

## PHOTON PROPAGATION IN A MAGNETIZED MEDIUM

SUSHAN KONAR\*

*Inter-University Centre for Astrophysics & Astronomy  
Pune, Maharashtra 411007, India  
\*E-mail: sushan@iucaa.ernet.in*

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Using the real time formalism of the finite temperature field theory we calculate the 1-loop polarization tensor in the presence of a background magnetic field in a medium. The expression is obtained to linear order in the background field strength. We discuss the Faraday rotation as well as the photon absorption probabilities in this context.

### 1. Introduction

The propagation of electro-magnetic waves, in a magnetised plasma, is of interest in systems ranging from laboratory plasma to astrophysical objects <sup>1</sup>. Yet, the expression for Faraday Rotation, for example, is derived assuming the medium to consist of non-relativistic and non-degenerate particles. Since, such assumptions may not be valid in every context we re-investigate this problem in a general framework.

Since, almost all the physical systems have magnetic fields smaller than the QED limit ( $eB < m_e^2$ ) a weak-field treatment is justified. Moreover, in compact astrophysical objects (white dwarfs/neutron stars) the Landau level spacings are negligible compared to the electron Fermi energy <sup>2</sup>. Hence, we can also assume that the field does not introduce any spatial anisotropy in the collective plasma behaviour.

Therefore, we calculate the polarization tensor ( $\Pi_{\mu\nu}$ ), at the 1-loop level, in the weak-field limit retaining terms up-to  $\mathcal{O}(B)$ . As expected, we recover Faraday rotation from the dispersive part of the polarization tensor and the absorptive part provides the damping/instability of the photons propagating in a plasma. These calculations have already been reported in detail in two recent articles by us (1999, 2001) <sup>3,4</sup>.

### 2. The Formalism

The presence of an external field or a medium introduces quantum corrections to the Lagrangian of an electro-magnetic field. In the momentum space the quadratic

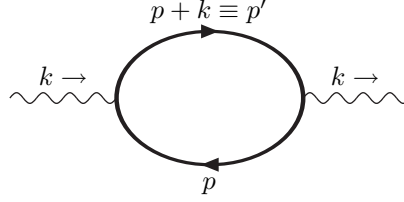


Figure 1: One-loop diagram for the vacuum polarization.

part of the Lagrangian, inclusive of such corrections, is given by,

$$\mathcal{L} = \frac{1}{2} \left[ -k^2 (g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) + \Pi_{\mu\nu}(k) \right] A^\mu(k) A^\nu(k), \quad (1)$$

where  $\Pi_{\mu\nu}(k)$  is the polarization tensor. Assuming the direction of photon propagation to be along the direction of the magnetic field, the dispersion relation for the two transverse components of the photon field  $A^\mu$ , is:

$$k^2 = \omega_0^2 \pm (a_{\text{disp}} + a_{\text{abs}}), \quad (2)$$

where  $a_{\text{disp}}$  and  $a_{\text{abs}}$  are the dispersive and the absorptive parts of  $\Pi_{\mu\nu}$ . Here we assume the Lorenz gauge condition ( $\partial_\mu A^\mu = 0$ ) and  $\omega_0$  is the plasma frequency. For a plane polarized electro-magnetic wave propagating with a frequency  $\omega$  the rate of rotation of the polarization angle, per unit length  $l$ , (i.e, the Faraday Rotation) is then given by:

$$\frac{d\Phi}{dl} = \frac{a_{\text{disp}}}{\sqrt{\omega^2 - \omega_0^2}}. \quad (3)$$

