

Bipolar Harmonic encoding of CMB correlation patterns

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Deviations from statistical isotropy can be modeled in various ways, for instance, anisotropic cosmological models (Bianchi models), compact topologies and presence of primordial magnetic field. Signature of anisotropy manifests itself in CMB correlation patterns. Here we explore the symmetries of the correlation function and its implications on the observable measures constructed within the Bipolar harmonic formalism for these variety of models. Different quantifiers within the Bipolar harmonic representation are used to distinguish between plausible models of breakdown of statistical isotropy and as a spectroscopic tool for discriminating between distinct cosmic topology.

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I. INTRODUCTION

The fluctuations in Cosmic Microwave Background (CMB) contain an amazing amount of information about our universe. Detailed measurements of anisotropy in the CMB reveal global properties, constituents and history of the universe. In standard cosmology, CMB anisotropy is assumed to be statistically isotropic and Gaussian. Gaussianity implies that the statistical properties of the temperature field can be completely characterized in terms of its mean $\langle \Delta T \rangle = 0$, and auto-correlation function $C(\hat{n}_1, \hat{n}_2) = \langle \Delta T(\hat{n}_1) \Delta T(\hat{n}_2) \rangle$, where $\hat{n} = (\theta, \phi)$, is a unit vector on the sphere. The angular brackets $\langle \dots \rangle$ denote ensemble expectation values, i.e, averages above are for all possible realizations of the field over a sphere. Since we have one CMB sky, that is just one out of all possible realizations, the ensemble expectation value $C(\hat{n}_1, \hat{n}_2)$ can be estimated in terms of sky averages only to a limited extent, depending on underlying symmetries in $C(\hat{n}_1, \hat{n}_2)$. Under the usual assumption of *Statistical Isotropy* (SI), implying essentially Einstein's cosmological principle for cosmological perturbations, the correlation function is invariant under rotations. It implies the correlation function $C(\hat{n}_1, \hat{n}_2) = C(\hat{n}_1 \cdot \hat{n}_2) \equiv C(\theta)$, can be readily estimated by averaging over all pairs of sky directions separated by an angle θ .

Spherical harmonics form a basis of the vector space of complex functions on a sphere, making them a natural choice for expanding the temperature anisotropy field,

$$\Delta T(\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n}). \quad (1)$$

Here ΔT is the temperature fluctuation around some average temperature T . The complex quantities a_{lm} are

drawn from a Gaussian distribution, related to the Gaussian temperature anisotropy as

$$a_{lm} = \int d\Omega_{\hat{n}} Y_{lm}^*(\hat{n}) \Delta T(\hat{n}). \quad (2)$$

The condition for SI now takes the form of a diagonal covariance matrix,

$$\langle a_{l_1 m_1} a_{l_2 m_2}^* \rangle = C_{l_1} \delta_{l_1 l_2} \delta_{m_1 m_2}. \quad (3)$$

Here C_l is the well known angular power spectrum. In the SI case, the angular power spectrum carries complete information about the Gaussian field, and the statistical expectation values of the temperature fluctuations are preserved under rotations in the sky. This property of CMB has been under scrutiny since the release of the first year of WMAP data. Tantalizing evidence for statistical isotropy violation in the WMAP data using a variety of statistical measures has also been claimed in recent literature [1–3]. However, the origin of these ‘deviations’ from SI remains to be modeled adequately. These deviations could be either genuinely cosmological, or statistical coincidence, or residual foreground contamination, or, a systematic error in the experiment and the data processing. Hence, it is important to carry out a systematic study of SI violations using statistical measures within a unified, mathematically complete, framework. Moreover, it is important to develop several independent statistical measures to study SI violations that can capture different aspects of any measured violation and provide hints toward its origin.

While testing a fundamental assumption, such as SI, is in itself a justifiable end, there are also strong theoretical motivation to hunt for SI violations in CMB on large scales. Topologically compact spaces [4–10] and anisotropic cosmological models [11–13], Cosmological magnetic fields generated during an early epoch of inflation [14] can also lead to violation of SI [15], are but a few examples.

This paper focuses on linking measures of SI violation to the reduced symmetries of the underlying cor-

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relation patterns¹ in the CMB map or the correlation function. While we present illustrative examples of the symmetries from various mechanisms of SI violation, this paper does not concern itself with a study of specific mechanisms. We define, within the framework of Bipolar harmonic representation of CMB sky maps, a number of observables that can be used to quantitatively test SI. We present a study of the properties of bipolar measures as one systematically reduces the rotational symmetries of the CMB correlations, as is expected in different theoretical scenarios. We recapitulate the bipolar harmonic representation and the definitions of a set of measurable quantities representing SI violation in section II. In Section III, these observable measures are computed for different levels of residual rotational symmetries of CMB correlations. This provides a clear understanding of the underlying symmetries revealed through the different bipolar measures. Section IV deals with bipolar formalism measurables using Bianchi template. Section V summarizing conclusions and discussions is followed by appendices where details of the calculations leading to results are presented.

II. BIPOLAR FORMALISM AND THE OBSERVABLE MEASURES

Any deviations from SI introduces *off-diagonal* terms in the covariance matrix Eq. (3), thereby making C_l an inadequate quantity to characterize the statistical properties of the temperature field [16]. Under such a situation Bipolar spherical harmonic expansion, proposed by Hajian and Souradeep [17, 19–23], proves to be the most general representation of the two point correlation function, where the angular power spectrum C_l is a subset of Bipolar spherical harmonic coefficients (BipoSH). Two point correlation function of CMB anisotropies can be expanded as

$$C(\hat{n}_1, \hat{n}_2) = \sum_{l_1, l_2, L, M} A_{l_1 l_2}^{LM} \{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{LM}, \quad (4)$$

here $A_{l_1 l_2}^{LM}$ are Bipolar Spherical Harmonic coefficients (BipoSH), $|l_1 - l_2| \leq L \leq (l_1 + l_2)$, $m_1 + m_2 = M$, and $\{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{LM}$ are Bipolar spherical harmonics [24]. Bipolar spherical harmonics form an orthonormal basis on $\mathbf{S}^2 \times \mathbf{S}^2$, with transformation properties under rotations similar to spherical harmonics. The tensor product in harmonic space can be explicitly written using Clebsch-Gordan coefficients $C_{l_1 m_1 l_2 m_2}^{LM}$ as,

$$\{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{LM} = \sum_{m_1 m_2} C_{l_1 m_1 l_2 m_2}^{LM} Y_{l_1 m_1}(\hat{n}_1) Y_{l_2 m_2}(\hat{n}_2). \quad (5)$$

¹ In this paper, we use the term ‘correlation patterns’ to interchangeably refer to SI violation

A. Bipolar spherical harmonic coefficients - BipoSH

BipoSH can be extracted by inverse transformation of Eq. (4), i.e., multiplying both sides of this equation by $\{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{L'M'}^*$, and using orthonormality of Bipolar spherical harmonics. Hence, given a real space correlation pattern BipoSH coefficients can be found using

$$A_{l_1 l_2}^{LM} = \int d\Omega_{\hat{n}_1} d\Omega_{\hat{n}_2} C(\hat{n}_1, \hat{n}_2) \{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{LM}^*.$$

Since $C(\hat{n}_1, \hat{n}_2)$ is symmetric under the exchange of \hat{n}_1 and \hat{n}_2 , this gives rise to following symmetry properties of BipoSH:

$$\begin{aligned} A_{l_1 l_2}^{LM} &= (-1)^{l_1 + l_2 - L} A_{l_2 l_1}^{LM} \\ A_{ll}^{LM} &= A_{ll}^{LM} \delta_{L, 2k} \quad . \quad k = 0, 1, 2, 3, \dots \end{aligned} \quad (6)$$

Hence, A_{ll}^{LM} exists for *even* L and vanishes otherwise. It was shown in [17] that the Bipolar Spherical Harmonic (BipoSH) coefficients $A_{l_1 l_2}^{LM}$ are a linear combination of elements of the harmonic space covariance matrix including the off-diagonal elements that encode SI violation,

$$A_{l_1 l_2}^{LM} = \sum_{m_1 m_2} \langle a_{l_1 m_1} a_{l_2 m_2}^* \rangle (-1)^{m_2} C_{l_1 m_1 l_2 - m_2}^{LM}. \quad (7)$$

When SI holds the covariance matrix is diagonal, Eq. (3) and Clebsch property (H6), therefore

$$A_{l_1 l_2}^{LM} = (-1)^{l_1} C_{l_1} (2l_1 + 1)^{1/2} \delta_{l_1 l_2} \delta_{L0} \delta_{M0}, \quad (8)$$

implying that A_{ll}^{00} contains all the information on the diagonal harmonic space covariance matrix given by C_l .

The well known power spectrum C_l thus forms a subspace of BipoSH [19]. Under SI, the only non-zero Bipolar spherical harmonic coefficient will be A_{ll}^{00} (equivalent of C_l), all the rest must vanish. The violation of SI thus implies A_{ll}^{00} are not sufficient to describe the field. Hence, BipoSH proves to be a better tool to test SI, as non-zero $A_{l_1 l_2}^{LM}$, other than $L = 0$ and $M = 0$, terms should confirm its violation.

It is impossible to measure all $A_{l_1 l_2}^{LM}$ from just one CMB map because of cosmic variance. Thus we need to combine them in different ways to diagnose different aspects of SI violations.

B. Bipolar power spectrum- BiPS

The *Bipolar Power Spectrum* (BiPS) is a rotationally invariant, quadratic measure that can be constructed out of BipoSH coefficients [17]. BiPS involves averaging over BipoSH that reduces cosmic variance in comparison to a single CMB map, however this does not erase all the SI signatures. BiPS is defined as

$$\kappa_L = \sum_{l_1, l_2, M} |A_{l_1 l_2}^{LM}|^2. \quad (9)$$

For statistically isotropic models $\kappa_L = \kappa_0 \delta_{L0}$, i.e., $\kappa_L = 0 \quad \forall L > 0$. Thus a breakdown of SI will imply non-zero components of BiPS. In real space, κ_L can be expressed as

$$\kappa_L = \left(\frac{2L+1}{8\pi} \right)^2 \int d\Omega_{\hat{n}_1} \int d\Omega_{\hat{n}_2} \left[\int dR \chi^L(R) C(R\hat{n}_1, R\hat{n}_2) \right]^2, \quad (10)$$

where $C(R\hat{n}_1, R\hat{n}_2)$ is the correlation function after rotating the coordinate system through an angle ω ($0 \leq \omega \leq \pi$), about an axis $\mathbf{n}(\Theta, \Phi)$. $R\hat{n}_1$ and $R\hat{n}_2$ are the coordinates of the pixels \hat{n}_1 and \hat{n}_2 in the rotated coordinate system. The rotation axis \mathbf{n} , is characterized by two parameters Θ ($0 \leq \Theta \leq \pi$), and Φ ($0 \leq \Phi \leq 2\pi$). χ^L , is the trace of finite rotation matrix in LM -representation called the *characteristic function*, and it is invariant under rotation of coordinate system,

$$\chi^L(R) = \sum_M D_{MM}^L(R). \quad (11)$$

Here dR is the volume element of the three-dimensional rotation group given by

$$dR = 4 \sin^2 \left(\frac{\omega}{2} \right) d\omega \sin \Theta d\Theta d\Phi. \quad (12)$$

A simplified expression for BiPS in real space is

$$\kappa_L = \frac{(2L+1)}{8\pi^2} \int d\Omega_{\hat{n}_1} \int d\Omega_{\hat{n}_2} C(\hat{n}_1, \hat{n}_2) \int dR \chi^L(R) C(R\hat{n}_1, R\hat{n}_2). \quad (13)$$

For statistical isotropic model condition $\kappa_L = \kappa_0 \delta_{L0}$ can be recovered using orthonormality of $\chi^L(R)$,

$$\int_0^\pi \chi^L(R) \chi^{L'}(R) \sin^2 \frac{\omega}{2} d\omega = \frac{\pi}{2} \delta_{LL'}.$$

The BiPS of CMB anisotropy computed from the maps measured by WMAP are consistent with SI, ruling out its radical violation [20]. An advantage of BiPS is that its rotational invariance allows for constraints to be placed on the presence of specific forms of CMB correlation patterns independent of the overall orientation in the sky.

