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Liouville field theory and the partition function for two-dimensional Newtonian gravity

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SUMMARY

I show that the grand partition function for a system of particles interacting via gravitational force can be related to the vacuum-to-vacuum amplitude of the Liouville field theory. Finite action solutions to the Liouville field equation can be expressed in closed form in two dimensions. This solution is used to construct the mean field approximation for the grand partition function. The accuracy of the mean field approximation is verified by comparing it with the exact partition function for a system with two particles.

1 INTRODUCTION

Consider a system of N particles interacting through Newtonian gravitational forces. For large N, say, $N = 10^5$–10^{11}, it is neither feasible nor useful to try to follow the trajectories of the individual particles. We will be more interested in the average, statistical properties of such a system rather than in the individual orbits. The physical behaviour of such a system turns out to be very different from the statistical behaviour of other, more familiar, many-body systems like neutral gases and plasmas. The central reason for the peculiarities exhibited by the gravitating systems can be traced back to the *unshielded, long-range* nature of the gravitational force. In contrast, gaseous systems have genuinely short-range interactions and plasmas have an effective short-range interaction due to Debye shielding. Because of this fundamental difference between these systems, the standard methods of statistical mechanics cannot be carried over in a direct manner to study gravitating systems. It is necessary to go back to the fundamentals of statistical mechanics and develop special techniques capable of handling the long-range nature of gravity (see, for a review, Padmanabhan 1990).

One of the fundamental concepts of statistical mechanics which fails in the context of gravitating systems is the extensive nature of the energy. Extensivity of energy does not hold good in the presence of long-range interactions. This has the consequence that, for gravitating systems, the microcanonical distribution and the canonical distribution will not lead to the same physical description.

The statistical behaviour also depends crucially on the spatial dimension. In 3D, the phase volume available for the system diverges due to the r^{-2} nature of the force and the theory can be defined only with a short distance cut-off. The situation is better – and more interesting – in 2D. In this case, the microcanonical description exists for all values of energies even in the absence of any artificial short distance cut-off. The canonical description, however, exists only above a critical temperature.

We study some properties of this two-dimensional system in this paper. In particular, we shall evaluate the grand partition function for this system in the mean-field limit. The procedure we adopt for this evaluation will also exhibit a simple, and conceptually useful, connection between the mean-field description of the Newtonian theory and the 'vacuum-to-vacuum' amplitude of a Liouville field theory. (This result turns out to be true in any dimension.) In fact, we will construct the grand partition function for the system using the classical solutions of the Liouville field equation. This *mean-field* partition function which we derive remains well defined for all values of the temperature. Since the *exact* partition function is known to exist only above a critical temperature, it follows that the mean-field approximation must break down at low temperatures. We investigate this effect by comparing the exact partition function with the mean-field partition function for the case of a two-particle system.

2 PARTITION FUNCTION FOR TWO-DIMENSIONAL GRAVITY

Consider a collection of N particles, interacting via 2D gravitational force and moving in a 2D region of radius R. We define the force in the 2D gravity as the force derived out of the potential $\phi(\boldsymbol{x})$ which satisfies the Poisson equation:

$$\nabla^2\phi = 2\pi G\rho. \tag{1}$$

The potential due to a point particle of mass m will be logarithmic; the potential energy of interaction between two

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such particles will be

$$U(\boldsymbol{x}_i, \boldsymbol{x}_j) = Gm^2 \ln\left(\frac{|\boldsymbol{x}_i - \boldsymbol{x}_j|}{L}\right), \qquad (2)$$

where L is a constant which sets the zero of the energy. [Such systems have been studied in the past by several people in the context of statistical mechanics of gravitating systems; e.g. Salsberg 1965; Katz & Lynden-Bell 1978. Some authors define two-dimensional gravity with 4π rather than 2π in the Poisson equation; the results derived below can be easily adapted for such a case by replacing m^2 by $2m^2$.]