### 3. 1-Loop Vacuum Polarization

At the 1-loop level, the polarization tensor arises from the diagram in fig. 1, with the dominant contribution coming from the electron line in the loop. To evaluate this diagram we use the electron propagator within a thermal medium in presence of a background electro-magnetic field. Since, we specialize to the case of a purely magnetic field, the field can be taken to be in the  $z$ -direction without any loss of generality. The magnitude of this field is denoted by  $\mathcal{B}$ . In presence of such a background field, the vacuum electron propagator is (following Schwinger<sup>5</sup>):

$$iS_B^V(p) = \int_0^\infty ds e^{\Phi(p,s)} C(p, s), \quad (4)$$

using the shorthands,

$$\Phi(p, s) \equiv is \left( p_\parallel^2 - \frac{\tan(e\mathcal{B}s)}{e\mathcal{B}s} p_\perp^2 - m^2 \right) - \epsilon|s|, \quad (5)$$

$$C(p, s) \equiv \left[ (1 + i\sigma_z \tan e\mathcal{B}s)(\not{p}_\parallel + m) - (\sec^2 e\mathcal{B}s)\not{p}_\perp \right]. \quad (6)$$

where,  $\sigma_z = i\gamma_1\gamma_2$ . To write  $\Phi(p, s)$  and  $C(p, s)$  we have used the following decomposition of the metric tensor :  $g_{\mu\nu} = g_{\mu\nu}^\parallel - g_{\mu\nu}^\perp$ , where,  $g_{\mu\nu}^\parallel = \text{diag}(1,0,0,-1)$  and

$g_{\mu\nu}^\perp = \text{diag}(0,1,1,0)$ . In the presence of a background medium, the above propagator is modified to <sup>6</sup>:

$$iS(p) = iS_B^V(p) + S_B^\eta(p), \quad (7)$$

where  $S_B^\eta(p) = -\eta_F(p) [iS_B^V(p) - i\bar{S}_B^V(p)]$  and  $\bar{S}_B^V(p) \equiv \gamma_0 S_B^{V\dagger}(p) \gamma_0$  for a fermion propagator. And  $\eta_F(p)$  contains the distribution function for the fermions and the anti-fermions, given by:

$$\eta_F(p) = \Theta(p \cdot u) f_F(p, \mu, \beta) + \Theta(-p \cdot u) f_F(-p, -\mu, \beta), \quad (8)$$

where  $u$  and  $\beta$  are the 4-velocity and the effective temperature of the background thermal medium. Here,  $f_F(p, \mu, \beta) = (e^{\beta(p \cdot u - \mu)} + 1)^{-1}$  is the Fermi-Dirac distribution function and  $\Theta$  is the step function. Rewriting eq.(7) in the following form:

$$iS(p) = \frac{i}{2} [S_B^V(p) + \bar{S}_B^V(p)] + i(1/2 - \eta_F(p)) [S_B^V(p) - \bar{S}_B^V(p)] \quad (9)$$

we recognise:

$$S_{\text{re}} = \frac{1}{2} [S_B^V(p) + \bar{S}_B^V(p)], \quad S_{\text{im}} = (1/2 - \eta_F(p)) [S_B^V(p) - \bar{S}_B^V(p)]. \quad (10)$$

The subscripts *re* and *im* refer to the real and imaginary parts of the propagator. Now, the amplitude of the 1-loop diagram of fig. 1 can be written as:

$$i\Pi_{\mu\nu}(k) = - \int \frac{d^4p}{(2\pi)^4} (ie)^2 \text{tr} [\gamma_\mu iS(p) \gamma_\nu iS(p')], \quad (11)$$

where, for the sake of notational simplicity, we have used  $p' = p + k$ . Then the dispersive part of the polarisation tensor is given by:

$$\Pi_{\mu\nu}^D(k) = -ie^2 \int \frac{d^4p}{(2\pi)^4} \text{tr} [\gamma_\mu S_B^\eta(p) \gamma_\nu iS_B^V(p') + \gamma_\mu iS_B^V(p) \gamma_\nu iS_B^\eta(p')], \quad (12)$$

and the (11)-component of the absorptive part, by:

$$\Pi_{\mu\nu}^A(k) = -ie^2 \int \frac{d^4p}{(2\pi)^4} \text{tr} [\gamma_\mu iS_{\text{im}}(p) \gamma_\nu iS_{\text{im}}(p')]. \quad (13)$$

