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# APPLIED MATHEMATICS : A NO MAN'S LAND ?*

*By*
JAYANT V. NARLIKAR

**Abstract**

Today applied mathematics in India finds itself in the no man's land between pure mathematics and theoretical physics. History shows that mathematics which has been invented in response to a specific problem brings a new vitality. This lecture describes the important role that applied mathematics has played and can continue to play and suggests ways of revitalising it in our teaching system.

The choice of my lecture topic today is by way of a tribute to the late Professor P.L. Bhatnagar, who was a luminary amongst applied mathematicians. Today, sadly, the breed of applied mathematicians is rapidly shrinking, and is in danger of becoming extinct in India.

When I was a student in the B.Sc. class of the Banaras Hindu University in the mid-fifties, mathematics in India used to be divided into two equal portions, pure and applied. Later I went to attend courses for the mathematical tripos examination at Cambridge. There too, the pure courses were grouped under the title 'pure mathematics' and the applied ones under 'natural philosophy'. The latter included statics, dynamics, electromagnetic theory, fluid dynamics, quantum mechanics and parts of astrophysics.

One may trace this tradition right back to Newtonian times. Newton's magnum opus was called 'Philosphiae Naturalis Principia Mathematica', that is, 'Mathematical Principles of Natural Philosophy'. Newton's professorship, the Lucasian chair was in the faculty of mathematics and continues to be so even today.

However, today applied mathematics *per se* finds itself on a shrinking base. The topics like classical or quantum mechanics, the electromagnetic theory, etc. are more commonly considered within the domain of theoretical physics and mathematics as such is increasingly taken to mean 'pure mathematics'. In India, however, theoretical physics does not form an important component in the physics departments of our universities. Where the above topics are included in physics depratments, the approach to them does not do justice to their mathematical content. On the other hand, the kind of mathematics associated with the old courses

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of statics and dynamics is not original, imaginative or exciting enough to compete with the more recent developments in pure mathematics, which is why applied mathematics of the kind I eneountered in my B.Sc. courses or of the kind that is found in the M.Sc. courses even today has a neglected or unwanted look about it. It is on a 'no man's land' where mathematicians don't want it and the physicists have no use for it.

The question therefore that needs to be asked and answered is: "Where does applied mathematics belong, if it does belong anywhere ?"

**The Newtonian Tradition**

Newton holds the key to that answer. His was a multifaceted genius. Physicists would like to claim him as one of them, vide his contributions to gravity, optics and sound. The mathematicians would also like to lay a claim on him through his use of geometrical methods and his discovery of the calculus. But if they do, they have to accept him as an applied mathematician. For, Newton's mathematical discoveries were motivated by his desire to solve problems of physics. The title of his book amply illustrates the fact. To him geometry or calculus were the tools for working out trajectories of planets under the law of gravitation. The real excitement in applied mathematics lies in finding a suitable technique to solve an outstanding problem.

Take for example Bernoulli's famous problem of the brachistochrone. Newton solved it and communicated the solution anonymously. Yet Bernoulli could recognize 'the lion from the paw'. Here the actual answer was not of great significance but the technique used to arrive at it was. This is the essence of applid mathematics.

Today a student would recognize it as a problem in calculus of variations. It is very easy to treat it as a particular problem of solving the Euler-Lagrange equation. It is very easy to treat the Euler-Lagrange equation as a particular case of functional analysis. This is how a pure mathematician would look at it. He would formulate an abstract variational problem in an abstract space which in a highly particular case would reduce to Bernoulli's problem. And in so doing he will miss all the excitement that Newton and Bernoulli must have felt in encountering the particular technique.

Neither of these two geniuses could have foreseen the fundamental role the variational technique plays in discovering the basic laws of nature. I will next illustrate that role to bring out another essential feature of applied mathematics.

**The Action Principle**

Consider Newton's laws of motion side by side with Maxwell' sequations. There is hardly any similarity between them. Yet, an applied mathematician can spot it, in terms of the technique of stationary action that is used to derive them. Let us look at this similarity in some detail.

Newton's laws of motion are written as

$$m\ddot{\mathbf{r}} = \mathbf{F}$$

in standard notation. These were subsequently seen as spacial cases of Lagrange's equations, again written in standard notation as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0 \qquad r = 1, 2, \ldots N$$

Here $q_1 \ldots q_N$ are the generalized coordinates. These equations, in turn, are derivable from Hamilton's principle of least (stationary) action:

$$\delta \int_{t_1}^{t_2} L(q_r, \dot{q}_r)\, dt = 0$$

Thus we see Newtonian dynamics as a special case of the variational principle.

Consider the next step. Instead of finite generalized coordinates, we have uncountably infinite number of them. The index $r$ then runs over a continuum. A special way of labelling the $q$'s is to associate a $q(\mathbf{r})$ with each spatial point with coordinate $\mathbf{r}$. This brings us to the notion of fields. Notice that the Lagrange equations and Hamilton's principle still continue to apply. But instead of dynamical equations, we now have 'field equations'.