C. Reduced Bipolar coefficients- rBipoSH

In order to extract information on the orientation of SI violation, or to detect correlation patterns in a specific direction in the sky, the *Reduced Bipolar coefficients* [23], obtained as

$$A_{LM} = \sum_{l_1=0}^{\infty} \sum_{l_2=|L-l_1|}^{L+l_1} A_{l_1 l_2}^{LM}, \quad (14)$$

provide another set of measures. The summation of BiPSH over spherical wave-numbers l_1 and l_2 , reduces the cosmic variance rendering these measurable from the single CMB sky map available.

Note that the summation involves both the terms $A_{l_1 l_2}^{LM}$, and $A_{l_2 l_1}^{LM}$, that are related via symmetry properties Eq. (6). Thus for any such combination where $l_1 + l_2 - L$ is odd, these two terms will cancel each other leaving no contribution to the summation. The reduced Bipolar coefficients A_{LM} , by definition have the following symmetry

$$A_{LM} = (-1)^M A_{L-M}^*, \quad (15)$$

which indicates A_{L0} are always real. When SI condition is valid, the ensemble average of A_{LM} vanishes for all non-zero values of L ,

$$\langle A_{LM} \rangle = 0, \quad \forall L \neq 0. \quad (16)$$

These A_{LM} coefficients fluctuate about zero in any given CMB anisotropy map. Therefore, a statistically significant deviation from zero would confirm violation of SI. Unlike BiPS, reduced Bipolar coefficients are sensitive to orientation, hence they can assign directions to correlation patterns of the map.

D. Bipolar map

It is possible to visualize correlation patterns using the *Bipolar map* constructed from the reduced Bipolar coefficients A_{LM} as [23],

$$\Theta(\hat{n}) = \sum_{LM} A_{LM} Y_{LM}(\hat{n}). \quad (17)$$

The Bipolar map from A_{LM} is computed in the same way as the temperature anisotropy map from a given set of spherical harmonic coefficients, a_{lm} . Bipolar map can also be represented in terms of *Tripolar Spherical Harmonics* of zero angular momentum (see appendix B for details),

$$\Theta(\hat{n}) = \sum_{L, l_1, l_2} \int d\Omega_{\hat{n}_1} d\Omega_{\hat{n}_2} C(\hat{n}_1, \hat{n}_2) (-1)^{l_1+l_2} \sqrt{(2L+1)} \delta_{\lambda L} \{Y_L(\hat{n}) \otimes \{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{\lambda}\}_{00}. \quad (18)$$

The tripolar spherical harmonics are expressed as [24]

$$\{Y_{l_1}(\hat{n}_1) \otimes \{Y_{l_2}(\hat{n}_2) \otimes Y_{l_3}(\hat{n}_3)\}_{l_{23}}\}_{LM} = \sum C_{l_1 m_1 l_2 m_2 l_3 m_3}^{LM} C_{l_2 m_2 l_3 m_3}^{l_{23} m_{23}} Y_{l_1 m_1}(\hat{n}_1) Y_{l_2 m_2}(\hat{n}_2) Y_{l_3 m_3}(\hat{n}_3),$$

where the summation is carried over m_1, m_2, m_3 , and m_{23} . The transformations under rotations of tripolar spherical harmonics are identical to spherical harmonics. In particular, the tripolar scalar harmonics, which are invariant under rotations, can be expressed as follows,

$$\{Y_{l_1}(\hat{n}_1) \otimes \{Y_{l_2}(\hat{n}_2) \otimes Y_{l_3}(\hat{n}_3)\}_{\lambda}\}_{00} = (-1)^{l_1+l_2+l_3} \delta_{\lambda l_1} \sum_{m_1 m_2 m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} Y_{l_1 m_1}(\hat{n}_1) Y_{l_2 m_2}(\hat{n}_2) Y_{l_3 m_3}(\hat{n}_3).$$

Orthogonality and normalization relation is as follows,

$$\int \int \int d\Omega_{\hat{n}_1} d\Omega_{\hat{n}_2} d\Omega_{\hat{n}_3} \{Y_{l_1}(\hat{n}_1) \otimes \{Y_{l_2}(\hat{n}_2) \otimes Y_{l_3}(\hat{n}_3)\}\}_{LM} \\ \{Y_{l'_1}(\hat{n}_1) \otimes \{Y_{l'_2}(\hat{n}_2) \otimes Y_{l'_3}(\hat{n}_3)\}\}_{L'M'} \\ = \delta_{l_1 l'_1} \delta_{l_2 l'_2} \delta_{l_3 l'_3} \delta_{\lambda \lambda'} \delta_{LL'} \delta_{MM'}.$$

From Eq. (18) its evident that under SI the Bipolar map is invariant under the rotations, since tripolar scalar harmonics are rotationally invariant and $C(\hat{n}_1, \hat{n}_2) = C(R\hat{n}_1, R\hat{n}_2)$. Hence, the map gets contribution only from the monopole term A_{00} ,

$$\Theta = \frac{1}{2} \sum_l (-1)^l \sqrt{\frac{(2l+1)}{\pi}} C_l. \quad (19)$$

Also, if the temperature map is rotated by a element of rotation group, “ R ” then Bipolar map also rotates identically (see Appendix B). For example, if you rotate the temperature map about the z-axis by some angle “ α ”,

$$\Delta T(R(\theta, \phi)) = \sum_{lm} a_{lm} Y_{lm}(\theta, \phi - \alpha),$$

the Bipolar map will also be rotated about z-axis through same angle “ α ”

$$\Theta(R(\theta, \phi)) = \sum_{LM} A_{LM} Y_{LM}(\theta, \phi - \alpha).$$

However, the Wigner-D matrices in the two cases will be different because of different m (or M) values.

III. BIPOLAR REPRESENTATION OF CMB CORRELATION SYMMETRIES

The homogeneity and isotropy of cosmic microwave background points to the Friedmann-Robertson-Walker(FRW) model of universe. Flat FRW model adequately describes the observed local properties of the universe, but the fact that universe with same local geometry can admit different global topology has been appreciated since the advent of post GR modern cosmology. This is because Einstein’s equations describe local properties of the spacetime and can only constrain, but not determine, the global topological structure.

Symmetries of the space are preserved in the correlation function and global topology modifies correlation function. The simply connected (topologically trivial) hyperbolic 3-space \mathcal{H}^3 , and the flat Euclidean 3-space \mathcal{E}^3 , are non-compact and have infinite volume. There are numerous theoretical motivations, however, to favor a spatially compact universe [4–7]. Compact topologies (more, generally, multiply connected space) break the statistical isotropy of CMB in characteristic patterns and induce a cutoff in the power spectrum because of finite spatial size [16, 25–27]. Theoretical possibilities include compact Euclidean and Hyperbolic 3-spaces which require

the space to be multiply connected. The compact hyperbolic manifolds are not globally homogeneous and they turn out to be not of much interest for the class of symmetries considered under the scope of this paper.

Simply connected universes are statistically isotropic, i.e. $C(\hat{n}_1, \hat{n}_2) = C(\hat{n}_1, \hat{n}_2)$. In contrast, all compact universe models with Euclidean or hyperbolic geometry $C(\hat{n}_1, \hat{n}_2)$ are statistically anisotropic. The isotropy of space is broken in multi-connected models; this breaking of symmetry may be apparent through the presence of some principal directions. In a cylinder, for instance, which is compact in one dimension and infinite in the other two, the metric tensor is exactly the same at every point hence it preserves local homogeneity. However, it is not globally isotropic and does not have the maximal symmetry. It is noteworthy that globally anisotropic models do not contradict observations, since the homogeneity of space and the local isotropy can ensure the observed isotropy of the CMB, however can influence the spectrum of density fluctuations. Multiply-connected models with zero or negative curvature can be compact in some, or all their dimensions. For instance a toroidal universe, despite its zero spatial curvature, has a finite volume which may in principle be measured. It contains a finite amount of matter. A cylindrical universe (in the sense that the spatial sections are cylinders), on the other hand, is noncompact in one dimension only and has an infinite volume, although with a finite circumference in the principal direction.

Homogeneity and isotropy are experimentally confirmed in the observations of distribution of luminous red galaxies [28], and the isotropy of CMB background [29, 30]. Most of the studies in CMB assume statistical isotropy of the universe (FRW model). However, indications for a preferred direction in CMB, have motivated the study of departures from statistical isotropy [1]. These deviations can arise from non-trivial spatial topologies [4, 5, 8–10], or departures from the background FRW metric [11, 31]. Alternatively, statistical anisotropies might also arise from coherent magnetic fields in the universe [14, 15, 32]. Anisotropic Cosmological models have been considered in the past and they lead to characteristic patterns in the CMB sky [13]. The Bianchi template is an example of SI violation due to departure from background FRW geometry. Here we will discuss the signature of anisotropy due to existence of preferred axis (axes) on BipoSH. Such SI violations can arise due to non-trivial topologies as well as coherent magnetic fields.

Since Bipolar formalism is sensitive to structures and patterns in the underlying two point correlation function, particularly the real space correlations, it is a novel tool to characterize statistical anisotropies [17, 19–23]. Rotational symmetry about a preferred axis (say \hat{z}) is the simplest way to break SI.

In general, the correlation function may be decomposed into isotropic and anisotropic parts [26],

$$C(\hat{n}_1, \hat{n}_2) = C^{(I)}(\hat{n}_1, \hat{n}_2) + C^{(A)}(\hat{n}_1, \hat{n}_2). \quad (20)$$

where

$$C^{(I)}(\hat{n}_1, \hat{n}_2) = C(n_1 \cdot n_2) = \sum_l \frac{2l+1}{4\pi} C_l P_l(\hat{n}_1 \cdot \hat{n}_2), \quad (21)$$

and the anisotropic part $C^{(A)}$ is orthogonal to the Legendre polynomials

$$\int d\Omega_{\hat{n}_1} \int d\Omega_{\hat{n}_2} C^{(A)}(\hat{n}_1, \hat{n}_2) P_l(\hat{n}_1 \cdot \hat{n}_2) = 0. \quad (22)$$

This decomposition is useful in our study of the symmetries of the CMB correlation patterns/structure that are explicit in real space.

