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CONFORMAL QUANTIZATION AND SPACE-TIME SINGULARITY

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This work generalizes earlier results of conformal quantization[1,2] that within the full range of space-times conformal to any singular space-time satisfying Einstein's field equations for minimally coupled matter, the singular solutions form a set of zero-probability measure. A wider definition of space-time singularity that includes the curvature singularity assumed in the earlier work (*op. cit.*) is adapted and it is shown that the previous conclusion stands even when the present state of the universe is defined by wave functionals that are not necessarily wave packets. Within the present framework of quantum gravity therefore it seems extremely unlikely that the universe had a singular origin.

1. Introduction

The work in classical general relativity in the 1960s[3–5] emphasized the inevitable occurrence of singularity in space-times governed by Einstein's field equations and satisfying certain reasonable physical and geometrical conditions. This result following the discovery of the cosmic microwave background radiation[6] has led to the general concept of the "origin" of the universe in a big bang—an epoch when there was a space-time singularity. The extent of confidence felt by the physicist in this picture is seen in the current work on the very early universe, when the universe was as young as $\sim 10^{-37}$ s.

Nevertheless it has to be remembered that the classical gravity that is used to determine the dynamics of the expanding universe cannot be trusted all the way to the original epoch $t = 0$. For cosmic time t in the range

$$0 \leq t \lesssim t_{\mathrm{P}}, \tag{1.1}$$

where

$$t_{\mathrm{P}} = \left(\frac{G\hbar}{c^5}\right)^{1/2} \tag{1.2}$$

is the Planck epoch, the appropriate theory to describe gravitational physics must necessarily be a quantum one.

There have been numerous attempts at quantizing gravity (for a partial survey of the state-of-the-art see Ref. 7). It is fair to say that as yet there is no theory of quantum gravity that (like its classical counterpart, general relativity) is both formally complete and applicable to practical problems. Curved space-time, nonlinearity, the insepara-bility of quantum gravitational effects from a background space-time and the paucity of observable effects have all contributed to the difficulty.

Nevertheless some progress can be made with limited goals. The question, "Did the universe have a singular origin?" needs to be looked at from the quantum gravitational point of view. To this end we may ask: "What is the most relevant degree of freedom of the tensorial gravitational field, to decide on the above question?" Without doubt the degree of freedom is the so-called conformal degree of freedom.

To fix ideas write the space-time metric as

$$ds^2 = g_{ik}\, dx^i\, dx^k \tag{1.3}$$

and define Q and h_{ik} by

$$Q = (-g)^{1/8}, \qquad g_{ik} = Q^2 h_{ik}, \tag{1.4}$$

where $g = \det \|g_{ik}\|$. We have used the signature $(+ - - -)$ with i, $k = 0, 1, 2, 3$. A conformal transformation

$$\tilde{g}_{ik} = \Omega^2 g_{ik}, \tag{1.5}$$

where Ω is a function of the four coordinates x^k leaves h_{ik} unchanged while Ω transforms as

$$\tilde{Q} = \Omega Q. \tag{1.6}$$

We identify Q with the conformal degree of freedom while the h_{ik} denote non-conformal degrees of freedom. Since we are dealing with the space-time continuum the actual number of degrees of freedom is ∞ (conformal) and ∞^5 (nonconformal) where ∞ is the infinity of functions on $\mathbb{R}^4$.

In the standard big bang cosmology it is the vanishing of Q that leads to singularity and it is the quantization of Q that will tell us whether the occurrence of singularity is a classical artefact. A complete theory of quantum gravity, when it becomes available, will include quantization of h_{ik} also.

An analogy with the quantum mechanical system describing the hydrogen atom will be pertinent here. Within the classical framework, the H-atom is unstable with the electron spiraling inwards to the proton in a time-scale of the order of about 10^{-23} s. The relevant dynamical variable in the classical problem is the electron-proton separation distance r. By quantizing r alone (and leaving the angular variables θ, ϕ dormant) we are able to recover the important result that the quantized H-atom has a non-singular stationary state.

Going ahead with partial quantization has a conceptual simplicity. Space-times conformal to one another have the same light-cone structure. Thus during quantum transitions of space-times with different values of Q, the causal relationship between any two space-time points is left intact. This invariance therefore leaves the "rest of physics" undisturbed which is an enormous simplification so far as interpretation of the result is concerned.

For the purpose of this paper therefore we will restrict ourselves to quantizing the conformal degree of freedom only, and we will call such quantization *conformal* quantization. The details of the formalism are described in Sec. 2 where it will be seen that *exact* quantitative results can be obtained within this framework when the Feynman path integral approach is used.

Earlier work with this approach[1,2] yielded an explicit quantum mechanical propagator giving the probability amplitude for transitions between two geometries conformal to the classical geometry (i.e., the geometry of space-time that satisfies Einstein's equations). Using this propagator it was possible to compute the probability that the universe had a singular conformal geometry at $t = 0$, and to show that this probability is vanishingly small. The present paper generalizes the above work on two fronts. First it extends the earlier conclusion to situations covered by a more generalized definition of singularity which includes the curvature singularity assumed in Ref. 2. Secondly it deals with wave functions and wave functionals of the universe that are more general than the wave packet forms assumed in Refs. 1 and 2.

In Sec. 2 we present brief descriptions of the results of Refs. 1 and 2. In Sec. 3 we outline the generalized definition of singularity followed, in Sec. 4, by a discussion of quantum conformal fluctuations near the classical singularity. Finally in Sec. 5 we assign a probability measure to the singular solutions.

2. Conformal Quantization

2.1. *Conformal action*

Let us denote by M space-time manifold whose geometry satisfies the classical Einstein's equations

$$R_{ik} - \tfrac{1}{2}g_{ik}R = -\kappa T_{ik}, \tag{2.1.1}$$

where $\kappa = 8\pi G/c^4$ and R, R_{ik}, T_{ik} have their usual meaning. We will later refer to the metric tensor g_{ik} by g. The energy tensor T_{ik} is taken as that for a system of particles minimally coupled to gravity—an assumption that may very well be justified in the very early universe if all the other basic interactions are less important than gravity in view of asymptotic freedom.

