{B}(k)$. Let us now assume that the Hubble exit of the modes occur at, say, $(k/a) = (zH)$. Then, the power spectrum corresponding to the above $\mathcal{B}(k)$ is given by

$$k^3 \mathcal{P}_\Phi(k) = \left(\frac{1+z^2}{2}\right) \left(\frac{\mathcal{H}^2}{2\pi^2}\right) \left[1 + \frac{\xi^2}{2} + \left(\frac{(z^2-1)\xi^2 - 4z\xi}{2(1+z^2)}\right) \cos\left(\frac{2}{\xi} - 2z\right) + \left(\frac{(z^2-1)\xi + z\xi^2}{1+z^2}\right) \sin\left(\frac{2}{\xi} - 2z\right) \right]. \quad (32)$$

Thus the k -dependence of the power spectrum remains unchanged when the initial condition on the k th oscillator is imposed at a time such that $[k/a(\eta_k)] = \text{constant}$. This expression is exact and shows that the spectrum is strictly scale-invariant, but is modified from the original result [viz. Eq. (24)] by a multiplicative factor independent of k . This factor depends on two parameters: (i) $\xi = (\mathcal{H}L_P)$ which measures the energy scale of inflation relative to the Planck scale and (ii) z which is the ratio of the Hubble radius and the physical wavelength of the perturbation. Let us now evaluate the power spectrum for $z = 1$ and

$z = (2\pi)$. For $z = 1$, the spectrum (32) reduces to

$$k^3 \mathcal{P}_\Phi(k) = \left(\frac{\mathcal{H}^2}{2\pi^2}\right) \left[1 + \frac{\xi^2}{2} - \xi \cos\left(\frac{2}{\xi} - 2\right) + \frac{\xi^2}{2} \sin\left(\frac{2}{\xi} - 2\right) \right], \quad (33)$$

while, for $z = (2\pi)$, we get

$$k^3 \mathcal{P}_\Phi(k) = \left(\frac{1 + 4\pi^2}{2}\right) \left(\frac{\mathcal{H}^2}{2\pi^2}\right) \left[1 + \frac{\xi^2}{2} + \left(\frac{(4\pi^2 - 1)\xi^2 - 8\pi\xi}{2(1 + 4\pi^2)}\right) \cos\left(\frac{2}{\xi}\right) + \left(\frac{(4\pi^2 - 1)\xi + 2\pi\xi^2}{1 + 4\pi^2}\right) \sin\left(\frac{2}{\xi}\right) \right]. \quad (34)$$

Moreover, as is quite often done in the literature, if the spectrum is evaluated at super Hubble scales, i.e. as $z \rightarrow 0$, then one finds that the spectrum is given by [16]

$$k^3 \mathcal{P}_\Phi(k) = \left(\frac{\mathcal{H}^2}{4\pi^2}\right) \left[1 + \frac{\xi^2}{2} - \left(\frac{\xi^2}{2}\right) \cos\left(\frac{2}{\xi}\right) - \xi \sin\left(\frac{2}{\xi}\right) \right]. \quad (35)$$

Notice that the numerical values of the modified amplitudes (33), (34), and (35) depends on the parameter ξ which is expected to be very small compared to unity for grand unified theory (GUT) scale inflation. For $\xi \ll 1$, one can easily obtain the leading order terms of polynomial expressions involving ξ , but determining $\cos(1/\xi)$ and $\sin(1/\xi)$ for $\xi \ll 1$ requires care. Since, one can easily replace $(1/\xi)$ by, say, $[(1/\xi) + (\pi/2)]$, in the arguments of trigonometric functions to the leading order, these expressions are intrinsically ambiguous. For $\xi \ll 1$ and $z = 1$, we get

$$k^3 \mathcal{P}_\Phi(k) \simeq \left(\frac{\mathcal{H}^2}{2\pi^2}\right) [1 - \xi F(2/\xi)], \quad (36)$$

where F is a rapidly oscillating cosine function. Similarly, when $\xi \ll 1$, for $z = (2\pi)$, we have