A. Statistical Isotropy (Rotational symmetry)

Under SI, the correlation function is a function only of θ , the angle between the two directions, say \hat{n}_1 and \hat{n}_2 . Hence, $C(\hat{n}_1, \hat{n}_2) \equiv C(\hat{n}_1 \cdot \hat{n}_2) = C(\theta)$, and the correlation function can be expanded in terms of Legendre polynomials

$$C^{(I)}(\theta) = \frac{1}{4\pi} \sum_{l=2}^{\infty} (2l+1) C_l P_l(\cos \theta), \quad (23)$$

where C_l is the angular power spectrum. The summation starts from $l=2$, since $l=0$ and $l=1$, respectively,

monopole and dipole, are usually subtracted out. For SI the angular power spectrum C_l contains all the information.

In Bipolar representation, the condition of SI for various observables, described in the previous section, can be summarized as follows [17, 23]:

- BipoSH : $A_{l_1 l_2}^{LM} = (-1)^{l_1} C_{l_1} (2l_1+1)^{1/2} \delta_{l_1 l_2} \delta_{L0} \delta_{M0}$,
- BiPS : $\kappa_L = \kappa_0 \delta_{L0}$,
- rBipoSH : $A_{LM} = \sum_{l_1} (-1)^{l_1} C_{l_1} (2l_1+1)^{1/2} \delta_{L0} \delta_{M0}$,
- Bipolar map: $\Theta = \frac{1}{2} \sum_l (-1)^l \sqrt{\frac{(2l+1)}{\pi}} C_l$.

Therefore, to test a CMB map for statistical isotropy, one should compute the BipoSH coefficients for the maps and look for non-zero BipoSH coefficients. Cosmic variance calculated for BipoSH under statistical isotropy is (see Appendix C) ,

$$\sigma_{SI}^2(\tilde{A}_{l_1 l_2}^{LM}) = C_{l_1} C_{l_2} [1 + (-1)^L \delta_{l_1 l_2}] \quad (24)$$

for rBipoSH is,

$$\sigma_{SI}^2(\tilde{A}_{LM}) = \sum_{l_1 l_2} C_{l_1} C_{l_2} [1 + (-1)^{l_1+l_2-L}] \quad (25)$$

and for BiPS [17, 18],

$$\begin{aligned} \sigma_{SI}^2(\kappa_L) = & \sum_{l:2l \geq L} 4C_l^4 \left[2 \frac{(2L+1)^2}{2l+1} + (-1)^L (2L+1) + (1+2(-1)^L) F_{ll}^L \right] + \sum_{l_1} \sum_{l_2=|L-l_1|}^{L+l_1} 4C_{l_1}^2 C_{l_2}^2 [(2L+1) + F_{l_1 l_2}^L] \\ & + 8 \sum_{l_1} \frac{(2L+1)^2}{2l_1+1} C_{l_1}^2 \left[\sum_{l_2=|L-l_1|}^{L+l_1} C_{l_2} \right]^2 + 16(-1)^L \sum_{l:2l \geq L} \frac{(2L+1)^2}{2l+1} \sum_{l_2=|L-l_1|}^{L+l_1} C_{l_1}^3 C_{l_2} \quad (26) \end{aligned}$$

where

$$\begin{aligned} F_{l_1 l_3}^L = & \sum_{m_1 m_2 = -l_1}^{l_1} \sum_{m_3 m_4 = -l_3}^{l_3} \sum_{M, M' = -L}^L C_{l_1 - m_1 l_3 - m_3}^{LM} C_{l_1 m_2 l_3 m_4}^{LM} \\ & \times C_{l_3 m_4 l_1 m_1}^{LM'} C_{l_3 - m_3 l_1 - m_2}^{LM'} \quad (27) \end{aligned}$$

and L is even. Statistically significant deviations from zero would mean violation of statistical isotropy.

B. Cylindrical symmetry

The correlation function must satisfy the symmetries of the underlying theory. In Friedman models the symmetry group is $SO(3)$, hence the correlation function is

invariant under rotations; any breakdown of SI will reduce this symmetry group. The simplest way to break SI is to introduce a favored direction in the sky, in such a case the reduced symmetry group is $SO(2)$ or cylindrical symmetry. Assuming the favored axis to be z-axis, the rotational symmetry about z-axis for any arbitrary $\Delta\phi$ will require,

$$C^{(A)}(\theta_1, \phi_1, \theta_2, \phi_2) = C^{(A)}(\theta_1, \phi_1 + \Delta\phi, \theta_2, \phi_2 + \Delta\phi).$$

where $n_1 \equiv (\theta_1, \phi_1)$ and $n_2 \equiv (\theta_2, \phi_2)$. The most general form of the correlation function in such a case is (see Appendix D)

$$C^{(A)}(\theta_1, \phi_1, \theta_2, \phi_2) = \sum_m f_m(\theta_1, \theta_2) \cos m(\phi_1 - \phi_2). \quad (28)$$

Further, if the correlation function is invariant under the reflection, i.e., looking at a correlation pattern in the

sky one cannot distinguish whether we are looking up or down the preferred direction, then

$$C(\pi - \theta_1, \phi_1, \pi - \theta_2, \phi_2) = C(\theta_1, \phi_1, \theta_2, \phi_2), \quad (29)$$

which leads to a condition that $l_1 + l_2$ is even. Biposh in such a case would be, or equivalently the covariance matrix will be [33],

$$\langle a_{l_1 m_1} a_{l_2 m_2}^* \rangle = \delta_{m_1 m_2} C_{|m_1|}^{l_1 l_2}, \quad (30)$$

where diagonal terms C_m^l of $C_{|m_1|}^{l_1 l_2}$ are called the cylindrical power spectrum, and $|m| > 0$ modes are the allowed frequencies for scale l . There may be correlations between various scales called connectivity of fluctuations. The expression for $C_{|m_1|}^{l_1 l_2}$ in terms of the correlation function is

$$C_{|m_1|}^{l_1 l_2} = \frac{1}{8\pi} \sqrt{\frac{(2l_1 + 1)(2l_2 + 1)(l_1 - m_1)!(l_2 - m_1)!}{(l_1 + m_1)!(l_2 + m_1)!}} \times \int_0^\pi P_{l_1 m_1}(\cos \theta_1) P_{l_2 m_1}(\cos \theta_2) f_{m_1}(\theta_1, \theta_2) d(\cos \theta_1) d(\cos \theta_2).$$

Using Eq. (D11) in the appendix, we obtain

$$A_{l_1 l_2}^{LM} = [1 + (-1)^{l_1 + l_2 - L}] \sum_m (-1)^m C_{|m|}^{l_1 l_2} C_{l_1 m l_2 - m}^{LM} \delta_{M0}.$$

When m is even, the functions P_l^m will be odd or even functions of their arguments, depending on whether l is odd or even respectively. Similarly, for the odd m 's. In both the cases when only one of l_1 and l_2 is odd, the integral vanishes. Therefore we have to consider cases when both of them are either odd or even. In such a case $l_1 + l_2$ is even and hence $A_{l_1 l_2}^{LM}$ vanishes for $L = \text{odd}$,

$$A_{l_1 l_2}^{LM} = A_{l_1 l_2}^{LM} \delta_{L, 2k} \delta_{M0}, \quad \text{where } k = 0, 1, 2, 3, \dots \quad (31)$$

$$A_{l_1 l_2}^{LM} = A_{l_1 l_2}^{LM} \delta_{L0} \delta_{M0} \delta_{l_1 l_2} + A_{l_1 l_2}^{LM} \delta_{M0}. \quad (32)$$

Therefore, the Biposh present under cylindrical symmetry are A_{ll}^{00} and $A_{l_1 l_2}^{L0}$ with even L . Using symmetry property of Biposh (6), under cylindrical symmetry we have $A_{l_1 l_2}^{LM} = A_{l_2 l_1}^{LM}$, i.e., the Biposh are symmetric under the exchange of l_1 and l_2 . There is another possibility that a_{lm} 's have a gaussian distribution with different variance for each m mode corresponding to a particular l . This implies breakdown of SI, as power in each m mode is different, $C_{|m_1|}^{l_1 l_2} = \delta_{l_1 l_2} C_{|m_1|}^{l_1}$, and the corresponding Bipolar coefficients are,

$$A_{l_1 l_2}^{LM} = A_{l_1 l_2}^{LM} \delta_{L0} \delta_{M0} \delta_{l_1 l_2} + A_{l_1 l_2}^{LM} \delta_{M0} \delta_{l_1 l_2}. \quad (33)$$

In such a case non-zero Biposh are A_{ll}^{00} and A_{ll}^{L0} , where multipole moment is even and ($L \geq 2$). Furthermore, it may happen that a given model displays the degeneracy

$C_{|m_1|}^{l_1} = C_{l_1}$ and the rotational symmetry $\text{SO}(3)$ of covariance matrix is restored. The rBipoSH for cylindrical symmetry are,

$$A_{LM} = \sum_{l_1 l_2} A_{l_1 l_2}^{LM} = A_{LM} \delta_{L, 2k} \delta_{M0} \quad \text{where } k = 0, 1, 2, 3, \dots \quad (34)$$

Hence the Bipolar map for such a symmetry will be,

$$\begin{aligned} \Theta(\hat{n}) &= \sum_{LM} A_{LM} Y_{LM}(\hat{n}) = \sum_L A_{L0} \delta_{L, 2k} Y_{L0}(\hat{n}) \\ &= \sum_L \sqrt{\frac{2L+1}{4\pi}} A_{L0} \delta_{L, 2k} P_L(\cos \theta) \end{aligned} \quad (35)$$

Thus the map here looks like a sphere which is divided into latitude bands, or zones, without any longitudinal variation.