The manifold with metric conformal to g will be denoted by $(M, \Omega^2 g)$ or simply by M_ϕ where,

$$\phi = \Omega - 1 \tag{2.1.2}$$

is the conformal fluctuation from the classical space-time metric. Under the transformation $g_{ik} \rightarrow \Omega^2 g_{ik}$ we get

$$R \rightarrow R(1 + \phi)^{-2} + 6(1 + \phi)^{-3}\Box\phi. \tag{2.1.3}$$

The classical Hilbert action accordingly transforms to

$$S_H = \frac{1}{16\pi}\int_V R\sqrt{-g}\,d^4x \rightarrow \frac{1}{16\pi}\int_V \{(1 + \phi)^2 R - 6\phi_i\phi^i\}\sqrt{-g}\,d^4x, \tag{2.1.4}$$

where V is the space-time 4-volume over which the action is defined, x^i are the four coordinates and we have set $c = 1$, $G = 1$, $\hbar = 1$. ϕ_i denotes the gradient $\partial\phi/\partial x^i$, and the indices are raised or lowered by the classical metric g.

Normally the conformal transformation in (2.1.4) should lead to the existence of second derivatives of ϕ arising from the $\Box\phi$ term in (2.1.3). However, Green's theorem is used to transform them to the form given in (2.1.4) together with a surface term defined over ∂V. This surface term cancels a similar term which comes from the conformal transform of the surface term introduced by Gibbons and Hawking[8] to effectively remove the second derivatives of the metric tensor from the Hilbert action. Henceforth we will ignore these surface effects.

To the Hilbert action must be added the term describing a system of noninteracting particles $a, b, \ldots$ of masses $m_a, m_b, \ldots$

$$S_m = \sum_a \int m_a\,ds_a, \tag{2.1.5}$$

where s_a is the proper time of a-th particle. It is clear that

$$S_m \rightarrow \sum_a \int m_a \Omega\,ds_a = S_m + \sum_a \int m_a \phi\,ds_a \tag{2.1.6}$$

under the conformal transformation.

Writing $S = S_H + S_m$ for the classical action, we therefore find that under the conformal transformation, S transforms to

$$S \rightarrow S_\phi = S + \frac{1}{16\pi}\int_V (R\phi^2 - 6\phi_i\phi^i)\sqrt{-g}\,d^4x. \tag{2.1.7}$$

This action functional of ϕ plays the key role in conformal quantization. Notice that there are no linear terms in ϕ because the classical action satisfies the stationarity condition $\delta S = 0$.

To proceed further we show how classical geometrodynamics are modified by quantum fluctuations.

2.2. *The quantum conformal propagator*

Suppose M is foliated by a sequence of space-like hypersurfaces $\{\Sigma\}$ and we define a time coordinate t to label Σ as $t =$ constant. Denote by ${}^3G$ the 3-geometry on Σ. Then according to Isenberg and Wheeler[9], the solution of Einstein's equations may be considered as a sequence of $\{{}^3G\}$ on $\{\Sigma\}$ sandwiched between two hypersurfaces $t = t_I$, $t = t_F$, $t_I > t_F$. The 4-volume V is thus the slab of space-time between these initial and final hypersurfaces which we shall label Σ_I and Σ_F respectively. The appropriate initial conditions to be specified on Σ_I are the conformal part of ${}^3G$ and the trace of the extrinsic curvature tensor, K.

In abstract terms we may identify any sequence of ${}^3G$ from Σ_I to Σ_F as a "path" Γ in function space. The classical path Γ_C denotes a particular sequence that satisfies the Einstein's equations and the given initial conditions. Any other path may not satisfy Einstein's equations and hence will not be an acceptable description of classical gravity.

This situation is altered in the following way in conformally quantized gravity. Let $\Omega^2 g$ be the new metric in V for an arbitrary Ω. Suppose that we allow Ω to vary as a function of x^i $(x^0 \equiv t)$ in $t_I \le t \le t_F$ subject to the requirement that $\Omega = \Omega_I$ $(\phi = \phi_I)$ on $t = t_I$ and $\Omega = \Omega_F$ $(\phi = \phi_F)$ on $t = t_F$. Such variation gives a restricted set S of paths Γ, over which we sum the Feynman path amplitude $\exp iS[\Gamma]$ to obtain the propagator

$$K[\phi_F, t_F; \phi_I, t_I] = \sum_{\Gamma \in S} \exp iS[\Gamma]. \qquad (2.2.1)$$

The classical path Γ_C corresponds, of course, to $\phi = 0$ and is not included in S. [We exclude here the trivial situation wherein a constant ϕ does give a uniformly scaled classical solution. Such a case would be included in S only if $\phi_I = \phi_F$.]

From (2.1.7) we can at once write down K as a path integral

$$K[\phi_F, t_F; \phi_I, t_I] = \exp iS_C \int \exp\left\{\frac{i}{16\pi}\int_V (R\phi^2 - 6\phi_i\phi^i)\sqrt{-g}\, d^4x\right\} D\phi. \qquad (2.2.2)$$

Here S_C is the action evaluated for the classical path Γ_C. The quantities R, g, etc. are supposed to be known from the classical solution while ϕ is the quantum input. K tells us how the quantum conformal fluctuations behave in the 4-volume V, and contains the entire content of conformal quantization.

The evaluation of Eq. (2.2.2) has been done.[1] Skipping the details given therein we state the final answer

$$K[\phi_F, t_F; \phi_I, t_I] = F(t_F, t_I) \exp \frac{3i}{8\pi} \chi, \tag{2.2.3}$$

where

$$\chi = \sum_{(P,Q=I,F)} \sum \iint A_{PQ}(\mathbf{r}, t_P; \mathbf{r}', t_Q) \phi_P(\mathbf{r}) \phi_Q(\mathbf{r}') \, d^3\mathbf{r} \, d^3\mathbf{r}'. \tag{2.2.4}$$

In the above expression **r** and **r**′ stand for the three space-like coordinates. Recall that since ϕ_I and ϕ_F have been specified respectively on the hypersurfaces $t = t_I$ and $t = t_F$, they are functions of three (space-like) coordinates only.