$$k^3 \mathcal{P}_\Phi(k) \simeq \left(\frac{1 + 4\pi^2}{2}\right) \left(\frac{\mathcal{H}^2}{2\pi^2}\right) \left[1 - \left(\frac{4\pi\xi}{1 + 4\pi^2}\right) F(2/\xi) + \left(\frac{(4\pi^2 - 1)\xi}{1 + 4\pi^2}\right) G(2/\xi) \right], \quad (37)$$

whereas, as $z \rightarrow 0$, we find that

$$k^3 \mathcal{P}_\Phi(k) = \left(\frac{\mathcal{H}^2}{4\pi^2}\right) [1 + (\xi^2/2) - (\xi^2/2) F(2/\xi) - \xi G(2/\xi)], \quad (38)$$

where G is a sine function and F , as above, is a cosine function. In each of these cases, a different choice for the subleading phase in the argument of the trigonometric function can make cosine into sine and vice versa.

Hence, only the profile of the oscillating functions are of relevance in the limit of $\xi \ll 1$, though the full expressions are often quoted in literature.

Evidently, the alternative prescription of specifying the initial condition when the physical wavelength of the mode is comparable to the Planck length (which sounds reasonable at first sight) is equivalent to assuming a quantum state at a $\eta = \text{constant}$ hypersurface, with very specific properties. It can be realized only if unknown physical effects of the trans-Planckian sector acts in a particular manner to populate the modes with a specific prescription. It is far from clear whether this will be possible in a generic context.

III. MIMICKING THE TRANS-PLANCKIAN EFFECTS

We shall now turn to the question of choosing an initial state such that a given power spectrum of perturbations is reproduced. Let us assume that, in the case of power-law inflation, a ‘‘trans-Planckian’’ theory leads to the following spectrum:

$$[k^3 \mathcal{P}_\Phi(k)]_M = C \left(\frac{\mathcal{H}^2}{2\pi^2}\right) \left(\frac{k}{\mathcal{H}}\right)^{2(\gamma+2)} M(k), \quad (39)$$

where $M(k)$ is the modification factor with, of course, $M(k) \geq 0$ for all k . (Alternatively, let us suppose that a future CMB observation leads to such a spectrum, starting a hectic flurry of theoretical activity to explain it.) Evidently, the modified spectrum (39) can be constructed from the spectrum (28) in the standard theory, provided we can find a positive definite function $B(k)$ that satisfies the condition

$$[1 + B^2(k) + 2B(k)d(k)] = [1 - B^2(k)]M(k). \quad (40)$$

This is a quadratic equation in $B(k)$ for each value of k and, if we choose $d(k) = \pm 1$, then, we find that the nontrivial and positive definite roots for $B(k)$ can be expressed in terms of $M(k)$ as follows:

$$B(k) = \frac{|M(k) - 1|}{M(k) + 1}. \quad (41)$$

Given a $M(k)$, this expression then allows us to reproduce the modified spectrum in the standard theory. However, note that the $B(k)$ we have constructed above is not unique as it corresponds to a particular value of the phase $b(k)$ in Eq. (25) such that $d(k) = \pm 1$. Clearly, a whole class of such functions can be constructed, with different choices for $d(k)$, but, as we shall see, this particular choice is sufficient for our purpose.

As we had mentioned in Section I, the Planck scale modifications to the standard inflationary perturbation spectrum have been obtained in the literature by considering various high energy models for the scalar field [7–20]. Most of these modified spectra deviate from the standard

scale-invariant spectrum at the ultraviolet end. On the other hand, the lower power in the quadrupole and the octopole moments of the CMB as measured by WMAP [6]—if it survives further releases of WMAP data—requires a suppression of power in the infrared end of the spectrum [35,36]. Such spectra have also been obtained in certain high energy models of the scalar field [19,22]. It is interesting to determine how these different modified spectra can be constructed by choosing suitable initial conditions on the scalar field in the standard theory.

In the following subsections, we shall explicitly construct the function $B(k)$ for four modified power spectra that have either been proposed to fit the observational data or have been obtained from a high energy model of the scalar field.

