Exact solutions for null fluid collapse in higher dimensions

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Abstract

A large family of inhomogeneous non-static spherically symmetric solutions of the Einstein equation for null fluid in higher dimensions has been obtained. It encompasses higher dimensional versions of many previously known solutions such as Vaidya, charged Vaidya and Husain solutions and also some new solutions representing global monopole or string dust. It turns out that physical properties of the solutions carry over to higher dimensions.

Keywords: Higher dimensional generalized Vaidya solutions, Gravitational collapse, Type II null fluid

1 Introduction

Inspired by work in string theory and other field theories, there has been a considerable interest in recent times to find solutions of the Einstein equation in dimensions greater than four. It is believed that underlying spacetime in the large energy limit of the Planck energy may have higher dimensions than the usual four. At this level, all the basic forces of Nature are supposed to unify and hence it would be pertinent in this context to consider solutions of the gravitational field equation in higher dimensions. Of course this consideration would be relevant when the usual four dimensional manifold picture of spacetime becomes inapplicable. This would happen as we approach singularity whether in cosmology or in gravitational collapse.

Chodos and Detweiler [1] have considered the Kasner vacuum solution in 5-dimansional Kaluza-Klein (K-K) theory and the question of dimensional reduction. There have been investigations of entropy and classification of homogeneous cosmologies in K-K theory [2-3]. For field of localized sources, higher dimensional versions of Schwarzschild, Reissner-Nordström, Vaidya [4-5] and Bonnor-Vaidya [6] have been considered by several authors [7-10]. Important issues like formation of primordial black holes have also been addressed to [8].

Gravitational collapse continues to occupy centre-stage in gravitational research ever since mid sixties. The question is what initial conditions lead to formation of black hole or naked singularity. There has been some interesting investigations in this direction which has been reviwed in [11]. The use of Vaidya solution was first made by Papapetrou [12] in studying collapse and he pointed out that it may lead to formation of naked singularity. Joshi et al [13] have done an extensive analysis of gravitational collapse using the Vaidya metric in the context of naked singularity. It is an important question, whether collapse always leads to singularity hidden behind a black hole event horizon or it is naked [14]? Vaidya [4] and Bonnor-Vaidya [6] solutions have been employed to study various aspects of black hole formation [15-16]. Recently Husain [17] has considered collapse of a Type II null fluid with an equation of state and has obtained some non-static spherically symmetric exact solutions of the Einstein equation. It turns out that the metrics representing null fluid collapse have multiple apparent horizons and in the large limit asymptotically flat metrics have short hair and they can be thought of "lying between" Schwarzschild and Reissner-Nordström metrics [18]. Wang [19] has very recently obtained a large family of inhomogeneous and non-static spherically symmetric solutions that includes all the previously known solutions. In this paper we wish to obtain the higher dimensional version of this large family and then consider a particular solution to show that all the collapse properties discussed by Husain carry over to higher dimensions. Further we also obtain solutions for a black hole with a global monopole charge [20-21] or sitting in a string dust [22] universe. It should be noted that this synthesis of solutions is made possible by the fact that energy-momentum tensor is linear in mass functions.

In section 2, we obtain the generalized Vaidya family in higher dimensions which would be followed in section 3 by discussion of various particular cases synthesized in the family. Finally we shall conclude with discussion of collapse properties of an analogue of the Husain solution.

2 Generalized Vaidya family

Let us consider $(n+2)$-dimensional spherical spacetime described by the metric,

$$ds^2 = 2dudr + (1 - \frac{2m(u,r)}{(n-1)r^{n-1}})du^2 - r^2 d\omega_n^2 \tag{1}$$

where

$$d\omega_n^2 = d\theta_1^2 + sin^2\theta_1 \, d\theta_2^2 + ... + sin^2\theta_{n-1} d\theta_n^2 \tag{2}$$

is the metric on n-space in spherical polar coordinates. Here m is an arbitrary function of retarded time u and radial coordinate r. When $m = m(u)$, it is the Vaidya solution in higher dimensions [9]. The usual Vaidya solution in 4-spacetime follows for $n = 2$.

Let us name the coordinates as follows:

$$x^0 = u, \; x^1 = r, \; x^i + 1 = \theta_i, \; i = 1, 2, ..., n. \tag{3}$$

In the standard way we compute the Einstein tensor and it reads as

$$\begin{aligned} G_{00} = \frac{n\dot{m}}{(n-1)r^n} - \frac{(nm'}{(n-1)r^n}[1 - \frac{2m}{(n-1)r^{n-1}}] \\ G_{01} = -\frac{nm'}{(n-1)r^n}, \\ G_{22} = \frac{m''}{(n-1)r^{n-3}}, \\ G_2^2 = G_3^3 = ... = G_{n+1}^{n+1}. \end{aligned} \tag{4}$$

Here and in what follows an overhead dash and dot denote derivative relative to r and u respectively.