The coefficients A_{II} and A_{FF} are related to the advanced and retarded Green's functions of the operator $\Box + R/6$ in a somewhat implicit form.[1] The cross coefficient $A_{IF} \equiv A_{FI}$ is however simpler to state:

$$A_{IF}{}^{-1} \equiv A_{FI}{}^{-1} = G^R(F, I). \tag{2.2.5}$$

G^R, the *retarded* Green's function between $(\mathbf{r}_F, t_F)$ and $(\mathbf{r}_I, t_I)$ satisfies the wave equation

$$\Box_x G^R(X, B) + \tfrac{1}{6} R(X) G^R(X, B) = [-\bar{g}(X)]^{-1/2} \delta_4(X, B), \tag{2.2.6}$$

δ_4 being the 4-dimensional delta function.

2.3. *The diverging uncertainty at classical singularity*

The propagator K obtained in (2.2.3) may be used in the following way. Let $\Psi_I(\phi_I)$ denote the wave functional characterizing the state of the geometry at $t = t_I$ in terms of quantum theory. Then the state at $t = t_F$ is given by

$$\Psi_F(\phi_F) = \int K[\phi_F, t_F; \phi_I, t_I] \Psi_I(\phi_I) D\phi_I. \tag{2.3.1}$$

Even without calculating this functional integral the following important conclusion can be drawn from it. First we note that if $G^R(X, B)$ diverges as X approaches the classical singular epoch in (M, g), then $A_{IF} \to 0$. Hence the cross term in χ tends to zero and the dependence of $\Psi_F(\phi_F)$ on $\Psi_I(\phi_I)$ is only through a constant factor. In other words the final state "totally loses memory" of its initial state.

This divergence of quantum uncertainty indicates that the classical solution itself is not a reliable indicator of quantum reality. In the set S of nonclassical paths, there is, however, a set of paths with geometries that are conformal to g and also singular. The complement of this set in S is made of paths with geometries that, while conformal to g, are nonsingular. In the diverging range of conformal fluctuations which set predominates?

This question was answered in Ref. 2 in favour of the latter set by defining a probability measure on the set S. Because of the exponential character of the propagator, it was easier to compute Ψ_F in terms of Ψ_I provided the latter is taken as a wave packet. Taking Ψ_I as a wave packet concentrated around the classical value $\phi_I = 0$, it was shown that Ψ_F is a wave packet centred around $\phi_F = 0$ but with a dispersion that diverges as the singular epoch is approached. The divergence is rapid enough to guarantee that the probability measure of singular geometries is zero.

A particularly simple illustration of this result is presented by the standard big bang model. Keeping the universe homogeneous and isotropic, the line element is given by the Robertson-Walker (RW) form

$$ds^2 = dt^2 - S^2(t)\left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right), \tag{2.3.2}$$

where $k = 0$, 1 or -1 and (r, θ, ϕ) are the co-moving coordinates of a typical fundamental observer.[10] The function $S(t)$ vanishes at the singular epoch $t = 0$.

It is worth noting that the most general fluctuations of (2.3.2) keeping the homogeneity and isotropy of the universe are the conformal fluctuations with ϕ a function of t only. Then functionals of ϕ and (2.3.1) are simplified to ordinary integrals:

$$\Psi_F(\phi_F) = \int K[\phi_F, t_F; \phi_I, t_I]\Psi_I(\phi_I)\, d\phi_I\,. \tag{2.3.3}$$

The explicit behaviour of K is given by Narlikar and Padmanabhan.[11] If Ψ_I is given by the wave packet

$$\Psi_I(\phi_I) = (2\pi\Delta_I{}^2)^{-1/4}\exp(-\phi_I{}^2/4\Delta_I{}^2), \tag{2.3.4}$$

then Ψ_F is given by a similar function of ϕ_F but with Δ_I replaced by

$$\Delta_F = \Delta_I\left\{A_{II}{}^2 + \frac{1}{16\Delta_I{}^4}\right\}^{1/2} A_{IF}{}^{-1}. \tag{2.3.5}$$

The divergence of Δ_F as $t \to 0$ arises from the fact that

$$A_{IF} \sim S_c(t) \qquad \text{as } t \to 0, \tag{2.3.6}$$

where $S_c(t)$ is the classical Friedmann expansion factor. Since all singular solutions of the form (2.3.2) must have[11] $\phi(t)/\Delta_F \to 0$ as $t \to 0$ the zero measure of singular solutions follows.

This special case of Robertson-Walker models is covered by the general result of Ref. 2. Here we further generalize the conditions under which the result there was obtained. The special example discussed above will, however, be needed again by way of illustration.

3. Singularities and Conformal Transformations

We shall now consider the general situation of a globally hyperbolic space-time (M, g) which may contain many singularities, either local or cosmological. Globally hyperbolic space-times form a general class with the only topological constraint that M be homeomorphic to $S \times \mathbb{R}$ where S is a 3-manifold. Such a space-time can be foliated by means of space-like hypersurfaces $\{\Sigma\}$ given by t = constant, that is, for each $t \in \mathbb{R}$, Σ_t will be a space-like hypersurface in M. We note that this general class includes most of the important cosmologies such as the Friedmann-Robertson-Walker types, the Bianchi models, steady state cosmologies as well as exact solutions like the Schwarzschild space-time. However, these are all special cases and one need no longer be limited by specific requirements such as homogeneity and isotropy.

It is well known that the presently available results within the classical general relativistic framework show that provided certain reasonable conditions are met, M must contain space-time singularities in the form of incomplete non-space-like geodesics which are not extendible to arbitrary values of their affine parameter. Further, it might be the case that what holds for freely falling observers might happen for accelerated observers as well in the sense that their trajectories are nongeodesic non-space-like curves with finite length (as measured by their proper space-time) but without an end point. Physically speaking such an observer would not be a part of the space-time after a finite lapse of time.[12]

Normally, one would like to think that a singularity of one type, such as say time-like geodesic incompleteness, would imply a singularity of other types also, such as null incompleteness. However, this does not necessarily happen and one can find examples of space-times which may be time-like geodesically complete but null incomplete or which are non-space-like geodesically complete but may contain finite length time-like curves without end points.[13] Thus, it would seem that the notion of geodesic incompleteness is not the same as what one would like to associate with in a genuine physical singularity, which is that the space-time curvatures there grow so large that the local laws of physics are drastically altered and even the usual ideas of space and time may break down. One would like to think of the geodesic incompleteness as an "effect" rather than a "cause". Thus, we shall deal here with curvature singularities which are genuine and physically all embracing in the sense that all observers, whether freely falling or accelerated, falling within the singularity are destroyed by infinite tidal forces and the curvature scalars or the Riemann tensor components should grow unboundedly along the trajectories falling into the singularity. This means that if S is a singularity in M (which is not a part of the space-time), then all the non-space-like geodesics falling within S are future incomplete at S and all the nongeodesic non-space-like curves ending up at S have finite length without an end point at S.