A. Modified spectrum I

Recently, the following spectrum was obtained in a power-law inflationary scenario using a Lorentz invariant modified theory [19]:

$$[k^3 \mathcal{P}_\Phi(k)]_M = C \left(\frac{\mathcal{H}^2}{2\pi^2} \right) \left(\frac{k}{\mathcal{H}} \right)^{2(\gamma+2)} \times \left[1 - \bar{C} \left(\frac{\mathcal{H}}{k_c} \right) \left(\frac{k}{\mathcal{H}} \right)^{(\gamma+2)} \right], \quad (42)$$

where k_c denotes the high energy scale, \bar{C} is given by

$$\bar{C} = [2C(\gamma + 1)^{3(\gamma+1)}]^{-1} \quad (43)$$

and it is assumed that $10^{-5} \lesssim (\mathcal{H}/k_c) \lesssim 10^{-3}$. (However, it should be mentioned here that this spectrum is not valid for arbitrarily small values of k [19].) In fact, a similar spectrum have also been obtained in noncommutative models of inflation [15,22]. These spectra exhibit a suppression of power at the large scales, a feature that could be relevant to the low quadrupole moment observed in the CMB [35]. It is straightforward to see that, for $z = 1$, the function $B(k)$ corresponding to the modified spectrum (42) is given by

$$B(k) = \left[\left(\frac{2}{\bar{C}} \right) \left(\frac{\mathcal{H}}{k_c} \right)^{-1} \left(\frac{k}{\mathcal{H}} \right)^{-(\gamma+2)} - 1 \right]^{-1}. \quad (44)$$

For $\gamma < -2$, we have $B \rightarrow k^{(\gamma+2)} \rightarrow 0$ as $k \rightarrow \infty$. This implies that, while, towards the infrared end, the initial state is different from the Bunch-Davies vacuum, the Bunch-Davies vacuum structure is retained at the ultraviolet end.

B. Modified spectrum II

Another modified primordial spectrum that has been proposed (see, for e.g., Refs. [36]) in order to account for the low quadrupole moment observed in the CMB, is the following:

$$[k^3 \mathcal{P}_\Phi(k)]_M = Ak^{(n_s-1)} [1 - e^{-(k/k_1)^\alpha}], \quad (45)$$

where A and n_s are the scalar amplitude and index of the standard spectrum. The pivot scale k_1 and the constant α (which turns out to be a positive number of the order of unity) are chosen to fit the CMB data. If we now assume that amplitude A and the index n_s of the above modified spectrum are the same as those in the case of power-law inflation in the standard theory [cf. Eq. (22)], then, we find that $B(k)$ is given by

$$B(k) = [2e^{(k/k_1)^\alpha} - 1]^{-1}. \quad (46)$$

Note that $B(k) \rightarrow 1$ as $k \rightarrow 0$ and $B_k \simeq e^{-(k/k_1)^\alpha} \rightarrow 0$ for large k . Once again, the vacuum structure at the ultraviolet end is not modified.

C. Modified spectrum III

Another modified spectrum whose effects on the CMB has also been analyzed is the following spectrum [21]

$$[k^3 \mathcal{P}_\Phi(k)]_M = \left(\frac{\mathcal{H}^2}{2\pi^2} \right) \left(1 - \xi \left(\frac{k}{k_2} \right)^{-\epsilon} \sin \left[\frac{2}{\xi} \left(\frac{k}{k_2} \right)^\epsilon \right] \right), \quad (47)$$

where $\xi \simeq 10^{-3}$, $\epsilon \simeq 10^{-2}$, and k_2 is a pivot scale. In order to match the leading term in the spectrum, let us assume that the inflating background undergoes exponential expansion and that the spectrum is evaluated at $z = 1$. It is then straightforward to construct $B(k)$ for the above spectrum and it is given by

$$B(k) = \left(\frac{\xi(k/k_2)^{-\epsilon} \sin[(2/\xi)(k/k_2)^\epsilon]}{2 - \xi(k/k_2)^{-\epsilon} \sin[(2/\xi)(k/k_2)^\epsilon]} \right). \quad (48)$$