The energy-momentum tensor for a Type II fluid is given by [23,18],

$$T_{ik} = \mu l_i l_k + (\rho + p)(l_i \eta_k + l_k \eta_i) - p g_{ik} \tag{5}$$

where

$$l_i l^i = \eta_i \eta^i = 0, \;\; l_i \eta^i = 1. \tag{6}$$

The null vector l_i is a double null eigenvector of T_{ik}. Physically occurring distribution is null radiation flowing in the radial direction correpsponding to $\rho = p = 0$, the Vaidya spacetime of radiating star. When $\mu = 0$, T_{ik} reduces to degenerate Type I fluid [23] and further it represents string dust for $\mu = 0 = p$.

The energy condition for such a distribution are as follows [23]:

(i) the weak and strong energy conditions,

$$\mu > 0, \;\; \rho \geq 0, \;\; p \geq 0 \tag{7}$$

(ii) the dominant energy condition,

$$\mu > 0 \; \dot{\rho} \geq p \geq 0. \tag{8}$$

In the case of $\mu = 0$, the energy conditions would become,

(iii) the weak condition

$$\rho + p \geq 0, \;\; \rho \geq 0 \tag{9}$$

(iv) the strong condition

$$\rho + p \geq 0, \;\; p \geq 0 \tag{10}$$

(v) the dominant

$$\rho \geq 0 \;\; -\rho \leq p \leq \rho. \tag{11}$$

The energy-momentum tensor (4) has support along both the two future pointing null vectors l_i and η_i, and it is exactly, as we shall show later, in the form to give Bonnor-Vaidya metric in higher diemnsions. We also note that $T_{ik} l^i l^k = 0$ and $T_{ik} \eta^i \eta^k = \mu$.

For the metric (1) we write

$$l_i = g_i^0, \;\; \eta_i = g_i^1 + \frac{1}{2}(1 - \frac{2m}{(n-1)r^{n-1}}) g_i^0. \tag{12}$$

We wish to solve the Einstein equation

$$G_{ik} = -8\pi T_{ik}. \tag{13}$$

Note that (11) satisfies the conditions (5).

Substituting (4) in (12) we obtain

$$8\pi\mu = -\frac{n\dot{m}}{(n-1)r^n},\ 8\pi\rho = \frac{nm'}{(n-1)r^n},\ 8\pi p = -\frac{m''}{(n-1)r^{n-1}}. \tag{14}$$

The part $\mu l_i l_k$ of T_{ik} in (4) is the component of matter field that moves along the null hypersurface $u = const.$. In particular when $p = \rho = 0$ we have Vaidya solution in higher dimensions. Thus the distribution in (13) represents Vaidya radiating star in Type II fluid universe in higher dimensions. Note that when $\mu = 0$, though we have ρ, p given by (13), T_{ik} does not reduce to the perfect fluid form. It would rather represent an imperfect fluid.

By proper choice of the mass function, the energy conditions can be satisfied. Without loss of generality we write

$$m(u,r) = \Sigma_{\infty}^{-\infty} a_i(u) r^i \tag{15}$$

where $a_i(u)$ are arbitrary functions of the retarded time u. The summation would go over to integral for continuous summation index. Substitution of (14) into (13) yields

$$\begin{aligned} 8\pi\mu &= -\frac{n}{n-1}\Sigma_{\infty}^{-\infty} a_i r^{i-n}, \\ 8\pi\rho &= \frac{n}{n-1}\Sigma_{\infty}^{-\infty} i a_i r^{i-n-1} \\ 8\pi p &= -\frac{1}{n-1}\Sigma_{\infty}^{-\infty} i(i-1) a_i r^{i-n-1} \end{aligned} \tag{16}$$

This goes over to Wang family [19] for the 4-dimensional spacetime when $n = 2$. Thus ours is the higher dimensional generalization. We recover Schwarzschild solution in higher dimensions for $\rho = \mu = p = 0$. Also note from (ref14) that $\rho = 0$ implies $p = 0$. That means Vaidya solution simply follows from $\rho = 0$. This family includes many previously known higher dimensional solutions, which we consider in the next section.

3 Particular cases

In the following we shall consider some of the particular cases.

(a) Let us choose the functions $a_i(u)$ as

$$a_i(u) = \begin{cases} a/2, & i = 1 \\ 0, & i \neq 1 \end{cases} \tag{17}$$

where a is an arbitrary constant. In this case we shall have

$$m(u,r) = \frac{a}{2}r, \qquad \mu = p = 0, \qquad 8\pi\rho = \frac{na}{2(n-1)r^n}. \tag{18}$$

Clearly the matter field is of Type I and satisfies all the energy conditions [8-9] for $a > 0$. The metric would read as (without the angular part)

$$ds^2 = 2dudr + (1 - \frac{a}{r^{n-2}})du^2 \tag{19}$$

which can be identified as higher dimensional description of field of a Schwarzschild particle with global monopole [20,21] or of a particle sitting in a string dust [22] universe. For $n = 2$, it reduces to the monopole solution [20].