This notion can be best formulated by means of the proper and terminal indecomposable past (future) sets in space-times which can characterize the singularities as well as the points at infinity of the space-time in terms of an additional boundary attached to M.[14] The construction of this boundary involves only the causal structure of space-time and hence no additional assumptions for M are needed. We shall first give a few definitions.

3.1. *Definitions*

By a *curve* γ in M we mean a smooth map from a general interval in $\mathbb{R}$. For a smooth non-space-like curve γ, the *length* of γ is defined as:

$$L(\gamma) = \int_p^q |g(\partial/\partial t, \partial/\partial t)|^{1/2}\, dt. \tag{3.1.1}$$

(If γ is piecewise smooth, a sum could be taken over the smooth pieces.) We shall use the standard notation here[12] in which the chronological past and future of events will be denoted as $I^{\pm}(x)$.

Let P be any nonempty subset of M with the property that there exists $A \subset M$ such that $I^-(A) = P$; then P is called a *past set.* If P cannot be expressed as the union of two proper past subsets then it is called an *indecomposable* past set (IP). If P is an IP and there is some $x \in M$ with $I^-(x) = P$ then P is called a *proper* IP or a PIP. When an IP set is not a PIP it is called a *terminal* IP or TIP.

One can define the future sets PIF and TIF in an analogous manner. It is easy to see that any $P \subset M$ is an IP if and only if there is a future directed curve γ such that $I^-(\gamma) = P$. Now let $M^\#$ be the union of $\hat{M}$ and $\check{M}$, which are the unions of all IP's and IF's in M respectively. Then, to avoid duplication in $M^\#$, we define M^* as the quotient space $M^\#/R_h$ (where R_h is the intersection of all equivalence relations $R \subset M^\# \times M^\#$) which makes $M^\#/R_h$ into a Hausdorff space. Now M^* can be viewed as a space-time with boundary; $M \subset M^*$ and the topology in M can be looked upon as the induced topology of M^*. This entire construction can be carried out provided M is at least distinguishing (that is, whenever $I^+(x) = I^+(y)$ or $I^-(x) = I^-(y)$ then $x = y$).

A point $x \in M^*$ is called a *regular point* if it is represented by a PIP or a PIF. All other points in M^* are represented by TIP's or TIF's and are called the *ideal points* of M. Clearly, the ideal boundary ∂M consists of points at infinity of M and singularities of the space-time.

As stated earlier, any TIP (or TIF) is generated by a future directed (past directed) non-space-like curve γ which has no future (past) end point. The singularities within the ideal boundary ∂M can now be distinguished in the following manner: Let S be a TIP of ∂M and Γ be the set of non-space-like curves generating S. If $L(\gamma) < \infty$ for all $\gamma \in \Gamma$, then S is a singularity of M. Thus, if S is a singularity, any $\gamma \in \Gamma$ will have finite length and no future end point (see Fig. 1(a)). On the other hand, if there exists at least one time-like curve with infinite length generating the TIP S, then S is a nonsingular

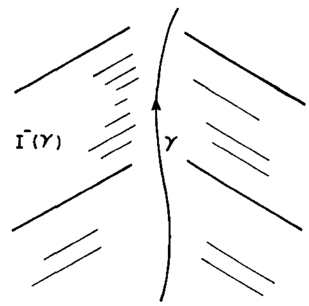

Fig. 1(a). Here γ is future incomplete non-space-like defining a singular TIP, $I^-(\gamma) = I^-(S)$, where S denotes the singularity (not in space-time). For any non-space-like curves γ_1, γ_2 such that $\gamma_i \subset I^-(\gamma)$, γ_i will have finite length in future but no future end point. Hence $I^-(\gamma_i)$ would define the same singular TIP $I^-(\gamma_i) = I^-(\gamma)$.

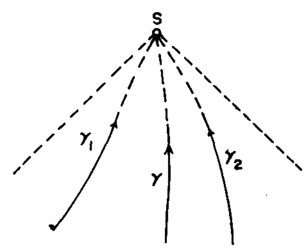

Fig. 1(b). Here γ is a future endless time-like curve with infinite length. $I^-(\gamma)$ defines an ideal point which is a point at infinity, i.e. a nonsingular TIP.

TIP (see Fig. 1(b)). Now the assumption stated earlier regarding singularities can be formulated as follows: [15]

All the singularities in M must be singular TIP's.

Thus, if γ is an incomplete non-space-like future directed geodesic, then the TIP $I^-(\gamma)$ defines a singularity S and our condition tells us that for any other future inextendible non-space-like curve $\lambda \subset I^-(\gamma)$, λ has finite length in future, no future end point, and $I^-(\lambda)$ is also a TIP defining S. Clearly, λ also terminates into the singularity S in future.

3.2. *Singularities and conformal changes*

With this framework in mind, we shall now consider the conformal transformations of the original globally hyperbolic geometry (M, g). When one considers arbitrary metric fluctuations in the original geometry (M, g), the causal future and past $I^{\pm}(x)$ are changed in M, the non-space-like geodesics of the new geometry are different and in general the entire causal structure of M is altered. However, as stated earlier, if the fluctuations are conformal to the original geometry, the causal relationships remain unchanged and the null geodesics are invariant under $g \to \Omega^2 g$. But the non-space-like geodesics of (M, g) and $(M, \Omega^2 g)$ could be quite different. Since the sets $I^{\pm}(p)$ are invariant under conformal fluctuations, the point set defined by any non-space-like geodesic in (M, g) will in general define some non-space-like curve for $(M, \Omega^2 g)$ which need not be a geodesic. As a consequence, the geodesic completeness properties of conformally transformed space-times are greatly altered and these have been studied by various authors in detail.