D. Modified spectrum IV

A more general modification of spectrum one can envisage is a spectrum which has corrections at both the infrared and the ultraviolet ends. If we now assume that the standard spectrum is modified in the infrared as in Eq. (45) and has the same correction at the ultraviolet end as in Eq. (47), then the complete spectrum will be given by

$$[k^3 \mathcal{P}_\Phi(k)]_M = Ak^{(n_s-1)} \left(1 - e^{-(k/k_1)^\alpha} - \xi \left(\frac{k}{k_2} \right)^{-\epsilon} \times \sin \left[\frac{2}{\xi} \left(\frac{k}{k_2} \right)^\epsilon \right] \right), \quad (49)$$

where A and n_s are the scalar amplitude and index of the standard spectrum and k_1 and k_2 are pivot scales such that $k_1 \ll k_2$. Also, as in the earlier case, let α be a positive constant of the order of unity. The corresponding $B(k)$ can be easily obtained to be

$$B(k) = \left(\frac{|e^{-(k/k_1)^\alpha} + \xi(k/k_2)^{-\epsilon} \sin[(2/\xi)(k/k_2)^\epsilon]|}{2 - e^{-(k/k_1)^\alpha} - \xi(k/k_2)^{-\epsilon} \sin[(2/\xi)(k/k_2)^\epsilon]} \right). \quad (50)$$

These examples demonstrate the fact that the modification of the spectrum due to “trans-Planckian considerations” is degenerate with the choice of the initial quantum state. Without further input, such as an assumption for the choice of initial quantum state, observations cannot distinguish between these two physical effects.

IV. MODIFIED SPECTRA AND BACKREACTION

An issue that remains unresolved in obtaining the modified spectra is whether the conditions that lead to modifications of the standard spectrum will also lead to a large backreaction on the inflating background (for a recent discussion on this issue, see Refs. [37]). In particular, will the energy in the quantum field dominate the inflaton energy thereby, possibly, terminating inflation? The approach we have adopted here allows us to address this issue along the following manner.

Since the modified spectra from a high energy theory can be obtained from the standard theory with a suitable choice of initial conditions on the modes, we are probably justified in using these modes to evaluate the energy density of the quantum scalar field (for an earlier discussion on this point, see the first reference listed under Ref. [12]). In what follows, we shall show that the energy density of the quantum scalar field *above* the Bunch-Davies vacuum is finite only if $|M(k) - 1|$ decays faster than k^{-2} for large k and $M(k)$ goes as k^λ with $-2 < \lambda < 2$ for small k . Also, we shall restrict our attention to the easily tractable case of exponential inflation.

The expectation value of the Hamiltonian operator corresponding to the oscillator $q_{\mathbf{k}}$, say, \mathcal{E}_k , can be evaluated using the wave function (9). We find that

$$\begin{aligned}\mathcal{E}_k &= \left(\frac{a^2}{2W(k)}\right)(|\mu'_k|^2 + k^2|\mu_k|^2) \\ &= \left(\frac{a^2}{2W(k)}\right)[|(f_k/a)'|^2 + k^2|(f_k/a)|^2]\end{aligned}\quad (51)$$

and hence, the energy density of the quantum scalar field is given by

$$\rho = \frac{1}{2\pi^2 a^4} \int_0^\infty dk k^2 \mathcal{E}_k. \quad (52)$$

In the case of exponential inflation, as we had mentioned earlier, the general solution for f_k can be expressed in terms of simple functions [cf. Eq. (29)]. On substituting the solution (29) in the expression (51), we find that the energy density per mode of the quantum field is given by

$$\begin{aligned}\mathcal{E}_k &= \left(\frac{1}{4k\eta^2}\right)[1 - B^2(k)]^{-1}([1 + B^2(k)](1 + 2k^2\eta^2) \\ &\quad - 2B(k)\cos[b(k) - 2k\eta] + 4B(k)(k\eta) \\ &\quad \times \sin[b(k) - 2k\eta]).\end{aligned}\quad (53)$$