It is easy to verify that the microcanonical distribution exists for such a 2D gravitating system while the canonical description exists only above a critical temperature. To see this, let us consider the phase volume $g(E)$, which is the volume of the constant energy surface in the phase space:

$$\begin{aligned} g(E) &= \int \prod_{i=1}^{N} d^2x_i d^2p_i \delta(E-H) \\ &= A \int \prod_{i=1}^{N} d^2x_i \left[E - \frac{1}{2}\sum_{i\neq j} Gm^2 \ln\frac{|\boldsymbol{x}_i - \boldsymbol{x}_j|}{L}\right]^{N-1}, \end{aligned} \qquad (3)$$

where we have used the Hamiltonian for a 2D gravitating system of N particles with logarithmic potential

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{1}{2}\sum_{i\neq j} Gm^2 \ln\frac{|\boldsymbol{x}_i - \boldsymbol{x}_j|}{L} \qquad (4)$$

and performed the momentum integrations. The integral is over a bounded region of space and hence can diverge – if at all it does – only near the origin. It is, however, easy to see that the integral does not diverge at short distances even though there is no cut-off in the potential: the expression in the square brackets in (3) does not contribute a divergent factor, $(\ln s)^{N-1}$, near the origin; but this divergence is suppressed by the s factors in the area elements $d^2x = s\,ds$. Thus $g(E)$ is finite, though not calculable in closed form.

Let us next consider the partition function which is given by

$$\begin{aligned} Z(\beta, N) &= \int_v \prod_{i=1}^{N} d^2p_i d^2x_i \exp(-\beta H) \\ &= \left(\frac{2\pi}{\beta}\right)^N \int \prod_{i=1}^{N} d^2x_i \exp - \beta\left(\frac{1}{2}\sum_{i\neq j} Gm^2 \ln\frac{|\boldsymbol{x}_i - \boldsymbol{x}_j|}{L}\right). \end{aligned} \qquad (5)$$

This quantity, however, does not exist for all β. Simple power cutting near the origin shows that the integral exists only if

$$\beta < \beta_c = 4(Gm^2N)^{-1}. \qquad (6)$$

In other words, the canonical description of the system exists only at sufficiently high temperatures (Salsberg 1965; Lynden-Bell & Katz 1978).

We shall now analyse the form of the partition function in greater detail. To begin with, notice that we can set the scale-length L to be equal to the radius of the confining volume R as far as the computation of Z is concerned. [The result for a general L can be obtained in the end by multiplying our result by $\exp[(\beta Gm^2/2)N(N-1)\ln(L/R)]$; see the L dependence of (5).] Thus our Hamiltonian and the partition function can be taken to be

$$\begin{aligned} H(\boldsymbol{p}_i, \boldsymbol{x}_i) &= \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{Gm^2}{2}\sum_{i\neq j}^{N} \ln\left(\frac{|\boldsymbol{x}_i - \boldsymbol{x}_j|}{R}\right) \\ Z(\beta, N) &= \frac{1}{(2\pi\hbar)^{2N} N!}\int_{|\boldsymbol{x}_i|<R} \prod_{i=1}^{N} d\boldsymbol{x}_i d\boldsymbol{p}_i \exp[-\beta H(\boldsymbol{p}_i, \boldsymbol{x}_i)]. \end{aligned} \qquad (7)$$

We shall define the generating function for $Z(\beta, N)$ by the relation

$$\mathscr{G}(\beta, t) = \sum_{N=1}^{\infty} (-t)^N Z(\beta, N), \qquad (8)$$

which is essentially the grand partition function for the system except for the additional minus sign in the parameter t. (The reason for the extra minus sign will become clear later on.) Substituting (7) in (8), and performing the momentum integrations, we get

$$\mathscr{G}(\beta, t) = \sum_{N=1}^{\infty} \frac{(-\mu)^N}{N!}\int_{|\boldsymbol{x}_i|<R} \prod_{i=1}^{N} d\boldsymbol{x}_i \exp\left[-\frac{\beta Gm^2}{2}\sum_{i\neq j} \ln\frac{|\boldsymbol{x}_i - \boldsymbol{x}_j|}{R}\right] \qquad (9)$$

with

$$\mu = \frac{2\pi m}{\beta\hbar}\, t. \qquad (10)$$

Once $\mathscr{G}(\beta, t)$ is computed, the partition function $Z(N, \beta)$ can be obtained by the power series expansion.