For null geodesics, the effect of conformal fluctuations is easy to see since they remain point-wise fixed. Using the geodesic equation, it is seen that under $\Omega \to \Omega^2 g$ the affine parameter λ along a null geodesic γ transforms as:

$$\lambda' = \int_0^\lambda \Omega(\gamma(\lambda))\, d\lambda \tag{3.2.1}$$

and hence the completeness properties of null geodesics will be changed. Similar effects can be analysed for non-space-like geodesics generally and Seifert[16] derived the following result which we shall use here:

Theorem: Let (M, g) be a globally hyperbolic space-time. Then there exists a conformal factor $\Omega_c > 0$ such that all the non-space-like geodesics in $(M, \Omega_c{}^2 g)$ are complete.

Later Clarke[17] constructed a conformal factor Ω_c using a weaker assumption of strong causality such that $(M, \Omega_c{}^2 g)$ will be null geodesically complete (briefly written as g-complete). A further improvement was given by Beem[18] by the construction of Ω_c for M satisfying the nonimprisonment condition, such that all non-space-like geodesics in $(M, \Omega_c{}^2 g)$ will be complete. Here the non-imprisonment condition means that whenever a non-space-like curve λ enters a compact space-time set K, then λ must leave K.

Now, let S be a singularity in (M, g), i.e., a singular TIP and let some future directed non-space-like γ be a generator of S; $I^-(\gamma) = I^-(S)$. Let the metric be conformally transformed as $g \to \Omega_c{}^2 g$. Then $(M, \Omega_c{}^2 g)$ is non-space-like g-complete and since the causality of M is invariant under Ω, it is globally hyperbolic. We show that as a result of this operation, the singularity S of the original space-time no longer remains a singular TIP but is thrown off as a point at infinity.

For, note first that $I^-(S)$ remains invariant as a point set and as a TIP under the conformal transformation. Let any non-space-like γ be a generator of this TIP. Choose any $p \in \gamma$. Let q_i be other event on γ to the future of p. Then by a well-known result (cf. Ref. 12) for globally hyperbolic space-times, there exists a maximal non-space-like geodesic γ_i of $(M, \Omega_c{}^2 g)$ from p to q_i. Let $q_i \to S$ along γ, then by the above construction we obtain a non-space-like geodesic γ' of $(M, \Omega_c{}^2 g)$ which is totally contained in $I^-(S)$ and such that $I^-(\gamma) = I^-(\gamma')$; i.e. it is a generator of the TIP S (see Fig. 2). However, all non-space-like geodesics of $(M, \Omega_c{}^2 g)$ are complete and hence γ' has infinite length in

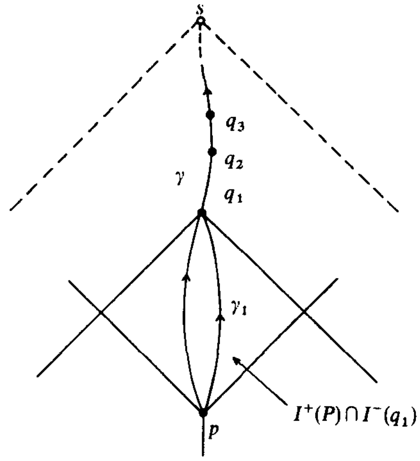

Fig. 2. Here $q_1 \in I^+(p)$ and γ_1 is a non-space-like geodesic maximising the lengths of all non-space-like curves from p to q_1. Taking $q_i \to S$ one can construct a maximal geodesic approaching the singularity S.

future as measured by its affine parameter. Thus, there exists a generator γ' for the TIP S which has infinite length and hence S is no longer a singularity, it is a nonsingular TIP in $(M, \Omega_c^2 g)$. It is now clear that all the singularities $\{S\}$ in the original (M, g) will be similarly transformed as points at infinity, though they are still within ∂M.

As shown above, the non-space-like γ' of $(M, \Omega_c^2 g)$ is contained within $I^-(S)$ and so it remains under conformal transformations, though its status as a geodesic may change to an arbitrary non-space-like curve. Thus, in (M, g), the point set of γ' will define a non-space-like curve γ'' totally contained in $I^-(S)$ and $I^-(\gamma'')$ defines the TIP S. However, S being a singular TIP in (M, g), γ'' has finite length in future:

$$L(\gamma'') = \int_p^q |g(\partial/\partial t, \partial/\partial t)|^{1/2}\, dt < \infty . \tag{3.2.2}$$

Using arc length as the parameter along γ'', $g_{ij}(dx^i/ds)(dx^j/ds) = \pm 1$ gives

$$L(\gamma'') = \int_p^q ds < \infty \tag{3.2.3}$$

in the limit as q tends to the singularity S along γ''. Now, under $g_{ij} \to \Omega_c^2 g_{ij}$, the non-space-like curve γ'' is mapped into the non-space-like geodesic γ' of $(M, \Omega_c^2 g)$, which being geodesically complete has infinite length:

$$L(\gamma') = \int_p^q \Omega_c\, ds = \infty , \tag{3.2.4}$$

as $q \to S$. Thus, as q tends to the singularity S along γ'', the conformal factor must blow up along γ''. We could have clearly chosen any other generator γ_1 of the TIP $I^-(S)$ and the construction of the non-space-like geodesic γ'_1 could have been repeated, deducing the blow-up of the conformal factor Ω_c along another non-space-like curve γ''_1 in (M, g) in $I^-(S)$ as the singularity S is approached along γ''_1, using the geodesic completeness of $(M, \Omega_c{}^2 g)$. We have thus shown that there is an "overall" divergence in Ω as the singularity S is approached from $I^-(S)$ which has the effect of throwing the singularity off to the infinity.

It may be noted in this connection that the various singularities in M would generally lie at various epochs in the space-time as defined by the foliation of M and that they would be normally covered by event horizons in order that the global hyperbolicity of M is preserved.