This is the case which is not considered by Wang [19].

(b) In this case we set

$$a_i = \begin{cases} m_0, & i = n-1 \\ 0, & i \neq n-1 \end{cases} \tag{20}$$

where m_0 is a constant. For this case, we have

$$m = m_0 r^{n-1}, \ \mu = 0, \ 8\pi\rho = nm_0/r^2, \ 8\pi p = -m_0(n-2)/r^2 \tag{21}$$

Here again it is Type I distribution satisfying the weak and dominant but not strong energy conditions for $m_0 > 0$. It violates the strong energy condition because $n \geq 2$. The metric would be given by

$$ds^2 = 2dudr + (1 - 2m_0/(n-1))du^2 \tag{22}$$

which would also give the global monopole spacetime as above for $n = 2$.

(c) For

$$a_i = \begin{cases} \frac{\wedge(n-1)}{n(n+1)}, & i = n+1 \\ 0, & i \neq n+1 \end{cases} \tag{23}$$

we find

$$m(u,r) = \frac{\wedge(n-1)}{n(n+1)} r^{n+1}, 8\pi\rho = -8\pi p = \wedge, \mu = 0 \tag{24}$$

where $\wedge$ is the cosmological constant. The metric would read as

$$ds^2 = 2dudr + (1 - \frac{2\wedge}{2(n+1)} r^2) du^2 \tag{25}$$

This is the de Sitter and anti de Sitter universe in higher dimensions for $\wedge g \gtrless 0$ and reducing to the familiar de Sitter metric when $n = 2$.

(d) We now obtain higher dimensional Bonnor-Vaidya solution of a radiating charged particle. For this we choose

$$a_i(u) = \begin{cases} f(u), & i = 0 \\ \frac{-4\pi e^2(u)}{n}, & i = l - n \\ 0, & i \neq 0, l - n \end{cases} \tag{26}$$

The two arbitrary functions $f(u)$ and $e(u)$ represent mass and electric charge at the retarded time u. the physical parameters would be given by

$$\begin{aligned} m &= f(u) - \frac{4\pi e^2(u)}{nr^{n-1}} \\ 8\pi\rho = 8\pi p &= \frac{4\pi e^2(u)}{nr^2} \\ 8\pi\mu &= -\frac{1}{(n-1)r^n}(n\dot{f} - \frac{8\pi e\dot{e}}{r^{n-1}}) \end{aligned} \tag{27}$$

Clearly ρ, p are always positive, the only condition that would restrict the functions $f(u)$ and $e(u)$ $\mu \geq 0$. If $\dot{f} > 0$ and $\dot{e} < 0$, then the energy conditions (7-8) will be satisfied. The metric would be given by

$$ds^2 = 2dudr + [1 - \frac{2f(u)}{(n-1)r^{n-1}} + \frac{8\pi e^2(u)}{n(n-1)r^{2(n-1)}} du^2 \tag{28}$$

This is the Bonnor-Vaidya solution in higher dimensions [10]. The electromagnetic field is given by

$$F_{ik} = \frac{e(u)}{r^n}\left(g_i^\prime g_k^0 - g_i^0 g_k^\prime\right) \tag{29}$$

with the 4-current vector,

$$4\pi J^i = -\frac{\dot{e}(u)}{r^n} g_1^i. \tag{30}$$

(e) Finally we consider the higher dimensional analogue of Husain solution [17], which would require

$$\mu_i(u) = \begin{cases} f(u), & i = 0 \\ -\frac{g(u)}{nk-1}, & i = 1 - nk, (k \neq l/n) \\ 0, & i \neq 0, 1 - nk \end{cases} \tag{31}$$

where $f(u)$ and $g(u)$ are arbitrary functions and k is a positive constant less than 1. The physical parameters would read as

$$\begin{aligned} m(u,r) &= f(u) - \frac{g(u)}{(kn-1)r^{kn-1}} \\ 8\pi\mu &= -\frac{n}{(n-1)r^n}\left(\dot{f} - \frac{\dot{g}}{(kn-1)r^{kn-1}}\right) \\ 8\pi\rho &= \frac{ng(u)}{(n-1)r^{n(k-1}}, p = k\rho \end{aligned} \tag{32}$$

If $g(u)$ is positive, obviously $\rho \geq 0$ and $\rho \geq p$. Similar to the previous case the main restriction would come from $\mu \geq 0$. This would mean either (i) $\dot{f} < 0, \dot{g} > g0$ and $k < g1/n$ or (ii) $\dot{f} < 0, \dot{g} < 0$ and $k/g > /n$.