A realistic example of cylindrical symmetry is a primordial homogeneous magnetic field which breaks statistical isotropy by inducing a preferred direction (\mathbf{e}). Therefore, the correlation function between two points (\mathbf{n} and \mathbf{n}') depends not only on the angular separation between two points (\mathbf{n}, \mathbf{n}') but also on their orientation with respect to the magnetic field. This dependence of correlation function on angles between \mathbf{n} and \mathbf{e} (as well as \mathbf{n}' and \mathbf{e}) leads to correlation between l and $l \pm 2$ modes [15]. The vector nature of the magnetic field induces off-diagonal correlations[15]

$$D_l(m) = \langle a_{l-1m}^* a_{l+1m} \rangle \equiv \langle a_{l+1m}^* a_{l-1m} \rangle. \quad (36)$$

Here D_l is the power spectrum of the off-diagonal elements of the covariance matrix, and the correlation function shows up as,

$$\langle a_{l_1 m_1} a_{l_2 m_2}^* \rangle = \delta_{m_1 m_2} \delta_{l_1 l_2} C_{l_1} + \delta_{m_1 m_2} (\delta_{l_1 + 1, l_2 - 1} + \delta_{l_1 - 1, l_2 + 1}) D_{l_1}. \quad (37)$$

The Biposh corresponding to this covariance matrix are [18],

$$\begin{aligned} A_{l_1 l_2}^{LM} &= (-1)^{l_1} (2l_1 + 1)^{1/2} C_{l_1} \delta_{l_1 l_2} \delta_{L0} \delta_{M0} \\ &+ D_{l_1} \delta_{l_1 l_2 \pm 2} \delta_{M0} \sum_m (-1)^m C_{l_1 m l_2 - m}^{L0}. \end{aligned} \quad (38)$$

The non-zero Biposh in this case are A_{ll}^{00} and $A_{l_1 l_2}^{L0}$. The reduced Bipolar spherical harmonic coefficients (rBipoSH) for this case are

$$\begin{aligned} A_{LM} &= \sum_l \delta_{M0} \delta_{L0} (2l + 1)^{1/2} C_l \\ &+ 2 \sum_{lm} (-1)^m D_l C_{l-1m l+1m}^{L0} \delta_{M0}. \end{aligned} \quad (39)$$

These coefficients are non-zero only for $l_1 + l_2 - L = \text{even}$, thus L can take only even values. Finally, the Bipolar map is,

$$\begin{aligned} \Theta(\hat{n}) &= \sum_L A_{L0} \delta_{L, 2k} Y_{L0}(\hat{n}), \quad k = 0, 1, 2, 3 \\ &= \frac{1}{2\sqrt{\pi}} A_{00} + \sum_L A_{L0} \delta_{L, 2a} P_L(\cos \theta), \quad a = 1, 2, \dots \end{aligned}$$

where A_{00} and A_{L0} are given by Eq.(39).

C. n-fold discrete Cylindrical symmetry

Violation of SI also manifests itself in compact universes with flat universal cover, which exhibits a n-fold rotational symmetry about an axis. There are six possible compact models of the universe having a flat universal cover (UC) [5]. These are visualized by identifying opposite sides of the fundamental polyhedra. The fundamental polyhedron (FP) may be a parallelepiped. The possible identifications then are (figure 1)

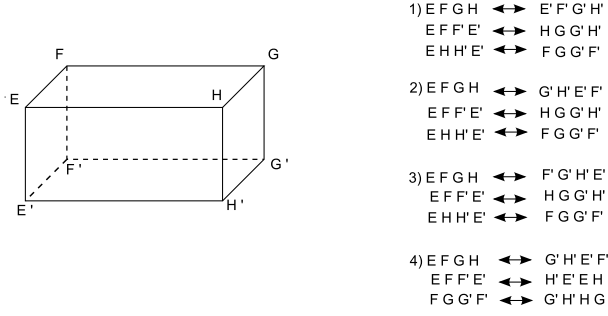


FIG. 1: *The locally Euclidean, closed, oriented 3-spaces.*

- 1) - opposite faces by translations.
- 2) - opposite faces, one pair being rotated by angle π .
- 3) - opposite faces, one pair being rotated by $\pi/2$.
- 4) - opposite faces, all three pairs being rotated by π .

The fundamental polyhedron can also be the interior of an hexagonal prism, with two possible identifications (figure 2) :

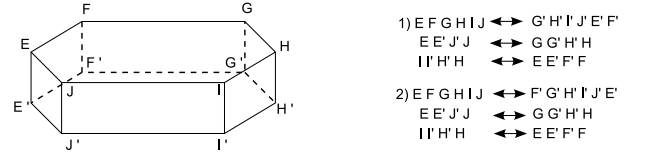


FIG. 2: *The locally Euclidean, closed, oriented 3-spaces.*

- 1) - opposite faces, the top face being rotated by an angle $2\pi/3$ with respect to the bottom face.
- 2) - opposite faces, the top face being rotated by an angle $\pi/3$ with respect to the bottom face.

Correlation function having a n-fold rotational symmetry about z-axis can be written as

$$C^{(A)}(\theta_1, \phi_1, \theta_2, \phi_2) = C^{(A)}\left(\theta_1, \phi_1 + \frac{2\pi}{n}, \theta_2, \phi_2 + \frac{2\pi}{n}\right). \quad (40)$$

This symmetry enforces $m_1 + m_2 = nk$, where n can be odd or even, depending upon the symmetry of the compact universe and $k = 0, 1, 2, 3, \dots$. Thus the general form of correlation function is (see Appendix E),

$$C^{(A)}(\theta_1, \phi_1, \theta_2, \phi_2) = \sum_{m_1, m_2} f_{m_1, m_2}(\theta_1, \theta_2) \times e^{i(m_1\phi_1 + m_2\phi_2)} \delta_{m_1 + m_2, nk}, \quad k = 0, \pm 1, \pm 2, \dots \quad (41)$$

Corresponding Bipolar spherical harmonic coefficients under the symmetry eq.(40) are,

$$A_{l_1 l_2}^{LM} = \frac{1}{4\pi} \sum_{m_1 m_2} \sqrt{\frac{(2l_1 + 1)(2l_2 + 1)(l_1 - m_1)!(l_2 - m_2)!}{(l_1 + m_1)!(l_2 + m_2)!}} \delta_{m_1 + m_2, nk} C_{l_1 m_1 l_2 m_2}^{LM} \int_0^{2\pi} \int_0^{2\pi} \int_0^\pi \int_0^\pi \times C(\theta_1, \phi_1, \theta_2, \phi_2) e^{-i(m_1\phi_1 + m_2\phi_2)} P_{l_1 m_1}(\cos \theta_1) P_{l_2 m_2}(\cos \theta_2) d(\cos \theta_1) d(\cos \theta_2) d\phi_1 d\phi_2. \quad (42)$$

All possible Euclidean models of compact universe exhibit reflection symmetry about the xy-plane. The correlation function under reflection symmetry is,

$$C(\theta_1, \phi_1, \theta_2, \phi_2) = C(\pi - \theta_1, \phi_1, \pi - \theta_2, \phi_2), \quad (43)$$

also under reflection of the coordinate system about x-y plane the spherical harmonics transform as,

$$Y_{lm}(\pi - \theta, \phi) = (-1)^{l+m} Y_{lm}(\theta, \phi). \quad (44)$$

Therefore, reflection symmetry demands,

$$P_{l_1}^{m_1}(\cos \theta_1) = P_{l_1}^{m_1}(\cos(\pi - \theta_1)) = P_{l_1}^{m_1}(-\cos \theta_1). \quad (45)$$

This implies $l_1 + m_1$ is even and similarly $l_2 + m_2$. Here we have used the symmetry property of Legendre polynomials, $P_{lm}(-x) = (-1)^{l+m} P_{lm}(x)$. Interestingly, from symmetries of spherical harmonics, one can show that n-fold symmetries are ruled out for odd n (see Appendix F).

Topologically compact universes exhibits even fold symmetry, but the emergence of this fact from the symmetry of two-point correlation pattern itself is nevertheless instructive. Therefore, we need to look at the cases when n is even.

D. Even-fold Cylindrical symmetry

Even fold symmetry refers to the case when n is even. For compact topologies this is always the case, for instance, Dirichlet domain (DD) of a T^2 toroidal universe [34], and a T^3 have a 4-fold symmetry, that of a hexagonal prism has a 6-fold symmetry and a squeezed torus has

2 fold symmetry. This symmetry puts another restriction on correlation function,

$$C(\theta_1, \phi_1, \theta_2, \phi_2) = C(\theta_1, -\phi_1, \theta_2, -\phi_2). \quad (46)$$

Hence most general correlation function under even-fold symmetry is (see Appendix E),

$$C^{(A)}(\theta_1, \phi_1, \theta_2, \phi_2) = \sum_{m_1, m_2} f_{m_1, m_2}(\theta_1, \theta_2) \delta_{m_1 + m_2, nk} \cos(m_1 \phi_1 + m_2 \phi_2).$$

Therefore Bipolar spherical harmonic coefficients are,

$$A_{l_1 l_2}^{LM} = [1 + (-1)^{l_1 + l_2 - L}] \sum_{m_1 m_2} \sqrt{\frac{(2l_1 + 1)(2l_2 + 1)(l_1 - m_1)!(l_2 - m_2)!}{(l_1 + m_1)!(l_2 + m_2)!}} C_{l_1 m_1 l_2 m_2}^{LM} \delta_{m_1 + m_2, nk} \int_0^{2\pi} \int_0^{2\pi} \int_0^\pi \int_0^\pi C(\theta_1, \phi_1, \theta_2, \phi_2) \cos(m_1 \phi_1 + m_2 \phi_2) P_{l_1}^{m_1}(\cos \theta_1) P_{l_2}^{m_2}(\cos \theta_2) d\phi_1 d\phi_2 d(\cos \theta_1) d(\cos \theta_2). \quad (47)$$

Reflection symmetry allows $A_{l_1 l_2}^{LM}$ only for even values of L . For odd indices coefficients vanish

$$A_{l_1 l_2}^{LM} = A_{l_1 l_2}^{LM} \delta_{Mnk} \delta_{L2a}, \quad k = 0, 1, 2, \dots, \quad a = 0, 1, 2, 3, \dots \quad (48)$$

Here nk is even, therefore for an even-fold symmetry i.e., ($n = 2, 4, 6, 8, \dots$) M is even and multipole moment $L \geq 2$ picks up only even values. Example: for a 2 fold symmetry possible values of $M = 2, 4, 6, 8, \dots$, similarly for 4-fold symmetry $M = 4, 8, 12, 16, \dots$ and so on. Another symmetry property of BipoSH is Eq. (6), so under even-fold rotational symmetry $l_1 + l_2 + m_1 + m_2 = \text{even}$ which implies $l_1 + l_2$ is even hence $A_{l_1 l_2}^{LM} = A_{l_2 l_1}^{LM}$ i.e., BipoSH are symmetric under the exchange of l_1 and l_2 . Note that for all possible compact flat spaces BipoSH vanish for odd indices, and the fact that two-point correlation function is invariant under reflection about xy-plane plays a pivotal role in restricting non-zero BipoSH only to even L 's. The rBipoSH for even universe with even fold symmetry are,

$$A_{LM} = A_{LM} \delta_{Mnk} \delta_{L2a}. \quad (49)$$

Bipolar map will be,

$$\Theta(\hat{n}) = \sum_{LM} A_{LM} Y_{LM}(\hat{n}) \delta_{Mnk} \delta_{L2a}, \quad (50)$$

L will take only even values and M will run from $-L$ to L , subsequently picking up even values.