The next step involves recognizing that the $q$'s are functions of time as well as space and when we write the Lagrangian $L$ it contains terms summed over all $\mathbf{r}$, i.e., an integral

$$L = \int \mathcal{L}\, d^3\mathbf{r}$$

The Hamiltonian principle thus becomes

$$\delta \iint \mathcal{L}\, d^3\mathbf{r}\, dt = 0$$

The field equations can therefore be obtained if we solve the 4-dimensional variational problem.

So the laws governing a field are essentially derivable once we know $\mathcal{L}$. The guess-work lies in this step.

To obtain Maxwell's equations we write $\mathcal{L}$ in the form a vector field $A_i$ ($i = 0, 1, 2, 3$; 0 representing time and 1, 2, 3 the three space dimensions). Just as the Lagrangian in particle dynamics contains squares of time derivatives, we expect to have here squares of $\partial A_i/\partial t$, $t = x^0$ = the time coordinate. But the symmetry of Lorentz invariance found in the electromagnetic theory tells us to write $\mathcal{L}$ in a Lorentz-invariant form. Thus spatial derivatives are also present on the same footing as $\partial A_i/\partial t$. The invariant form that leads to Maxwell's equations is

$$(A_{i,k} - A_{k,i})\,(A^{i,k} - A^{k,i})$$

To the applied mathematician the most exciting thing in the above derivation is the universality of the variational technique that links Newton's laws of motion with Maxwell's equations. The physical consequences of Maxwell's equations belong to the domain of interest of the theoretical physicist.

The common application of a technique to various fields is something that interests the applied mathematician. The vitality of applied mathematics lies in the wide range of applicability of its techniques, and this is the aspect that should be conveyed to the student of applied mathematics.

## Discovering the Appropriate Techniques

It is stated that abstract mathematics has results that find serendipitous applications in real life. This has been factually borne out on several occasions and I will not elaborate this point. However, a pure mathematician is not an ideal person to whom one may approach for guidance on a specific issue. If a physicist comes with a problem to the mathematician (of the abstract variety), the latter may not be interested in the problem or may point to some result which turns out not to be relevant to the solution needed.

I may tell an intresting story I heard in this connection from a particle physicist. A lone rider in Texas entered a frontier town, tired and covered with dust. He found a shop which had the sign "Laundry ! Your clothes cleaned while you wait." He entered and removed his shirt and trousers. Dumping them on the counter he said "Please clean these ! since I have no spares, I will wait while you do the needful." The shop assistant was horrified and said "No Sir! we don't clean clothes here." "What about the sign outside ?" asked the angry customer. "Oh, that ? We are sign painters and that is a sign painted by us."

It has very often happenned that the physicist has to work out his own technique. It may serve his purpose although it may not be rigorous or general by the standards of pure mathematics. A noted example of this is the delta function invented by Dirac. It is not a function in the strict sense of the word. Yet is serves a very useful purpose in describing a point source. The idea can be made rigorous and mathematically respectable. But the applications aspects are served by the way Dirac introduced it.

A few years ago, while developing a path integral formalism for quantum cosmology I needed quantities that were 'inverses' of the Green's functions of wave equations. Following partly intuition and partly a superficial knowledge of functional analysis I introduced these inverse operators and arrived at a physical result that was fully 'sensible'. Again, the strict mathematical criteria may disallow some of the operations done with these entities. But I feel that the cosmetics of rigour can be eventually applied to these operations without altering their physical content.

As a post-doc at caltech in 1964 I had attended the lectures by Richard Feynman on path integrals. On saveral occasions during his lectures Feynman carried out operations which would be frowned upon by the pure mathematician. He made no apologies for it, however! Instead he would say "I know the physics is right—so the mathematical operations can be made rigorous. I am not interested in the latter part: I am interested in the physical result."

## The Diversity of Applied Mathematics

Being a professional physicist, I have been so far referring to the physics/mathematics interface. However, the applications of mathematics go beyond physics. Indeed, to avoid being trapped in the no man's land between physics and mathematics, applied mathematics teaching and research has to look beyond applications to physics.

Anybody who has seen Newman's books "The World of Mathematics" or D'Arcy Thompson's "Growth and Form" will get a wide angled view of mathematical applications. They range from soap bubbles to novel warfare, from beehives to the correct biological sizes of different species, from gambling strategies to economic models. Although these descriptions are superficial, many of these applications have been developed to more sophisticated levels. More recently, there has been considerable interest in mathematical modelling, fractals, chaos, numerical algorithms for computing, etc.

It is my personal feeling that mathematics that has been created with specific applications in view by persons of intuition has inherent vitality that puts it on par with any branch of abstract mathematics. To restore applied mathematics to its former position of parity with its pure counterpart it is essential to bring these new vibrant inputs to our curricula. Rather than be treated as poor relations by the pure mathematicians and ignored by theoretical physicists, the applied mathematicians should migrate from their present no man's land and go for richer pastures elsewhere.

**Tata Institute of Fundamental Research**
**Homi Bhabha Road**
**Bombay–400005,**
**and**
**Inter-University Centre for Astronomy and Astrophysics**
**Pune–411007.**