We will now show that $\mathscr{G}(\beta, t)$ is related to the vacuum-to-vacuum amplitude of a Liouville field theory. To this end, we begin with the standard result in the theory of functional integration:

$$\begin{aligned} &\int \mathscr{D}\phi \exp\left\{-\int d\boldsymbol{x}\left[\frac{1}{2}|\nabla\phi|^2 + J(\boldsymbol{x})\,\phi(\boldsymbol{x})\right]\right\} \\ &\quad = Q \exp -\frac{1}{2}\int d\boldsymbol{x}\, d\boldsymbol{y}\, J(\boldsymbol{x})\, G(\boldsymbol{x}-\boldsymbol{y})\, J(\boldsymbol{y}), \end{aligned} \qquad (11)$$

where Q is a normalization factor related to the determinant of the Laplacian operator and G is the Green's function for the Laplacian operator. If we now choose

$$J(\boldsymbol{x}) = q\sum_{i=1}^{N} \delta(\boldsymbol{x} - \boldsymbol{x}_i) \qquad (12)$$

and use the explicit form of the 2D Green's function

$$G(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2\pi}\ln\left(\frac{|\boldsymbol{x} - \boldsymbol{y}|}{R}\right), \qquad (13)$$

we get

$$\begin{aligned} &Q \exp\left[-\frac{q^2}{4\pi}\sum_{i\neq j} \ln\frac{|\boldsymbol{x}_i - \boldsymbol{x}_j|}{R}\right] \\ &\quad = \int \mathscr{D}\phi \exp\left[-q\sum_{i=1}^{N} \phi(\boldsymbol{x}_i) - \int d\boldsymbol{x}\left(\frac{1}{2}|\nabla\phi|^2\right)\right]. \end{aligned} \qquad (14)$$

Setting

$$\frac{q^2}{4\pi}=\frac{Gm^2\beta}{2} \tag{15}$$

and comparing (9) and (14), we can express $\mathscr{G}(\beta, t)$ in the form

$$\mathscr{G}(\beta, t)=\sum_{N=1}^{\infty}\frac{(-t)^N}{N!}\int_{|x_i|<R}\prod_{i=1}^{N}dx_i\langle\exp q\phi(x_i)\rangle, \tag{16}$$

where the functional average $\langle\ldots\rangle$ is defined as

$$\langle F(\phi)\rangle=Q^{-1}\int\mathscr{D}\phi F[\phi]\exp\left[-\frac{1}{2}\int dx|\nabla\phi|^2\right]. \tag{17}$$

The summation in (16) can be easily performed, leading to the exponential form; thus we get the functional integral form for the generating function:

$$\begin{aligned}\mathscr{G}(\beta, t)&=\left\langle\exp\int dx(-t)\exp q\phi(x)\right\rangle\\&=Q^{-1}\int\mathscr{D}\phi\exp-\int dx\left[\frac{1}{2}|\nabla\phi|^2+te^{q\phi}\right].\end{aligned} \tag{18}$$

This is precisely the vacuum-to-vacuum amplitude of a Liouville field theory with the action functional

$$A[\phi]=\int dx\left[\frac{1}{2}|\nabla\phi|^2+te^{q\phi}\right]. \tag{19}$$

It should be obvious that the result is valid in any number of dimensions. The choice of sign in (8) is motivated by this analogy; the functional integral is well defined only for positive t. The 'sum' is over all functions satisfying the boundary condition $\phi(R)=0$.