IV. BIPOLAR MAP: EXAMPLE OF THE BIANCHI TEMPLATE

Now we will consider a Bianchi template as an example to show how a Bipolar map looks like for a given temperature map. The choice of Friedmann-Robertson-Walker (FRW) model as a model of our universe was initially due to its simplicity, and later because of observational evidence which strongly suggests universe to be homogeneous and isotropic at large scales. However, the presently observed isotropy may not necessarily hold in the past and the universe may have been anisotropic in its early stages and tends to FRW only later as it evolves. Bianchi models are the simplest examples which have the property to isotropise as they evolve in future. Bianchi classification contains 10 equivalent classes giving generic description of a homogeneous and anisotropic cosmology [12]. The most general Bianchi type which admits FRW at late time are VII_h and IX . However, the type IX re-collapses after a finite time hence do not come arbitrarily close to isotropy. Spiral pattern are characteristic signatures of VII_0 and VII_h models [12, 35, 36]. Jaffe et. al. proposed Bianchi VII_h models as an explanation of WMAP anomalies. Since class VII_h models resembles a universe with vorticity and hence can lead to bounds on the universal rotation in cosmological (CMB) data [37]. They proposed correction for some anomalies in the first year maps from WMAP, however, introducing such corrections induces other features like preferred direction and violation of SI. Pontzen et al. calculated various temperature and polarisation anisotropy patterns which may be formed in Bianchi cosmologies [38]. Ghosh

et. al. analyzed the temperature map for Bianchi VII_h template [13]. Given the temperature map for Bianchi VII_h template, here we see how a Bipolar map actually looks like.

The temperature map for Bianchi VII_h template is of the form

$$\Delta T^B(\theta, \phi) = f_1(\theta) \sin \phi + f_2(\theta) \cos \phi, \quad (51)$$

BipoSH for Bianchi template are,

$$\begin{aligned} A_{l_1 l_2}^{LM} = & \int_0^\pi \int_0^\pi \{ W_{l_1 l_2}(\theta_1, \theta_2) C_{l_1-1 l_2 1}^{LM} \delta_{M0} + X_{l_1 l_2}(\theta_1, \theta_2) C_{l_1-1 l_2-1}^{LM} \delta_{M-2} + \\ & Y_{l_1 l_2}(\theta_1, \theta_2) C_{l_1 1 l_2-1}^{LM} \delta_{M0} + Z_{l_1 l_2}(\theta_1, \theta_2) C_{l_1 1 l_2 1}^{LM} \delta_{M2} \} d(\cos \theta_1) d(\cos \theta_2). \end{aligned} \quad (52)$$

Therefore, rBiposh are,

$$\begin{aligned} A_{LM} = & \sum_{l_1 l_2} \int_0^\pi \int_0^\pi \{ W_{l_1 l_2}(\theta_1, \theta_2) C_{l_1-1 l_2 1}^{LM} \delta_{M0} + X_{l_1 l_2}(\theta_1, \theta_2) C_{l_1-1 l_2-1}^{LM} \delta_{M-2} + \\ & Y_{l_1 l_2}(\theta_1, \theta_2) C_{l_1 1 l_2-1}^{LM} \delta_{M0} + Z_{l_1 l_2}(\theta_1, \theta_2) C_{l_1 1 l_2 1}^{LM} \delta_{M2} \} d(\cos \theta_1) d(\cos \theta_2), \end{aligned} \quad (53)$$

where

$$\begin{aligned} W_{l_1 l_2} = & \pi^2 \sqrt{\frac{(2l_1+1)(2l_2+1)(l_1+1)!(l_2-1)!}{(4\pi)^2(l_1-1)!(l_2+1)!}} \{ f_1(\theta_1) f_1(\theta_2) + i(f_1(\theta_1) f_2(\theta_2) - f_2(\theta_1) f_1(\theta_2)) + \\ & f_2(\theta_1) f_2(\theta_2) \} P_{l_1}^{-1}(\cos \theta_1) P_{l_1}^1(\cos \theta_2), \\ X_{l_1 l_2} = & \pi^2 \sqrt{\frac{(2l_1+1)(2l_2+1)(l_1+1)!(l_2+1)!}{(4\pi)^2(l_1-1)!(l_2-1)!}} \{ -f_1(\theta_1) f_1(\theta_2) + i(f_1(\theta_1) f_2(\theta_2) + f_2(\theta_1) f_1(\theta_2)) + \\ & f_2(\theta_1) f_2(\theta_2) \} P_{l_1}^{-1}(\cos \theta_1) P_{l_1}^{-1}(\cos \theta_2), \\ Y_{l_1 l_2} = & \pi^2 \sqrt{\frac{(2l_1+1)(2l_2+1)(l_1-1)!(l_2+1)!}{(4\pi)^2(l_1+1)!(l_2-1)!}} \{ f_1(\theta_1) f_1(\theta_2) + i(-f_1(\theta_1) f_2(\theta_2) + f_2(\theta_1) f_1(\theta_2)) + \\ & f_2(\theta_1) f_2(\theta_2) \} P_{l_1}^1(\cos \theta_1) P_{l_1}^{-1}(\cos \theta_2), \\ Z_{l_1 l_2} = & \pi^2 \sqrt{\frac{(2l_1+1)(2l_2+1)(l_1-1)!(l_2-1)!}{(4\pi)^2(l_1+1)!(l_2+1)!}} \{ -f_1(\theta_1) f_1(\theta_2) - i(f_1(\theta_1) f_2(\theta_2) + f_2(\theta_1) f_1(\theta_2)) + \\ & f_2(\theta_1) f_2(\theta_2) \} P_{l_1}^1(\cos \theta_1) P_{l_1}^1(\cos \theta_2). \end{aligned}$$

Hence,

$$A_{LM} = A_{LM} \delta_{M,k} \quad k = 0, \pm 2. \quad (54)$$

Possible values of M are $0, \pm 2$. Note that A_{LM} exists only for $l_1 + l_2 - L = \text{even}$, and vanishes otherwise. Keeping in mind the reality of two-point correlation function, i.e., $A_{LM} = (-1)^M A_{L-M}^*$, here we have $A_{L2} = A_{L-2}^*$. Since A_{LM} coefficients are complex numbers we can define, $X_{LM} = \Re(A_{LM})$ and $Z_{LM} = \Im(A_{LM})$. Therefore,

where super-script B signifies Bianchi, and $f_1(\theta)$ and $f_2(\theta)$ are parameters of the model which should be calculated numerically [12].

Bipolar map for a Bianchi template looks like (see appendix G),

$$\begin{aligned} \Theta(\hat{n}) = & \sum_L A_{L0} Y_{L0}(\theta, \phi) + 2 \sum_L X_{L-2} G_L(\theta) \cos 2\phi \\ & - 2 \sum_L Z_{L-2} G_L(\theta) \sin 2\phi, \end{aligned} \quad (55)$$

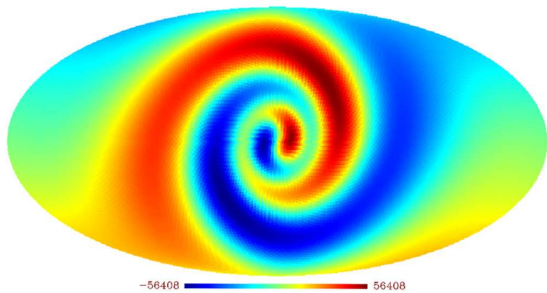


FIG. 3: *Temperature map for Bianchi VII_h.*

where

$$G_L(\theta) = \frac{1}{(\sin \theta)^2} \sqrt{\frac{(L-1)L(L+1)(L+2)}{4\pi(2L+1)}} \quad (56)$$

$$\left[\frac{P_{L-2}(\cos \theta)}{2L-1} - \frac{2(2L+1)P_L(\cos \theta)}{(2L-1)(2L+3)} + \frac{P_{L+2}(\cos \theta)}{2L+3} \right].$$

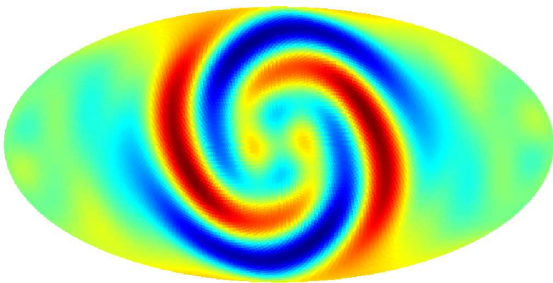


FIG. 4: *Bipolar map for Bianchi VII_h.*

Thus, a spiral pattern in temperature map will show up as double spiral pattern in Bipolar map.

V. CONCLUSION AND DISCUSSION

Representation of correlation function of CMB anisotropy in terms of Bipolar spherical harmonics provides a novel approach to study violations of SI. Very recently the Bipolar representation has been used to quantify anomalies in the analysis of WMAP seven - year data [42]. These anisotropies can arise due to departure from FRW metric (eg. Bianchi models), non-trivial spatial topologies (compact spaces) or from primordial magnetic fields, among others. Here we have studied various measurable quantities of Bipolar formalism to quantify breakdown of SI.

We studied anisotropic homogeneous cosmologies which leave a characteristic pattern on CMB. Like Bianchi VII_h temperature map which has a spiral pattern of a pair of cold and hot spots with a dipole in

azimuthal space. We found that the corresponding pattern in Bipolar space becomes a double spiral having a quadrupole in azimuthal space of the Bipolar map.

Another application is in case of homogeneous isotropic models where an anisotropic topological identification has been imposed. As an example, if the space is compact in one (or more) direction(s), the statistical isotropy is broken due to introduction of preferred direction(s).

We calculate BipoSH when this preferred direction is introduced. We have shown here that for compact topologies, symmetry requirements can restrict BipoSH to even multipole moments, i.e., BipoSH vanish for odd indices for all kind of physically plausible models of flat multi-connected universe. Hyperbolic manifolds do not have the desired symmetry and hence we expect odd multipoles to be non-zero in these manifolds. Hence, we have a tool to distinguish different topologies. In case of homogeneous magnetic fields we have shown that BipoSH's are restricted to even L and $M = 0$.

A new representation of Bipolar map has been proposed. Further work needs to be done in this direction to extract new information from this representation.

This technique can be applied to polarization maps and it may prove to be a powerful method to decipher the topology of the universe, something on which general relativity is completely silent. The Bipolar formalism can also be applied to various anisotropic universes and can be used as a tool to distinguish various types of SI breakdown.

Acknowledgments

The authors deeply regret the untimely demise of their colleague, Himan Mukhopadhyay, and acknowledge her valuable contribution in this work that originates and builds upon the excellent work carried out during a graduate school project at IUCAA [39]. We also thank Tuhin Ghosh for discussions at several stages of this work. SJ and NJ would like to thank IUCAA for its hospitality and computational facilities.

Appendix A: A review of topologically compact spaces

Topologically compact spaces break the statistical isotropy, thereby introducing signatures in CMB correlation patterns. A compact cosmological model, \mathcal{M} , is a *Quotient space*, constructed by identifying points of standard FRW space under the action of suitable discrete subgroup of motions Γ , of the full isometry group G of the FRW space. The isometry group G is the group of motions which preserves the distance between points. The simply connected infinite FRW spatial hypersurface with same constant curvature geometry is the universal cover (UC), \mathcal{M}^u , tiled by the copies of the compact space,

\mathcal{M} . It can be spherical (\mathcal{S}^3), Euclidean (\mathcal{E}^3) or hyperbolic (\mathcal{H}^3). The compact space for a given location of observer is represented as *Dirichlet domain* (DD), with the observer at its basepoint. Any point \mathbf{x} of the compact space has an image, $x_i = \gamma_i \mathbf{x}$, in each copy of DD on the universal cover, where $\gamma_i \in \Gamma$. By construction DD represents the compact space as a *convex polyhedron* with even number of faces identified pairwise under Γ . In cosmology, DD around the observer represents the universe as seen by the observer and the symmetries of the correlation function are nothing but the symmetries of the corresponding DD.