Now, for the terms odd in powers of  $\mathcal{B}$ , the explicitly gauge invariant expressions are (see <sup>3,4</sup> for details):

$$\begin{aligned} \Pi_{\mu\nu}^D(k) &= 4ie^2 \varepsilon_{\mu\nu\alpha\parallel} \beta k^\beta \int \frac{d^4p}{(2\pi)^4} \eta_-(p) \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_0^{\infty} ds' e^{\Phi(p',s')} \\ &\times \left[ p^{\tilde{\alpha}\parallel} \tan e\mathcal{B}s + p'^{\tilde{\alpha}\parallel} \tan e\mathcal{B}s' - \frac{\tan e\mathcal{B}s \tan e\mathcal{B}s'}{\tan e\mathcal{B}(s+s')} (p+p')^{\tilde{\alpha}\parallel} \right], \quad (14) \end{aligned}$$

and,

$$\begin{aligned} \Pi_{\mu\nu}^A(k) &= -ie^2 \varepsilon_{\mu\nu\alpha\parallel} \beta k^\beta \int \frac{d^4p}{(2\pi)^4} X(\beta, k, p) \int_{-\infty}^{\infty} ds e^{\Phi(p,s)} \int_{-\infty}^{\infty} ds' e^{\Phi(p',s')} \\ &\times \left[ p^{\tilde{\alpha}\parallel} \tan e\mathcal{B}s + p'^{\tilde{\alpha}\parallel} \tan e\mathcal{B}s' - \frac{\tan e\mathcal{B}s \tan e\mathcal{B}s'}{\tan e\mathcal{B}(s+s')} (p+p')^{\tilde{\alpha}\parallel} \right]. \quad (15) \end{aligned}$$

In writing the above expressions we have used the notation of  $p^{\tilde{\alpha}}$ , for example. This signifies a component of  $p$  which can take only the "parallel" indices, i.e., 0 or 3 and is moreover different from the index  $\alpha$  appearing elsewhere in the expression. We have also used the shorthands,  $\eta_-(p) = \eta_F(p) - \eta_F(-p)$  and  $X(\beta, k, p) = (1 - 2\eta_F(p))(1 - 2\eta_F(p'))$ .

#### 4. Results

Our results are obtained in the rest frame of the background medium. We also take the long wavelength limit, i.e,  $K \ll \omega$ , where ( $k_0 = \omega$ ). Finally, we assume the magnetic field to be small such that we can retain terms only linear in  $\mathcal{B}$ . In this limit, the dispersive part of the polarization tensor is given by:

$$\begin{aligned} \Pi_{\mu\nu}^D(k) &= 8ie^2 \varepsilon_{\mu\nu\alpha\parallel\beta} \mathcal{B} \omega \int \frac{d^4 p}{(2\pi)^4} \eta_-(p) p_0 \int_{-\infty}^{\infty} ds e^{is(p^2 - m^2) - \epsilon|s|} \\ &\times \int_0^{\infty} ds' e^{is'(p'^2 - m'^2) - \epsilon|s'|} \left\{ s + s' - \frac{ss'}{s + s'} \right\}, \end{aligned} \quad (16)$$

where we have made a further assumption that  $\omega \ll m_e$  (see <sup>3</sup> for details). Surprisingly, in the above mentioned limit, the absorptive part of the polarization tensor has two terms with different signs which opens up the possibility that for a given magnetic field, depending on the chemical potential and the external photon momentum  $\Pi_{\mu\nu}^A$  can be either positive or negative giving rise to damping or instability of the propagating photon (see <sup>4</sup> for details). Also, the absorption of the photons happen between two limiting values of  $p$  given by,

$$P_{\min} = -\frac{K}{2} + \frac{k_0}{2} \left(1 - \frac{4m^2}{k^2}\right)^{1/2}, \quad P_{\max} = \frac{K}{2} + \frac{k_0}{2} \left(1 - \frac{4m^2}{k^2}\right)^{1/2}. \quad (17)$$

Since,  $P$  is real, the condition  $k^2 \geq 4m^2$  must be satisfied. This is an important kinematic constraint which ensures the conservation of energy-momentum in the weak-field limit.

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