4. Quantum Fluctuations near a Space-Time Singularity

It is now possible to use the results developed in Sec. 3 to examine the behaviour of the quantum conformal fluctuations in the vicinity of a classical singularity. Here we shall no longer assume that the state of the universe is represented by a Gaussian wave packet. Though it is reasonable to demand that the classical situation $\phi = 0$ should be average of the quantum description, the Gaussian packet assumption is obviously too strong a demand for the wave function of the universe: it is a sufficient condition that may not be necessary. We shall thus choose the state of the universe to be represented by a general wave functional $\Psi(\phi, t)$ which satisfies the normal requirements such as $\int \Psi\bar{\Psi} D\phi = 1$, etc. Next, we shall prove that the divergence of the Green's function which is seen explicitly in the cases discussed in Refs. 1 and 2 is a general feature of the behaviour of Green's function in the vicinity of space-time singularities in any general globally hyperbolic space-time. However, before dealing with the general case, we shall first return to the quantum treatment of the homogeneous and isotropic exact situation where the state of the universe will be represented by a completely general wave function $\Psi(\phi, t)$. An analysis of this case may yield some insights for the general situation to be dealt with later.

4.1. *Homogeneous and isotropic space-times*

As pointed out by Narlikar and Padmanabhan[11] the propagator in this case is given as:

$$K(\phi_F, t_F; \phi_I, t_I) = F(t_F, t_I) \exp \frac{3i}{8\pi} \{A_{II}\phi_I{}^2 + 2A_{IF}\phi_I\phi_F + A_{FF}\phi_F{}^2\}, \qquad (4.1.1)$$

where we have $A_{IF} \to 0$ as $t \to 0$; i.e. in the limit as the classical singularity is approached. Let $\Psi(\phi, t)$ be the general wave function of the universe whose time evolution is given by (2.3.3). We will continue to impose the requirement that the classical state $\phi = 0$ should, at all times be the average of the quantum description. Therefore we have at $t = t_I$

$$\langle \phi_I \rangle \equiv \int_{-\infty}^{\infty} \bar{\Psi}_I(\phi_I)\phi_I \Psi_I(\phi_I)\, d\phi_I = 0. \tag{4.1.2}$$

Further, the requirement $\langle \phi \rangle = 0$ at all times means that the "momentum" conjugate to ϕ averages to zero. At $t = t_I$ this condition implies that

$$\int_{-\infty}^{\infty} \bar{\Psi}_I'(\phi_I)\Psi_I(\phi_I)\, d\phi_I = 0 = \int_{-\infty}^{\infty} \bar{\Psi}_I(\phi_I)\Psi_I'(\phi_I)\, d\phi_I, \tag{4.1.3}$$

where overhead prime denotes differentiation.

The average $\langle \phi_F \rangle$ at any later time t_F can now be shown to be zero provided (4.1.2) and (4.1.3) hold. For, by the relation (2.3.3) we obtain

$$\langle \phi_F \rangle = \int_{-\infty}^{\infty} \bar{\Psi}_I(\phi_1)\bar{K}(\phi_F, t_F; \phi_1, t_I)\phi_F K(\phi_F, t_F; \phi_2, t_I)\Psi_I(\phi_2)\, d\phi_1\, d\phi_2\, d\phi_F. \tag{4.1.4}$$

If one uses the δ-function representations

$$\delta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\, dx, \tag{4.1.5}$$

$$\delta'(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} ixe^{i\omega x}\, dx, \quad \text{etc.}, \tag{4.1.6}$$

and

$$\int_{-\infty}^{\infty} G(x)\delta'(x)\, dx = -G'(0), \tag{4.1.7}$$

then (4.1.4) can be simplified as

$$\langle \phi_F \rangle = \frac{\pi i}{2A_{IF}^{\,2}}|F(t_F, t_I)|^2 \int_{-\infty}^{\infty} \{\bar{\Psi}_I(\phi, t_I)\Psi_I'(\phi, t_I) + 2iA_{II}\phi\bar{\Psi}_I(\phi, t_I)\Psi_I(\phi, t_I)\}\, d\phi. \tag{4.1.8}$$

Hence, using (4.1.2) and (4.1.3) we get

$$\langle \phi_F \rangle = 0, \tag{4.1.9}$$

i.e. at all times the classical state is the average of the quantum ensemble.

Now the dispersion Δ_F at time t_F is given by

$$\Delta_F{}^2 \equiv \langle \phi_F{}^2 \rangle \equiv \int_{-\infty}^{\infty} \bar{\Psi}_F(\phi_F)\phi_F{}^2\Psi_F(\phi_F)\,d\phi_F$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(t_F, t_I)|^2 \exp\{iA_{II}(\phi_1{}^2 - \phi_3{}^2)\}\bar{\Psi}_I(\phi_3, t_I)\Psi_I(\phi_1, t_I)$$

$$\times\ \phi_F{}^2 \exp\{2i\phi_F A_{IF}(\phi_1 - \phi_3)\}\,d\phi_1\,d\phi_3\,d\phi_F. \tag{4.1.10}$$

Again, using (4.1.6) we get

$$\int_{-\infty}^{\infty} \phi_F{}^2 \exp\{2i\phi_F A_{IF}(\phi_1 - \phi_3)\}\,d\phi_F = \frac{2\pi}{8A_{IF}{}^3}\delta''(\phi_1 - \phi_3). \tag{4.1.11}$$

Using the normalization condition

$$\frac{\pi|F(t_2, t_1)|^2}{A_{IF}} = 1, \tag{4.1.12}$$

implied by

$$\int_{-\infty}^{\infty} \bar{\Psi}_F(\phi_F)\Psi_F(\phi_F)\,d\phi_F = 1, \tag{4.1.13}$$

we finally get for (4.1.10)

$$\Delta_F{}^2 = \frac{1}{4A_{IF}{}^2}\int_{-\infty}^{\infty} [\bar{\Psi}_I\Psi_I'' - 4A_{II}{}^2\phi^2\bar{\Psi}_I\Psi_I + 2iA_{II}\phi\bar{\Psi}_I\Psi_I']\,d\phi. \tag{4.1.14}$$

Since the integral itself is independent of t_F we see that as $t_F \to 0$, $\Delta_F{}^2 \sim A_{IF}{}^{-2}$. This dependence is the same as (2.3.5) for wave packets.

The above derivation uses the path integral method to propagate the wave functions. It is also possible to use the Hamiltonian formulation to examine the behaviour of the dispersion Δ where one can proceed by constructing the Hamiltonian and a differential equation to evolve the dispersion. Again the results are valid for a general wave function and Δ_F is seen to diverge near the singularity. The details of this approach will be given in a later publication.