Correlation function of a scalar field, Φ , on a compact manifold, \mathcal{M} , can be expressed as [40],

$$\xi_{\Phi}^C(\mathbf{x}, \mathbf{x}') = \sum_i \sum_{j=1}^{m_i} P_{\Phi}(k_i) \Psi_{ij}(\mathbf{x}) \Psi_{ij}^*(\mathbf{x}'), \quad (\text{A1})$$

where

$$(\nabla^2 + k_i^2) \Psi_{ij} = 0. \quad (\text{A2})$$

Ψ_i are orthonormal set of eigenfunctions of Laplacian on the hypersurface, having positive and discrete set of eigenvalues $\{k_i^2\}$ ($k_o^2 = 0$ and $k_i^2 < k_{i+1}^2$) with multiplicities m_i . On a compact manifold, \mathcal{M} , the set of eigenfunctions and eigenvalues are not always easy to obtain in closed form (even numerically, for compact hyperbolic spaces). On the other hand, eigenfunctions $\Psi_j^u(k, \mathbf{x})$ of the universal cover (UC), \mathcal{M}^u , of a compact manifold usually known because of their simplicity (e.g., \mathcal{H}^3 , \mathcal{S}^3 and \mathcal{E}^3), hence they can be used to compute the correlation functions $\xi_{\Phi}^C(\mathbf{x}, \mathbf{x}')$ on the UC. For flat and hyperbolic UC's the set of eigenvalues are continuous. The function $P_{\Phi}(k_i)$ is the *rms* amplitude of the eigenmode expansion of the field Φ , whose information lies in the physical mechanism responsible for the generation of Φ . The *regularized method of images* [25], describes how correlation function on the compact manifold can be calculated once the correlation function on the universal cover is known [16, 25, 26], which is expressed as,

$$\xi_{\Phi}^C(\mathbf{x}, \mathbf{x}') = \widetilde{\sum_{\gamma \in \Gamma}} \xi_{\Phi}^u(\mathbf{x}, \mathbf{x}'). \quad (\text{A3})$$

This implies that correlation function on a compact space, \mathcal{M} , can be expressed as sum over the correlation function on its universal covering space, \mathcal{M}^u , calculated between \mathbf{x} and the images $\gamma \mathbf{x}'$ ($\gamma \in \Gamma$) of \mathbf{x}' . The local homogeneity and isotropy demands that the correlation function on the UC is only a function of the distance between two points \mathbf{x} and \mathbf{x}' i.e $r \equiv d(\mathbf{x}, \mathbf{x}')$. The correlation function on a compact universe with flat UC is,

$$\xi_{\Phi}^C(\mathbf{x}, \mathbf{x}') = \sum_i \int \frac{dk}{k} P_{\Phi}(k) \frac{\sin kd_i}{kd_i}, \quad (\text{A4})$$

Here, $P_{\Phi}(k)$ can be determined from the early universe physical mechanism and d_i is the distance between the

images of \mathbf{x} and \mathbf{x}' (d_o is the distance between original points). Summation implies summing over all images. Hence, correlation function depends not only on the distance between two points and the distance of their images but symmetry defines both the pair to have identical distance from their images i.e., take any two points on the last scattering surface and their corresponding images about xy-plane, correlation function will turn out to be invariant under this reflection. Figure (5) illustrates this

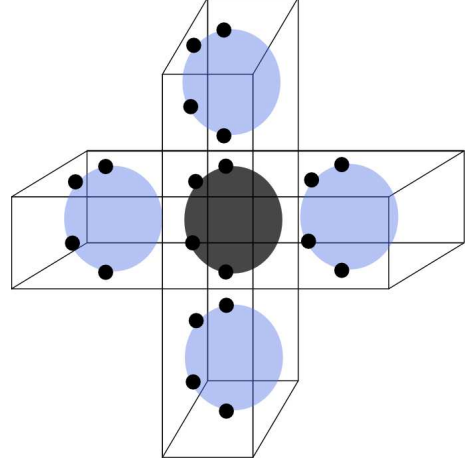


FIG. 5: Images of two-pairs of point which are mirror images of each other about the XY plane in a T^3 space (one dimension suppressed).

point for a T^3 universe. The DD of a squeezed torus is shown in figure (6). The choice of the axes here is a little bit more non-trivial. The xy-plane is not parallel to any of the faces of the DD or the FP, but still it would cut the LSS into two halves in such a way that there will be symmetry under reflection about the xy-plane and on the xy-plane there will be 2-fold rotational symmetry. However,

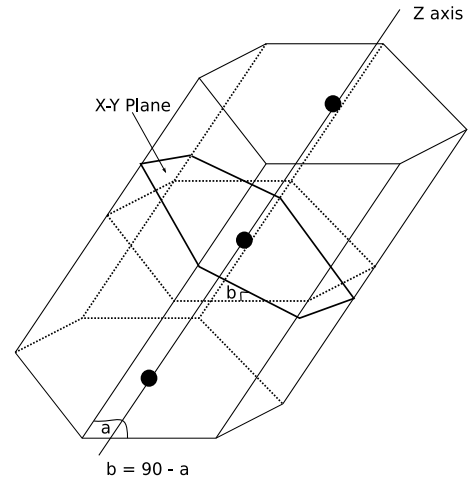


FIG. 6: Dirichlet domain of a squeezed torus.

here we point out that reflection symmetry does not hold

good for the compact spaces for which the opposite faces are glued together with a twist [39].

Topology of the universe leaves characteristic signatures on CMB. If the universe is finite and smaller than the distance to the last scattering surface(LSS), then the signature of the topology of the universe is imprinted on the CMB. For such a small universe LSS can wrap around the universe and will self-intersect. The intersection of the LSS, which is a 2-sphere with itself is a circle that will appear twice in the cosmic microwave background. Hence, there might exist pairs of circles which share correlated patterns of temperature fluctuations. This *circles in the sky* [41] method is a powerful and direct probe for detecting non-trivial spatial topology. The correlated patterns would be matching perfectly if the temperature fluctuation did not depend on the direction of observation and if the patterns were not distorted. However, the observed temperature fluctuations has direction dependent components, i.e. the *Doppler effect* and the *integrated Sachs-Wolfe effect*. Also observationally, galaxy cut and foreground removals can also distort the matching. However, one can search for such patterns in CMB correlation function statistically. In a multi-connected space, there exist preferred direction(s) so that global isotropy is broken. The angular correlation will then depend on two directions of observations and can also depend on the position of the observer. This induces correlations between a_{lm} 's of different l and m . Thus, another indirect probe is to search such patterns or signatures in the statistics of CMB temperature fluctuations.

Appendix B: Bipolar map representation in terms of tripolar spherical harmonics

Bipolar map is defined as

$$\Theta(\hat{n}) = \sum_{LM} A_{LM} Y_{LM}(\hat{n}), \quad (\text{B1})$$

where $A_{LM} = \sum_{l_1 l_2} A_{l_1 l_2}^{LM}$, therefore

$$\Theta(\hat{n}) = \sum_{LM} \sum_{l_1 l_2} A_{l_1 l_2}^{LM} Y_{LM}(\hat{n}) \quad (\text{B2})$$

Now using the expansion of $A_{l_1 l_2}^{LM}$ we get

$$\Theta(\hat{n}) = \sum_{LM} \sum_{l_1 l_2} \int d\Omega_{\hat{n}_1} \int d\Omega_{\hat{n}_2} C(\hat{n}_1, \hat{n}_2) \times \{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{LM}^* Y_{LM}(\hat{n})$$

tripolar scalar spherical harmonics are defined as

$$\{Y_{l_1}(\hat{n}_1) \otimes \{Y_L(\hat{n}) \otimes Y_{l_2}(\hat{n}_2)\}_{\lambda}\}_{00} = (-1)^{l_1+l_2+L} \quad (\text{B3})$$

$$\delta_{\lambda 1} \sum_{m_1 m_2 M} \begin{pmatrix} l_1 & L & l_2 \\ m_1 & M & m_2 \end{pmatrix} Y_{l_1 m_1}(\hat{n}_1) Y_{LM}(\hat{n}) Y_{l_2 m_2}(\hat{n}_2)$$

where $\begin{pmatrix} l_1 & L & l_2 \\ m_1 & M & m_2 \end{pmatrix}$ are Wigner-3j symbols and are related to Clebsch Gordan coefficients in the following way,

$$C_{l_1 m_1 l_2 m_2}^{l_3 m_3} = (-1)^{l_1-l_2+m_3} \sqrt{2l_3+1} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & -m_3 \end{pmatrix}$$

Hence Bipolar map can be represented in terms of tripolar scalar spherical harmonics,

$$\Theta(\hat{n}) = \sum_{L, l_1, l_2} \int d\Omega_{\hat{n}_1} d\Omega_{\hat{n}_2} C(\hat{n}_1, \hat{n}_2) (-1)^{l_1+l_2} \sqrt{2L+1} \delta_{\lambda L} \{Y_{l_1}(\hat{n}_1) \otimes \{Y_L(\hat{n}) \otimes Y_{l_2}(\hat{n}_2)\}_{\lambda}\}_{00} \quad (\text{B4})$$

The representation of the Bipolar map in terms of tripolar harmonic function makes the transformation properties of the bipolar map under rotations explicit.

In a rotated sky map,

$$\Theta'(\hat{n}) = \sum_{LM l_1 l_2} \int [\sum_{l'_1, l'_2, L', M'} A_{l'_1 l'_2}^{L' M'} \sum_{M''} D_{M'' M'}^{L'}(R) \{Y_{l'_1}(\hat{n}_1) \otimes Y_{l'_2}(\hat{n}_2)\}_{L' M''}] \times \{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{LM}^* Y_{LM}(\hat{n}) d\Omega_{\hat{n}_1} d\Omega_{\hat{n}_2} \quad (\text{B5})$$

Using orthogonality of Bipolar spherical harmonics

$$\Theta'(\hat{n}) = \sum_{LM l_1 l_2} \sum_{l'_1, l'_2, L', M'} A_{l'_1 l'_2}^{L' M'} \sum_{M''} D_{M'' M'}^{L'}(R) Y_{LM} \delta_{l_1 l'_1} \delta_{l_2 l'_2} \delta_{LL'} \delta_{MM'} \quad (\text{B6})$$

$$\Theta'(\hat{n}) = \sum_{LM'} A_{LM'} \sum_M D_{MM'}^L(R) Y_{LM}(\hat{n}) = \Theta(R\hat{n}) \quad (\text{B7})$$

Thus when correlation pattern is rotated by “R”, Bipolar map also rotates by “R”.