In the Bunch-Davies vacuum, which corresponds to

$B(k) = 0$, this expression for \mathcal{E}_k reduces to

$$\mathcal{E}_k^{\text{BD}} = \frac{k}{2} + \frac{1}{2k\eta^2} \quad (54)$$

and it is known that the corresponding energy density (when regularized by point-splitting) is given by [27,38]

$$\rho_{\text{BD}} = -\left(\frac{29}{960\pi^2}\right)\mathcal{H}^4. \quad (55)$$

This energy density is much smaller than the energy density in the classical field that drives inflation. The energy density of the scalar field *above* the Bunch-Davies vacuum is then given by Eq. (52) with \mathcal{E}_k replaced by $\bar{\mathcal{E}}_k$, where

$$\begin{aligned}\bar{\mathcal{E}}_k &= \mathcal{E}_k - \mathcal{E}_k^{\text{BD}} \\ &= \left(\frac{1}{2k\eta^2}\right)[1 - B^2(k)]^{-1}(B^2(k)(1 + 2k^2\eta^2) - B(k) \\ &\quad \times \cos[b(k) - 2k\eta] + 2B(k)(k\eta)\sin[b(k) - 2k\eta]).\end{aligned}\quad (56)$$

In order to understand the behavior of the above $\bar{\mathcal{E}}_k$ for large k , we can ignore the terms containing the sine and the cosine functions as they will oscillate rapidly in this limit. On neglecting these terms and, on making use of the relation (41) between $B(k)$ and $M(k)$, we find that, for large k , the above expression for $\bar{\mathcal{E}}_k$ can be written in terms of $M(k)$ as follows:

$$\bar{\mathcal{E}}_k \simeq \left(\frac{[M(k) - 1]^2}{4M(k)}\right)k. \quad (57)$$

For the energy density ρ corresponding to this $\bar{\mathcal{E}}_k$ to be finite in the ultraviolet limit, the integral of $(k^2\bar{\mathcal{E}}_k)$ over k should converge at large k . It is easy to see that this condition cannot be satisfied if $M(k) \propto k^\nu$ for any value of ν . For all values of ν , this expression varies as $k^{(3+\nu)}$ at large k and hence the integral is divergent. To obtain a finite result, we need $M(k) \rightarrow 1$ at large k . If we now assume that $M(k) \propto (1 \pm k^{-\delta})$ for large k , it is then clear that the energy density corresponding to the above $\bar{\mathcal{E}}_k$ will converge only if $\delta > 2$. Thus, the deviations from the standard spectra should die down faster than k^{-2} for large k .

One can also investigate the infrared limit in a similar fashion. As $k \rightarrow 0$, we find that the leading divergence arises due to the first term in the expression (56) for $\bar{\mathcal{E}}_k$ so that, in this limit, we have

$$\bar{\mathcal{E}}_k \propto \left(\frac{[M(k) - 1]^2}{M(k)k}\right). \quad (58)$$

If we now assume that $M(k) \propto k^\lambda$ as $k \rightarrow 0$, then the finiteness of the energy density ρ corresponding to this $\bar{\mathcal{E}}_k$ requires that $-2 < \lambda < 2$. It is therefore possible to enhance or reduce power in the infrared limit within a range and still maintain finite energy density. Modifications of the form $[M(k) - 1] \propto k^\varepsilon$ with $\varepsilon > 0$ are also

allowed and there are no restrictions on ε in this case. (We stress that the finiteness of ρ is a necessary condition for ignoring backreaction, but it is not sufficient. The latter will require comparing the energy density in the quantum field with the background energy density which is difficult to do without assuming a specific model.) Clearly, amongst the four modified spectra that we have considered in the last section, only the second spectrum (provided $\alpha < 2$) will lead to a finite energy density for the case of exponential inflation.