4.2. *The divergence of Green's function in general*

It is well accepted that the above assumptions of homogeneity and isotropy represent very specialized situations and in general in the actual universe small or large deviations from these must be admitted. We shall now return to the general framework of the globally hyperbolic space-times which fully allow for this possibility and examine the behaviour of Green's functions in the vicinity of singularities within this general scenario. Since we will consider universes with arbitrary distribution of particles and allow inhomogeneities and anisotropies, we no longer have $\phi = \phi(t)$ only. Instead, ϕ becomes a general function of space-time points: $\phi = \phi(x^\mu, t)$, where $\mu = 1, 2, 3$ denotes space coordinates. The evolution of the general wave functional $\Psi[\phi(x^\mu), t]$ in this case is governed by Eq. (2.3.1).

Suppose, (M, g) is a general globally hyperbolic space-time satisfying Einstein's equations and with singularities distributed at various epochs as formulated in Sec. 3. Let Ω_c be a conformal factor such that the geometry $(M, \Omega_c{}^2 g)$ is geodesically complete. Denote by $G(X, B)$ and $G_c(X, B)$ the Green's functions for the geometries (M, g) and $(M, \Omega_c{}^2 g)$ respectively. It is known that the Green's functions of two conformally related space-times are connected by

$$G(X, B) = \Omega_c(X)\Omega_c(B)G_c(X, B). \tag{4.2.1}$$

Let a space-time singularity S in (M, g) be approached along some generator γ in $I^-(S)$. Then, as shown earlier, as $X \to S$ along γ, $\Omega_c(X) \to \infty$ in $I^-(S)$; i.e. Ω_c blows up as the singularity is approached. Now in the conformally transformed $(M, \Omega_c{}^2 g)$ the singularity S is thrown off to infinity and hence $G_c(X, B)$ will be regular for any X along γ within the space-time. Thus (4.2.1) implies that the Green's function $G(X, B)$ must diverge as Ω_c in the limit of approach to the singularity. From the arguments given in Sec. 2.3 we therefore conclude that quantum uncertainty diverges in this limit.

5. The Probability Measure of the Class of Singular Solutions

Using the results developed so far, we have been able to draw the conclusion that the quantum conformal fluctuations diverge near the epoch where the classical geometry admits a space-time singularity. This means that the classical picture can no longer be taken as a representative near this epoch and conclusions concerning the "inevitable" occurrence of singularities have to be revised. The quantum conformal framework may admit geometries which are no longer infected with the singularity problem. However, this raises the important question mentioned briefly in Sec. 2.3. Given the full range of possible geometries within the quantum conformal framework, what is the probability measure of those that are singular at the classically singular epoch? If it turns out that after quantizing the conformal fluctuations the set of singular geometries has a nonzero measure of probability, then it means that the singularity problem has not been avoided in the new framework. On the other hand, if the probability measure for the occurrence of singular solutions were to have a zero measure, we would have effectively eliminated the singularity problem by quantizing the conformal fluctuations.

5.1. *Homogeneous and isotropic space-times*

Again we begin by analysing the simple situation of homogeneous and isotropic space-times where ϕ depends on t only and the quantum state of the universe is described by a completely general wave function

$$\Psi = \Psi(\phi, t, \Delta). \tag{5.1.1}$$

We have singled out Δ above to indicate the dispersion of Ψ. Let the singularity be denoted by S, which is characterised by the TIP $I^-(S)$ and generated by some future endless non-space-like curve γ of finite length. Recalling the results from Sec. 3, we see that as $X \to S$ along γ, if ϕ behaves as $\phi = K\phi_c$ with $\phi_c = \Omega_c - 1$, and $K > 0$ is a constant, then ϕ blows up along γ and the lengths of all non-space-like curves in $I^-(S)$ can no longer be finite, which is a contradiction. Thus, for the singularity S to be admitted in M, we must have as $X \to S$,

$$\phi < K\phi_c, \quad \text{for all } K > 0, \tag{5.1.2}$$

i.e. $\phi/\phi_c \equiv \alpha$ must tend to zero.

Defining a new variable $u = \phi/\Delta$ if we now rescale Ψ as

$$\chi(u, t) = \sqrt{\Delta}\, \Psi(u\Delta, t, \Delta), \tag{5.1.3}$$

then the new wave function $\chi(u, t)$ is seen to have a unit dispersion at all times and the condition for singularity becomes

$$\frac{u\Delta}{\phi_c} \to 0. \tag{5.1.4}$$

Thus the singular solutions are defined by those values of u such that

$$u = \alpha\frac{\phi_c}{\Delta}, \qquad \alpha \to 0. \tag{5.1.5}$$

Now, as the analysis of Sec. 4.1 shows, $\Delta \sim A_{IF}{}^{-1}$. Further, (2.2.5) and (4.2.1) show that A_{IF}, which is the inverse of the Green's function G goes as $\phi_c{}^{-1}$. Thus we have $\Delta \sim \phi_c$ in this case and (5.1.5) becomes

$$u < \alpha\beta, \tag{5.1.6}$$

where $\beta =$ constant and $\alpha \to 0$. Since the probability density is given by

$$|\chi(u, t)|^2 = \bar{\chi}(u, t)\chi(u, t), \tag{5.1.7}$$

the probability measure of the set of geometries which will be singular, is given by

$$P_S = \int_{|u|<\alpha\beta} |\chi(u,t)|^2 \, du, \qquad \alpha \to 0. \tag{5.1.8}$$

The only circumstance under which P_S would remain nonzero in this limit is if $|\chi|^2$ approaches a multiple of $\delta(u)$. However, this is impossible since the dispersion of χ is unity—unless the limiting form of χ is such as to give for $|\chi|^2$ a delta function superposed on a function of support extending beyond $|u| = 1$. Excluding such pathological cases, we therefore conclude that among the set S the singular geometries will occur only with a vanishing probability. Note also that the geometries in the non-singular class defined by $|u| > \alpha\beta$ contribute predominantly to the probability measure as $X \to S$. We have thus shown that within the quantum conformal framework the measure of the set of those geometries which are likely to occur with singularities is zero.