Appendix C: Cosmic Variance of Bipolar Quantities

Cosmic variance is defined as the variance of estimator of an observable constructed from a single sky map. In particular for BipoSH

$$\sigma^2(\tilde{A}_{l_1 l_2}^{LM}) = \langle (\tilde{A}_{l_1 l_2}^{LM})^2 \rangle - \langle \tilde{A}_{l_1 l_2}^{LM} \rangle^2. \quad (\text{C1})$$

Using Gaussianity of ΔT , one can analytically compute the variance of $\tilde{A}_{l_1 l_2}^{LM}$.

$$\tilde{A}_{l_1 l_2}^{LM} = \sum_{m_1 m_2} a_{l_1 m_1} a_{l_2 m_2}^* (-1)^{m_2} C_{l_1 m_1 l_2 - m_2}^{LM} \quad (\text{C2})$$

therefore

$$\langle \tilde{A}_{l_1 l_2}^{LM} \tilde{A}_{l_1 l_2}^{*LM} \rangle = \sum_{m_1 m_2} \sum_{m'_1 m'_2} \langle a_{l_1 m_1} a_{l_2 m_2}^* a_{l_1 m'_1}^* a_{l_2 m'_2} \rangle (-1)^{m_2+m'_2} C_{l_1 m_1 l_2 - m_2}^{LM} C_{l_1 m'_1 l_2 - m'_2}^{*LM} \quad (\text{C3})$$

Considering temperature field to be a Gaussian random field, one can expand the four-point correlation function in terms of two-point correlation function. Further, considering the fact that under statistical isotropy the covariance matrix is diagonal eq.(3) the above equation reduces to

$$\begin{aligned} \langle \tilde{A}_{l_1 l_2}^{LM} \tilde{A}_{l_1 l_2}^{*LM} \rangle &= C_{l_1} C_{l_2} \delta_{l_1 l_2} (2l_1 + 1) \delta_{L0} \delta_{M0} \\ &+ C_{l_1} C_{l_2} [1 + (-1)^L \delta_{l_1 l_2}] \end{aligned} \quad (\text{C4})$$

Also, we have

$$\langle \tilde{A}_{l_1 l_2}^{LM} \rangle = (2l_1 + 1)^{1/2} C_{l_1} \delta_{l_1 l_2} \delta_{L0} \delta_{M0} \quad (\text{C5})$$

Hence the cosmic variance is

$$\sigma_{SI}^2(\tilde{A}_{l_1 l_2}^{LM}) = C_{l_1} C_{l_2} [1 + (-1)^L \delta_{l_1 l_2}] \quad (\text{C6})$$

Similarly for rBipoSH,

$$\sigma_{SI}^2(\tilde{A}_{LM}) = \sum_{l_1 l_2} C_{l_1} C_{l_2} [1 + (-1)^{l_1 + l_2 - L}] \quad (\text{C7})$$

Appendix D: Correlation function for Cylindrical symmetry

Expansion of correlation function in terms of Bipolar spherical harmonics is,

$$\begin{aligned} C^{(A)}(\hat{n}_1, \hat{n}_2) &= \sum_{l_1, l_2, L, M} A_{l_1 l_2}^{LM} \sum_{m_1 m_2} C_{l_1 m_1 l_2 m_2}^{LM} \\ &Y_{l_1 m_1}(\hat{n}_1) Y_{l_2 m_2}(\hat{n}_2) \end{aligned} \quad (\text{D1})$$

Now rotational symmetry about z-axis for any arbitrary $\Delta\phi$ implies,

$$C^{(A)}(\theta_1, \phi_1, \theta_2, \phi_2) = C^{(A)}(\theta_1, \phi_1 + \Delta\phi, \theta_2, \phi_2 + \Delta\phi) \quad (\text{D2})$$

Therefore

$$\begin{aligned} &\sum_{l_1, l_2, L, M, m_1, m_2} A_{l_1 l_2}^{LM} C_{l_1 m_1 l_2 m_2}^{LM} Y_{l_1 m_1}(\theta_1, \phi_1) Y_{l_2 m_2}(\theta_2, \phi_2) \\ &= \sum_{l_1, l_2, L, M, m_1, m_2} A_{l_1 l_2}^{LM} C_{l_1 m_1 l_2 m_2}^{LM} Y_{l_1 m_1}(\theta_1, \phi_1 + \Delta\phi) \\ &Y_{l_2 m_2}(\theta_2, \phi_2 + \Delta\phi) \end{aligned}$$

which means

$$e^{i(m_1 + m_2)\Delta\phi} = 1 \quad (\text{D3})$$

therefore

$$m_1 + m_2 = \frac{2k\pi}{\Delta\phi}, \quad k = 0, \pm 1, \pm 2, \dots \quad (\text{D4})$$

for zero fold symmetry $m_1 + m_2 = 0$ which means $m_1 = -m_2$, hence

$$\begin{aligned} C^{(A)}(\hat{n}_1, \hat{n}_2) &= \sum_{l_1, l_2, L, M, m_1, m_2} A_{l_1 l_2}^{LM} C_{l_1 m_1 l_2 m_2}^{LM} Y_{l_1 m_1}(\hat{n}_1) \\ &Y_{l_2 m_2}(\hat{n}_2) \delta_{m_1, -m_2} \end{aligned} \quad (\text{D5})$$

using the expansion of spherical harmonics in terms of associated Legendre polynomials,

$$Y_{lm}(\theta, \phi) = e^{im\phi} \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos\theta) \quad (\text{D6})$$

therefore correlation function will be,

$$C^{(A)}(\hat{n}_1, \hat{n}_2) = \sum_m f_m(\theta_1, \theta_2) e^{im(\phi_1 - \phi_2)} \quad (\text{D7})$$

where

$$\begin{aligned} f_m(\theta_1, \theta_2) &= \frac{1}{4\pi} \sum_{l_1, l_2, L} A_{l_1 l_2}^{LM} C_{l_1 m l_2 -m}^{LM} \delta_{M0} \\ &\sqrt{\frac{(2l_1+1)(2l_2+1)(l_1-m)!(l_2+m)!}{(l_1+m)!(l_2-m)!}} \\ &P_{l_1}^m(\cos\theta_1) P_{l_2}^{-m}(\cos\theta_2) \end{aligned} \quad (\text{D8})$$

Symmetry ensures,

$$C^{(A)}(\theta_1, \phi_1, \theta_2, \phi_2) = C^{(A)}(\theta_1, -\phi_1, \theta_2, -\phi_2) \quad (\text{D9})$$

Imposing this symmetry we get,

$$C^{(A)}(\theta_1, \phi_1, \theta_2, \phi_2) = \sum_m f_m(\theta_1, \theta_2) \cos m(\phi_1 - \phi_2) \quad (\text{D10})$$

This is the most general correlation function under zero fold rotational symmetry. Using

$$A_{l_1 l_2}^{LM} = \int d\Omega_{\hat{n}_1} d\Omega_{\hat{n}_2} C(\hat{n}_1, \hat{n}_2) \{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{LM}^*$$

and eq.(D10) we get,

$$\begin{aligned} A_{l_1 l_2}^{LM} &= [1 + (-1)^{l_1 + l_2 - L}] \sum_m (-1)^m \frac{\sqrt{(2l_1+1)(2l_2+1)(l_1-m)!(l_2-m)!}}{(4\pi)^2 (l_1+m)!(l_2+m)!} C_{l_1 m l_2 -m}^{LM} \delta_{M0} \\ &\int_0^\pi \int_0^\pi P_{l_1 m}(\cos\theta_1) P_{l_2 m}(\cos\theta_2) f_m(\theta_1, \theta_2) d(\cos\theta_1) d(\cos\theta_2). \end{aligned} \quad (\text{D11})$$

Appendix E: n-fold cylindrical symmetry

Correlation function in such a case is,

$$C^{(A)}(\theta_1, \phi_1, \theta_2, \phi_2) = C^{(A)}\left(\theta_1, \phi_1 + \frac{2\pi}{n}, \theta_2, \phi_2 + \frac{2\pi}{n}\right). \quad (\text{E1})$$

This implies,

$$\begin{aligned} \sum_{l_1 l_2 m_1 m_2 LM} A_{l_1 l_2}^{LM} C_{l_1 m_1 l_2 m_2}^{LM} Y_{l_1 m_1}(\theta_1, \phi_1) Y_{l_2 m_2}(\theta_2, \phi_2) = \\ \sum_{l_1 l_2 m_1 m_2 LM} A_{l_1 l_2}^{LM} C_{l_1 m_1 l_2 m_2}^{LM} Y_{l_1 m_1}\left(\theta_1, \phi_1 + \frac{2\pi}{n}\right) Y_{l_2 m_2}\left(\theta_2, \phi_2 + \frac{2\pi}{n}\right) \end{aligned}$$

where

$$f_{m_1 m_2}(\theta_1, \theta_2) = \frac{1}{4\pi} \sum_{l_1 l_2 LM} \sqrt{\frac{(2l_1 + 1)(2l_2 + 1)(l_1 - m_1)!(l_2 - m_2)!}{(l_1 + m_1)!(l_2 + m_2)!}} A_{l_1 l_2}^{LM} C_{l_1 m_1 l_2 m_2}^{LM} P_{l_1}^{m_1}(\cos \theta_1) P_{l_2}^{m_2}(\cos \theta_2)$$

Demanding explicitly the two fold symmetry that holds for all even-fold symmetry,

$$C(\theta_1, \phi_1, \theta_2, \phi_2) = C(\theta_1, -\phi_1, \theta_2, -\phi_2) \quad (\text{E2})$$

this symmetry rules out the presence of sine terms in correlation function. Hence for even-fold symmetry correlation function reduces to,

$$C^{(A)}(\theta_1, \phi_1, \theta_2, \phi_2) = \sum_{m_1, m_2} f_{m_1, m_2}(\theta_1, \theta_2) \delta_{m_1 + m_2, nk} \cos(m_1 \phi_1 + m_2 \phi_2) \quad (\text{E3})$$

Appendix F: Absence of odd-fold symmetries for compact spaces

Let us take a compact space with reflection symmetry. It would demand that

$$C(\pi - \theta_1, \phi_1, \pi - \theta_2, \phi_2) = C(\theta_1, \phi_1, \theta_2, \phi_2). \quad (\text{F1})$$

It can be shown that

$$C(\theta_1, \phi_1, \theta_2, \phi_2) = \sum_{l_1 l_2 m_1 m_2 LM} (-1)^{m_2} \langle a_{l_1 m_1} a_{l_2 m_2}^* \rangle \times C_{l_1 m_1 l_2 - m_2}^{LM} \{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{LM}. \quad (\text{F2})$$

The symmetry of spherical harmonics would ensure that

$$\begin{aligned} \{Y_{l_1}(\pi - \theta_1, \phi_1) \otimes Y_{l_2}(\pi - \theta_2, \phi_2)\}_{LM} \\ = (-1)^{m_1 + m_2} \{Y_{l_1}(\theta_1, \phi_1) \otimes Y_{l_2}(\theta_2, \phi_2)\}_{LM}. \quad (\text{F3}) \end{aligned}$$