When $k = 1$, the above solution reduces to Bonnor-Vaidya solution discussed in the previous case. The metric explicitly reads as

$$ds^2 = 2dudr + \left[1 - \frac{2f(u)}{(n-1)r^{n-1}} + 2g(u)(n-1)(kn-1)r^{n(k+1)-2}\right] du^2 \tag{33}$$

This is Husain solution [17] in higher dimensions. It is asymptotically flat for $k < 1/n$ representing a bounded source while it is cosmological for $k > 1/n$. When $kn = 1, m(u,r) = f(u) + g(u) ln r$, it can be seen that the enrgy conditons are always violated for sufficiently small r and hence this case is ruled out. We shall therefore consider the other two cases, one for bounded source and other for cosmological model.

Case $k > 1/n$. For simplicity let us first set $k = 1$. Then we write $2f = A(1 - tanhu)$ and $2g = 1 + Btanhu$ with the constants $A \geq 0, 0 \leq B \leq 1$, then all the energy conditions are satisfied. The metric then takes the form

$$ds^2 = 2dudr + \left[1 - \frac{A(1 - \tan hu)}{(n-1)r^{n-1}} + \frac{1 + B \tan hu}{(n-1)^2 r^{2(n-1}}\right] du^2 \tag{34}$$

In the limit $u \to \infty$, it has a naked singularity at $r = 0$. However for $u \to -\infty$, it may have horizons depending upon the realtive values of A and B. Specifically horizons are given by

$$(n-1)r^{n-1} = A \pm \sqrt{A^2 + B - 1} \tag{35}$$

In the other case $k < \frac{1}{n}$, we write $2f = C + A(1 - tanhu), 2g = B(1 - tanhu)$ satisfying the energy conditions. The metric

$$ds^2 = 2dudr + \left[1 - \frac{A(1 - \tan hu)}{(n-1)r^{n-1}} + \frac{B(1 - \tan hu}{(n-1)(kn-1)r^{n(k+1)-2}}\right] du^2 \tag{36}$$

where $A, B \geq 0$. In the limit $u \to \infty$, the metric will either have naked singularity at $r = 0$ for $C \neq 0$ or it would be flat for $C = 0$. On the other side as $u \to -\infty$ with $C = 0$, there would occur apparent horizons given by

$$R^{\frac{n(k+1)-2}{n-1}} - \frac{2A}{n-1} R^{\frac{nk-1}{n-1}} + \frac{2B}{(n-1)(nk-1)} = 0, R = r^{n-1} \tag{37}$$

In particular when $n = 2$ and $k = 1/3$, the above equation takes the form

$$(r - 2A)^3 = (3B)^3 r \tag{38}$$

which would always have a real root for permissible values of A and B.

4 Discussion

We see that the 4-dimensional spherically symmetric solutions describing Type II fluid go over to $(n+2)$-dimensional spherically symmetric solutions and essentially retaining their physical behaviour. In particular higher dimensional version of Husain solution [17] that describes gravitational collapse leading to asymptotically flat black hole solutions for $k > 1/n$. The general metric depends upon the parameter k and two arbitrary functions of retarded coordinate u, which are constrained by the enrgy conditions. Also the long retarded time limit of the asymptotically flat solutions would fall between Schwarzschild and Reissner-Nordetröm solutions as in [17] in the sense that $1/nlkleq1$ in (ref31,32).

In general the $k/g1/n$ solutions represent evolution of naked singularity into itself or into black holes and the constant parameters in (ref33) determine which of these two possibilities would occur. In the other case of $kl1/n$, it is evolution of naked singularity or flat space into black hole in a cosmological background. Of course in all the cases the cosmic censorship is respected. As in [17], we also have $T_{ik}l^il^k = 0$ even though the distribution is along both the null directions.

It would however be possible to find more exact solutions of the similar kinds by imposing the equation of state $p = k\rho$. The main reason for clubbing together of so many cases is that physical parameters involve only linear derivatives of the mass function $m(u, r)$. Thus linear combinations of all the cases discussed above would also be a solution. For example if we take the equation of state $p = \rho - \Lambda/4\pi$, the resulting solution is the higher dimensional Bonnor-Vaidya metric with the cosmological constant, which is a linear combination of the cases (a) and (d). Also note that it is trivially possible to include a global monopole or string dust in all the above cases. It turns out that solutions with global monopole could be considered as dual to the Einstein solutions in a sense that they are solutions of dual equation [24]. The dual solutions simply imbibe global monopole or string dust in the original spacetime, and they are generally not asymptotically flat.

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