The exact vacuum-to-vacuum amplitude for Liouville field theory is not known. However, we can evaluate the generating function in the saddlepoint approximation as

$$\mathscr{G}(\beta, t)\cong\exp-A[\phi_{\rm cl}] \tag{20}$$

where $\phi_{\rm cl}(x)$ is the solution to the 'classical' Liouville equation:

$$\nabla^2\phi_{\rm cl}=qt\exp(q\phi_{\rm cl}). \tag{21}$$

This equation has the same structure as that of the isothermal sphere, clearly demonstrating the fact that the mean-field approximation for a gravitating gas corresponds to the saddlepoint limit of the grand partition function. Fortunately, one knows the general, spherically symmetric, solution to the Liouville equation in 2D (see, e.g. Gervais & Neveu 1982). It is given by two-parameter family of functions

$$\phi_{\rm cl}(r; n, l)=q^{-1}\ln\left[\frac{2n^2}{tq^2l^2}\frac{(r/l)^{n-2}}{[1-(r/l)^n]^2}\right], \tag{22}$$

parametrized by n and l. The action $A[\phi_{\rm cl}]$ will be finite only if $n=2$. In this case, the action becomes

$$A[\phi_{\rm cl}]=\frac{8\pi}{q^2}\left[\ln\left(1-\frac{R^2}{l^2}\right)+\frac{2(R/l)^2}{1-(R/l)^2}\right]. \tag{23}$$

The value of the parameter l can be fixed by the boundary condition $\phi_{\rm cl}(r=R)=0$. This condition gives, after some algebra, the relation

$$\rho\equiv\left(\frac{R}{l}\right)^2=\frac{1}{2k}(2k+1-\sqrt{4k+1}),\qquad k=\frac{tq^2R^2}{8}, \tag{24}$$

between l and other parameters of the problem. The generating function can be now expressed as

$$\mathscr{G}(\beta, t)=(1-\rho)^{-8\pi/q^2}\exp\left[-\frac{16\pi}{q^2}\frac{\rho}{1-\rho}\right], \tag{25}$$

where ρ is given by (24).

It is interesting to note that the grand partition function for this system (in the mean-field limit), can be expressed in such a simple and closed form. In principle, this expression can be expanded as a Taylor series in t; the coefficient of $(-t)^N$ will give then the partition function for the N-particle system. However, the partition function does not seem to have a simple closed form.

From the form of the grand partition function, it is easy to verify that it remains well defined for all positive values of β. But we know from the earlier analysis that the theory does not exist below a critical temperature. It follows that the mean-field approximation should break down at low temperatures. To see how this result comes about, we should compare the mean-field partition function with the exact partition function. This can be easily done for a toy model with $N=2$. We shall now perform this comparison.

We can easily expand $\mathscr{G}(\beta, t)$ up to quadratic order in t and determine $Z_2(\beta)$, the partition function for a two-particle system. Straightforward calculation gives (with the notation $V=\pi R^2$)

$$Z_2(\beta)=\frac{1}{2}\left[\frac{mV}{2\pi\beta\hbar^2}\right]^2\left(1+\frac{\beta Gm^2}{4}\right)\left(\frac{L}{R}\right)^{Gm^2\beta}\qquad\text{(saddlepoint).} \tag{26}$$

We have multiplied by the $(L/R)^{Gm^2\beta}$ factor to obtain the result for arbitrary L. For this case of $N=2$, it is also possible to evaluate the partition function exactly. To do this we need to compute the spatial part of the partition function

$$\begin{aligned}I_2&=\int dx_1\,dx_2\exp-\beta Gm^2\ln\frac{|x_1-x_2|}{R}\\&=\int\frac{dx_1\,dx_2}{|x_1-x_2|^{\beta Gm^2}}R^{\beta Gm^2}.\end{aligned} \tag{27}$$