This put together with equations (F1) and (F2) indicates that m_1 must be an even number. Since they are dummy indices, m_2 would be even too. Now let us consider such a space with n fold symmetry. Evidently

$$C\left(\theta_1, \phi_1 + \frac{2\pi}{n}, \theta_2, \phi_2 + \frac{2\pi}{n}\right) = C(\theta_1, \phi_1, \theta_2, \phi_2) \quad (\text{F4})$$

Hence, $e^{i(m_1 + m_2)\frac{2\pi}{n}} = 1$, which implies $m_1 + m_2 = nk, k = 0, \pm 1, \pm 2, \pm 3, \dots$

Most general form of correlation function will be,

$$C^{(A)}(\theta_1, \phi_1, \theta_2, \phi_2) = \sum_{m_1 m_2} f_{m_1 m_2}(\theta_1, \theta_2) e^{i(m_1 \phi_1 + m_2 \phi_2)} \delta_{m_1 + m_2, nk}$$

since we know

$$\begin{aligned} \left\{Y_{l_1}\left(\theta_1, \phi_1 + \frac{2\pi}{n}\right) \otimes Y_{l_2}\left(\theta_2, \phi_2 + \frac{2\pi}{n}\right)\right\}_{LM} = \\ \exp\left(\frac{2\pi(m_1 + m_2)}{n}\right) \{Y_{l_1}(\theta_1, \phi_1) \otimes Y_{l_2}(\theta_2, \phi_2)\}_{LM}. \quad (\text{F5}) \end{aligned}$$

For Eq. (F4) to hold $(m_1 + m_2)/n$ must be even. Since m_1 and m_2 are even, n has to be even too.

Appendix G: Bianchi template

The temperature map for Bianchi template is written as

$$\Delta T(\theta, \phi) = f_1(\theta) \sin \phi + f_2(\theta) \cos \phi. \quad (\text{G1})$$

Bipolar map can be expressed as,

$$\Theta(\hat{n}) = \sum_{LM} \sum_{l_1 l_2} \sum_{m_1 m_2} \int d\Omega_{\hat{n}_1} d\Omega_{\hat{n}_2} \langle \Delta T(\hat{n}_1) \Delta T(\hat{n}_2) \rangle C_{l_1 m_1 l_2 m_2}^{LM} Y_{l_1 m_1}^*(\hat{n}_1) Y_{l_2 m_2}^*(\hat{n}_2) Y_{LM}(\hat{n}).$$

The integrals over ϕ contribute only for $m = \pm 1$ otherwise it vanishes. Constraint on the values of $m_1 = \pm 1$ and $m_2 = \pm 1$, admits only $M = 0, \pm 2$. Reduced Bipolar coefficient is then

$$\begin{aligned} A_{LM} = \sum_{l_1 l_2} \sum_{m_1 m_2} \int d\Omega_{\hat{n}_1} d\Omega_{\hat{n}_2} \langle \Delta T(\hat{n}_1) \Delta T(\hat{n}_2) \rangle \\ (-1)^{m_1 + m_2} C_{l_1 m_1 l_2 m_2}^{LM} Y_{l_1 - m_1}(\hat{n}_1) Y_{l_2 - m_2}(\hat{n}_2). \end{aligned}$$

A_{LM} exists only for $l_1 + l_2 - L = \text{even}$, and vanishes otherwise and the reality condition demands A_{LM} , i.e., $A_{LM} = (-1)^M A_{L-M}^*$. Now Bipolar map is

$$\Theta(\hat{n}) = \sum_{LM} A_{LM} Y_{LM}(\hat{n}) \quad (\text{G2})$$

but for Bianchi template it will be

$$\Theta(\hat{n}) = \sum_L A_{L0} Y_{L0}(\hat{n}) + \sum_{L \geq 2} A_{L2} Y_{L2}(\hat{n}) + \sum_{L \geq 2} A_{L-2} Y_{L-2}(\hat{n})$$

which can be written as

$$\Theta(\hat{n}) = \sum_L A_{L0} Y_{L0}(\hat{n}) + \sum_{L \geq 2} A_{L-2}^* Y_{L-2}^*(\hat{n}) + \sum_{L \geq 2} A_{L-2} Y_{L-2}(\hat{n}) \quad (\text{G3})$$

Since A'_{LM} s are complex numbers, we define

$$A_{LM} = X_{LM} + iZ_{LM} \text{ and } A_{LM}^* = X_{LM} - iZ_{LM} \quad (\text{G4})$$

and the Bipolar map (G3) can then be written as

$$\theta(\hat{n}) = \sum_L A_{L0} Y_{L0}(\hat{n}) + \sum_L X_{L-2} (Y_{L-2}^*(\hat{n}) + Y_{L-2}(\hat{n})) + i \sum_L Z_{L-2} (Y_{L-2}(\hat{n}) - Y_{L-2}^*(\hat{n})).$$

Defining,

$$G_L(\theta) = \frac{1}{(\sin \theta)^2} \sqrt{\frac{(L-1)L(L+1)(L+2)}{4\pi(2L+1)}} \left[\frac{P_{L-2}(\cos \theta)}{2L-1} - \frac{2(2L+1)P_L(\cos \theta)}{(2L-1)(2L+3)} + \frac{P_{L+2}(\cos \theta)}{2L+3} \right]$$

the Bipolar map is represented as

$$\Theta(\theta, \phi) = \sum_L A_{L0} Y_{L0}(\theta, \phi) + \sum_L X_{L-2} G_L(\theta) 2 \cos 2\phi - \sum_L Z_{L-2} G_L(\theta) 2 \sin 2\phi \quad (\text{G5})$$

where we have used expansion of spherical harmonics in terms of associated Legendre polynomials

$$Y_{lm}(\theta, \phi) = e^{im\phi} \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos \theta)$$

and

$$Y_{l \pm 2}(\theta, \phi) = \frac{e^{i \pm 2\phi}}{(\sin \theta)^2} \sqrt{\frac{(L-1)L(L+1)(L+2)}{4\pi(2L+1)}} \left[\frac{P_{L-2}(\cos \theta)}{2L-1} - \frac{2(2L+1)P_L(\cos \theta)}{(2L-1)(2L+3)} + \frac{P_{L+2}(\cos \theta)}{2L+3} \right].$$

Appendix H: Useful Mathematical Relations

Orthonormality of spherical harmonics

$$\int d\Omega_{\hat{n}} Y_{l_1 m_1}(\hat{n}) Y_{l_2 m_2}^*(\hat{n}) = \delta_{l_1 l_2} \delta_{m_1 m_2} \quad (\text{H1})$$

Symmetry property of spherical harmonics

$$Y_{lm}^*(\hat{n}) = (-1)^m Y_{l-m}(\hat{n}). \quad (\text{H2})$$

Spherical harmonic expansion of Legendre polynomials

$$P_l(\hat{n} \cdot \hat{n}') = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_{lm}^*(\hat{n}) Y_{lm}(\hat{n}'). \quad (\text{H3})$$

Property of legendre polynomial

$$P_l^{-m} = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m. \quad (\text{H4})$$

Symmetry properties of Clebsch-Gordan coefficients

$$C_{a\alpha b\beta}^{c\gamma} = (-1)^{a+b-c} C_{b\beta a\alpha}^{c\gamma}, \quad (\text{H5})$$

$$C_{a\alpha b\beta}^{c\gamma} = (-1)^{a+b-c} C_{a-\alpha b-\beta}^{c-\gamma}.$$

Summation rules of Clebsch-Gordan coefficients

$$\sum_{\alpha\beta} C_{a\alpha b\beta}^{c\gamma} C_{a\alpha b\beta}^{c'\gamma'} = \delta_{cc'} \delta_{\gamma\gamma'} \{abc\} \{abc'\}$$

$$\sum_{a\gamma} C_{a\alpha b\beta}^{c\gamma} C_{a\alpha b'\beta'}^{c\gamma} = \frac{2c+1}{2b+1} \delta_{bb'} \delta_{\beta\beta'} \{abc\} \{ab'c\}$$

$$\sum_{c\gamma} C_{a\alpha b\beta}^{c\gamma} C_{a\alpha' b\beta'}^{c\gamma} = \delta_{\alpha\alpha'} \delta_{\beta\beta'} \{abc\}$$

$$\sum_b (-1)^{a-b} C_{ab a-b}^0 = \prod_a \delta_{c0} \quad (\text{H6})$$

where

$$\prod_{abc\dots} = [(2a+1)(2b+1)\dots(2c+1)]^{1/2} \quad (\text{H7})$$

and where $\{abc\}$ is defined by

$$\{abc\} = \begin{cases} 1 & \text{if } a+b+c \text{ is an integer} \\ 0 & \text{otherwise} \end{cases} \quad (\text{H8})$$

and where a, b and c satisfy triangle inequality $|a-b| \leq c \leq (a+b)$.

Tripolar spherical harmonics are expressed as,

$$\{Y_{l_1}(\hat{n}_1) \otimes \{Y_{l_2}(\hat{n}_2) \otimes Y_{l_3}(\hat{n}_3)\}_{l_23}\}_{LM} = \sum_{m_1 m_2 m_3} C_{l_1 m_1 l_2 m_2 l_3 m_3}^{LM} C_{l_2 m_2 l_3 m_3}^{l_23} Y_{l_1 m_1}(\hat{n}_1) Y_{l_2 m_2}(\hat{n}_2) Y_{l_3 m_3}(\hat{n}_3)$$

Tripolar scalar spherical harmonics are defined as

$$\{Y_{l_1}(\hat{n}_1) \otimes \{Y_L(\hat{n}) \otimes Y_{l_2}(\hat{n}_2)\}_\lambda\}_{00} = (-1)^{l_1+l_2+L} \quad (\text{H9})$$

$$\delta_{\lambda l_1} \sum_{m_1 m_2 M} \begin{pmatrix} l_1 & L & l_2 \\ m_1 & M & m_2 \end{pmatrix} Y_{l_1 m_1}(\hat{n}_1) Y_{LM}(\hat{n}) Y_{l_2 m_2}(\hat{n}_2).$$

where $\begin{pmatrix} l_1 & L & l_2 \\ m_1 & M & m_2 \end{pmatrix}$ are Wigner-3j symbols.

Orthogonality of tripolar spherical harmonics, is given as

$$\int \int \int d\Omega_{\hat{n}_1} d\Omega_{\hat{n}_2} d\Omega_{\hat{n}_3} \{Y_{l_1}(\hat{n}_1) \otimes \{Y_{l_2}(\hat{n}_2) \otimes Y_{l_3}(\hat{n}_3)\}_\lambda\}_{LM} \{Y_{l_1'}(\hat{n}_1) \otimes \{Y_{l_2'}(\hat{n}_2) \otimes Y_{l_3'}(\hat{n}_3)\}_{\lambda'}\}_{L'M'}^* = \delta_{l_1 l_1'} \delta_{l_2 l_2'} \delta_{l_3 l_3'} \delta_{\lambda \lambda'} \delta_{LL'} \delta_{MM'}.$$

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