5.2. *General space-times*

Next, let us consider the classical situation of a general globally hyperbolic space-time with arbitrary distribution of material particles and many singularities spread over at various epochs. One could also have here all—embracing cosmological singularities in the future or past in the case of "closed" universes, i.e. those which admit a compact Cauchy surface. A globally hyperbolic M admits what is called a "cosmic time function" which is a real valued function f from $M \to \mathbb{R}$ such that f always increases along any future directed non-space-like curve and ∇f is always future pointing and time-like. We can foliate M using a Cauchy cosmic time function which has the property that for each $c \in \mathbb{R}$, $f^{-}(c)$ is a Cauchy surface in M. Thus, cosmic time function provides a natural choice of foliation which we shall use here.

Let $\{S_1, S_2, \ldots, S_n, \ldots\}$ be the singularities in M lying at various epochs t_i, which are characterised by the TIP's $I^-(S_1)$, $I^-(S_2)$, $\ldots$, $I^-(S_n)$, $\ldots$, etc. generated by future directed non-space-like curves γ_1, γ_2, etc. that have finite lengths in future but no future end points. Now, consider a singularity S_i lying on a space-like hypersurface Σ_{t_i}. Let the TIP $I^-(S_i)$ be generated by γ_i. As shown above, as $X \to S_i$ along γ_i, we must have again $\phi/\phi_c = \alpha \to 0$ and hence the condition for the geometry to be singular at an epoch $t = t_i$ is given as

$$u < \alpha \frac{\phi_c}{\Delta} \quad \text{along } \gamma_i. \tag{5.2.1}$$

Now, in the above general situation under consideration, the state of the universe at any epoch t is no longer described by a single variable $\phi \in \mathbb{R}$ as in the homogeneous, isotropic case, but is described by an infinite set of continuum variables $\phi = \phi(x^\mu, t)$. In this situation, the evolution of the wave function is given by (2.3.1) where the propagator in this general case is given by (2.2.3). We adopt here the following simplifying notation used in Refs. 1 and 2. A repeated continuous variable J in any expression would mean an integration over the entire range. Thus, the propagation equation for a wave function can be expressed as:

$$\Psi_F(\phi_F) = K[\phi_F, t_F; \phi_I, t_I]\Psi_I(\phi_I). \tag{5.2.2}$$

With this notation, the general propagator (2.2.3) can be written as

$$K[\phi_F, t_F; \phi_I, t_I] = F(t_F, t_I)\exp i[A_{FF}(X_F, X_F')\phi_F(X_F)\phi_F(X_F') \\ + A_I(X_I)\phi_I(X_I)^2 + 2A_{IF}(X_I, X_F)\phi_I(X_I)\phi_F(X_F)], \tag{5.2.3}$$

where, in the second term, we have diagonalised the quadratic form $A_{II}\phi_I(X_I)\phi_I(X_F')$. (In Ref. 2 this diagonalization is discussed in detail). Now, suppose the wave functional describing the state of the universe is given by,

$$\Psi_I[\phi_I(X)] = \frac{1}{[2\pi\Delta_I(X)]^{1/4}}\exp\left\{-\frac{[\phi_I(X)]^2}{4[\Delta_I(X)]^2}\right\}. \tag{5.2.4}$$

The approximation that the present state of the universe is almost classical means that $|\Delta_I| \ll 1$. Using (2.3.1) and (5.2.4) we can evolve Ψ_I to the wave functional Ψ_F at any other later epoch $t = t_F$ to get

$$\Psi_F[\phi_F(X)] \propto \exp\left(\frac{A_{IF}{}^2\phi_I(X)^2}{\left(iA_I - \dfrac{1}{4\Delta_I{}^2}\right)} + iA_{FF}(X, X')\phi_F(X)\phi_F(X')\right). \tag{5.2.5}$$

The normalising constant of proportionality does not depend on ϕ_I. Now, (5.2.5) can be used to obtain the probability density

$$|\Psi_F|^2 = f\exp\left(-\frac{\phi_F{}^2}{2\Delta_F{}^2}\right), \quad f = \text{constant} \tag{5.2.6}$$

where, in analogy to (2.3.5)

$$\Delta_F{}^2 = \Delta_I{}^2\left(A_I{}^2 + \frac{1}{16\Delta_I{}^4}\right)A_{IF}{}^{-2}. \tag{5.2.7}$$

Thus, we see that as $X \to S_i$, $\Delta \sim A_{IF}{}^{-1}$. However, as seen earlier, $A_{IF} \sim \bar{G}(X_2, X_1)^{-1} \sim \Omega_c{}^{-1}$ and hence we have $\Delta \sim \phi_c$. As a result, we can write

$$\Delta = \sigma(X^\mu, t)\phi_c, \tag{5.2.8}$$

where σ is a well-behaved function along γ_i. Thus, the condition (5.2.1) for the singularity now becomes

$$u < \alpha\beta(X^\mu, t). \tag{5.2.9}$$

Now we can write the probability measure for the class of geometries of S which will be singular at $t = t_i$, as

$$P_S = \int_{|u|<\alpha\beta} |\Psi(u,t)|^2 \, Du = \int_{|u|<\alpha\beta} f \exp(-u^2/2) \, Du. \tag{5.2.10}$$

Clearly, the integral (5.2.10) becomes arbitrarily small in the limit as $\alpha \to 0$ and hence we deduce that in the limit of approach to the classically singular epoch t_i, the probability that space-time might admit a singularity at the epoch $t = t_i$ is vanishingly small.

Finally, it is not difficult to see that the analysis of Sec. 5.1 can be extended to wave functionals and analogous conclusions drawn for Ψ_I not necessarily describable as wave packets.

6. Conclusion

We have thus established under very general assumptions that within the set of all geometries conformal to a given singular classical geometry, the subset of singular geometries occurs with zero probability. The probability in this work is defined in terms of conformal quantization of Einstein's equations.

This work, of course, does not deal with nonconformal degrees of freedom, and to this extent the overall question of how probable singularities are in general quantum cosmology remains open. Nevertheless, the direct connection between conformal transformation and singularities discussed in this paper prompts us to hazard the conjecture that singular geometries would turn out to be exceptions rather than the rule in the full theory of quantum gravity.

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