It is clear that the integral exists only for $\beta Gm^2<2$. In this case, we can use the standard expansion

$$\frac{1}{(1+x^2-2x\cos\theta)^n}=\sum_{m=0}^{\infty}C_m^n(\theta)\,x^m, \tag{28}$$

where C_m^n are the Gegenbauer polynomials,

$$C_m^n(\theta)=\sum_{l,k=0;\,l+k=m}^{m}\frac{\Gamma(n+k)\,\Gamma(n+l)}{k!\,l!\,\Gamma^2(n)}\cos[(k-l)\,\theta] \tag{29}$$

and perform the angular integration. Only the $k=l$ term survives the angular integration allowing us to do the radial integration. This leads to the result

$$I_2(\beta)=\frac{\pi^2 R^4}{(1-Gm^2\beta/4)}\sum_{k=0}^{\infty}\frac{1}{k+1}\left[\frac{\Gamma(k+Gm^2\beta/2)}{k!\Gamma(Gm^2\beta/2)}\right]^2. \qquad (30)$$

We can do the summation in the above expression in the following way: from the definition of hypergeometric function ${}_2F_1(\alpha, \beta, \gamma; z)$ [see Gradshteyn & Ryzhik 1965; p. 1039], it follows that (for $\alpha<1$)

$$\int_0^1 {}_2F_1(\alpha, \alpha; 1; z)\, dz$$

$$=1+\frac{\alpha^2}{1}\frac{1}{2}+\frac{[\alpha(\alpha+1)]^2}{[1.2]^2}\frac{1}{3}+\frac{[\alpha(\alpha+1)(\alpha+2)]^2}{[1.2.3]^2}\frac{1}{4}+\dots \qquad (31)$$

$$=\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left[\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\,k!}\right]^2.$$

The integral on the left-hand side can be re-expressed in terms of gamma functions using the formula

$$\int_0^1 dz\, F(\alpha, \alpha; 1; z)=\frac{\Gamma[2(1-\alpha)]}{[\Gamma(2-\alpha)]^2}; \qquad (\alpha<1), \qquad (32)$$

[see p. 849, equation 4 of Gradshteyn & Ryzhik 1965]. Thus we get, for $\alpha<1$,

$$\sum_{k=0}^{\infty}\frac{1}{(k+1)}\left[\frac{\Gamma(\alpha+k)}{k!\Gamma(\alpha)}\right]^2=\frac{\Gamma[2(1-\alpha)]}{[\Gamma(2-\alpha)]^2}. \qquad (33)$$

Using this result, and including the momentum integrations, we can write the final answer as

$$Z_2^{\text{exact}}=\frac{1}{(1-\beta Gm^2/4)}\frac{\Gamma(2-Gm^2\beta)}{\Gamma^2(2-Gm^2\beta/2)}\frac{1}{2}\left(\frac{mV}{2\pi\beta\hbar^2}\right)^2\left(\frac{L}{R}\right)^{Gm^2\beta}. \qquad (34)$$

The $(L/R)^{Gm^2\beta}$ scaling is added so that the result is valid for arbitrary L.

Comparing this expression with the one obtained by the saddlepoint approximation, we see that the saddlepoint approximation is accurate at high temperatures; i.e. for $Gm^2\beta \ll 1$. Further, as the temperature is reduced, the approximation deviates more and more from the exact expression. In fact, $Z_2^{(\text{exact})}$ diverges at

$$Gm^2\beta_c=2 \qquad (35)$$

and the system cannot exist at $T<T_c$. Such a result, of course, was known previously from the scaling argument.

It would be interesting to see whether the analysis of this paper can be combined with several previously known properties of the system (existence of the critical temperature, form of the equation of state,...) to arrive at the *exact* partition function for the system for arbitrary N. If it can be done, then this system will be a realistic prototype for an exactly solvable model with long-range interactions. This issue is under investigation.

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