Our expression in Eq. (57) shows that a constant $M(k)$ independent of k leads to a divergent energy density above the Bunch-Davies vacuum. In particular, the state obtained by giving initial conditions for each of the modes at $[k/a(\eta_k)] = \text{constant}$ (leading to a $\mathcal{B}(k)$ that is independent of k), produces a divergent contribution to the energy density and hence is suspect as a valid quantum state. It is sometimes argued in the literature that this state has the same energy density as the Bunch-Davies vacuum along the following lines: When \mathcal{B} is independent of k , we are dealing with mode functions of the form $(f_k + \mathcal{B}f_k^*)$, with f_k given by Eq. (29), which belong the set of so-called α -vacua [39]. It is possible to construct a regularization scheme such that the divergences which arise at the coincidence limit of the Greens function in this case, is the same as that in the case of $\mathcal{B} = 0$. Such a subtraction will lead to $\rho = \rho_{BD}$ for these states. In our approach, this is equivalent to ignoring the contributions in Eq. (57) when $M(k) = M_0$ is a constant different from unity. Since this leads to $\bar{\mathcal{E}}_k \propto k$, one can think of M_0 as the (constant) occupation number $\langle n_k \rangle$ for all k ; the regularization involves ignoring all these ‘‘particles’’ in measuring the energy density.

We believe this argument is spurious for two reasons. Firstly, a transparent and direct discussion in terms of harmonic oscillators presented above gives a different result showing that $M(k) = M_0 = \text{constant}$ leads to a divergent energy density. So, at the least, the results depend on the explicit regularization procedure adopted. Secondly, in a realistic model, we will have to deal with a weak dependence of \mathcal{B} on k rather than no dependence. Then, the argument based on α -vacua will no longer hold, irrespective of how weak this dependence is. The analysis given above, however, is completely transparent and clear.

V. DISCUSSION

The fact that the measurements of the CMB anisotropies strongly indicate a primordial spectrum that is nearly scale-invariant already implies that the initial state of the quantum scalar field is a state that is ‘‘very close’’ to the Bunch-Davies vacuum. Clearly, sufficiently precise measurements of the anisotropies in the CMB can provide us with the form the inflationary perturbation spectrum to a good accuracy. However, this information can at most help us determine the initial state of the quantum scalar field in the

standard theory and it is not sufficient to aid us in discriminating between the various Planck scale models of matter fields. The reason being that, given a perturbation spectrum, we should be able to obtain the spectrum from *any* high energy model of the matter field by simply choosing a suitable state in the modified theory just as we had done in the standard theory.

The above result can be obtained either in the Heisenberg picture or in the Schrodinger picture since these descriptions are mathematically equivalent. We have, however, adapted the Schrodinger picture since the problem we are discussing is essentially that of a harmonic oscillator with a time-dependent frequency for which the intuition available in the Schrodinger picture is of some value. Our discussion clearly shows that virtually any power spectrum which is either observed in future or suggested by phenomenological models, can be reproduced by a suitable choice of the quantum state. Evidently, without additional assumptions one cannot disentangle the dynamics from the initial conditions and hence, the CMB observations alone cannot act as a discriminator between different theoretical models.

There are several possible avenues for future work arising from this discussion. One particularly interesting question will be the evolution of the quantum state of the Universe into the future. Several recent observations [40] (as well as not so recent observations, see Refs. [41]) suggest that the Universe has just entered an accelerating phase dominated by dark energy with an equation of state $P \approx -\rho$. While the nature of this dark energy is unclear, it seems likely that at least at sufficiently large scales it will act like a cosmological constant leading to a late time de Sitter phase [42]. It will be interesting to study the late time evolution of the quantum wave function of the scalar field. The de Sitter phase in the future should lead to its own thermal fluctuations with a characteristic temperature and it will be interesting to see how that can emerge.

Another issue of interest is the study of correlations across the horizon in the case of de Sitter spacetime. It has been shown that the quantum entanglement of modes across the horizon can lead to a holographic description of gravity [43]. The effects of this entanglement are easy to calculate when de Sitter spacetime is described in the static coordinates. On the other hand, the time-dependent Gaussian state used in this paper is more naturally tuned to the Friedmann coordinates of the de Sitter spacetime. It will be worthwhile to relate these two descriptions and understand how the correlations across the horizon arises in the time-dependent background.

Finally, the description in terms of the wave function can be easily translated to one in terms of the path integral kernel using the Feynman-Kac formula. This will allow one to provide a purely path integral derivation of the results presented in this paper. All these issues are currently under investigation.

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