

Holography in action

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The Einstein-Hilbert action and its natural generalizations to higher dimensions (like the Lanczos-Lovelock action) have certain peculiar features. All of them can be separated into a bulk and a surface term, with a specific (“holographic”) relationship between the two, so that either term can be used to extract information about the other. Further, the surface term leads to entropy of the horizons on shell. It has been argued in the past that these features are impossible to understand in the conventional approach but find a natural explanation if we consider gravity as an emergent phenomenon. We provide further support for this point of view in this paper. We describe an alternative decomposition of the Einstein-Hilbert action and the Lanczos-Lovelock action into a new pair of surface and bulk terms, such that the surface term becomes the Wald entropy on a horizon and the bulk term is the energy density (which is the Arnowitt-Deser-Misner Hamiltonian density for Einstein gravity). We show that this new pair also obeys a holographic relationship, and we give a thermodynamic interpretation of this relation in this context. Since the bulk and surface terms, in this decomposition, are related to the energy and entropy, the holographic condition can be thought of as analogous to inverting the expression for entropy given as a function of energy $S = S(E, V)$ to obtain the energy $E = E(S, V)$ in terms of the entropy in a normal thermodynamic system. Thus the holographic nature of the action allows us to relate the descriptions of the same system in terms of two different thermodynamic potentials. Some further possible generalizations and implications are discussed.

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I. MOTIVATING THE HOLOGRAPHIC ACTIONS FOR GRAVITY

The peculiar relationship between thermodynamics and dynamics of horizons, known for four decades [1], is now being slowly recognized as indicating a more fundamental principle in which gravity can be viewed as an emergent phenomenon like fluid mechanics or elasticity. (For a recent review of this approach, see [2]; for a sample of papers, implementing and discussing this paradigm in different ways, see Ref. [3].) Such a point of view draws support from several pieces of evidence, of which we mention the following:

(a) The field equations of gravity reduce to a thermodynamic identity on the horizons in a wide variety of models much more general than just Einstein’s gravity [4,5]. As pointed out first in [6], and confirmed later by several works, the thermodynamic paradigm seems to be applicable to a wide class of theories that are much more general than Einstein gravity in four dimensions.

(b) It is possible to obtain the field equations of gravity—again for a wide class of theories—from purely thermodynamic considerations (see, e.g., [7]).

(c) One can obtain an equipartition law analogous to $E = (1/2)nk_B T$ for the density n of microscopic degrees of freedom in any static geometry [8], providing a direct

window to microscopic physics in the thermodynamic limit.

In such an approach, geometrical variables like the metric, etc., are derived concepts (similar to pressure, density, etc. of a gas), and the dynamical equations governing them can be derived from the thermodynamic limit of an underlying microstructure, say, by extremizing a suitably defined entropy functional [7]. But, on the other hand, we know from a standard textbook description that one can obtain the field equations for gravity from a action functional in which the metric is varied. This raises the following question: If gravity is indeed an emergent phenomenon, should not the conventional action functionals contain some signature of this fact?

After all, field equations “know” that there exists an alternative, emergent, interpretation for the dynamics. Hence it seems reasonable to assume that this information must be embedded in the action functionals describing theories of gravity in some manner. There is evidence that this is indeed the case [9]. There are peculiar holographic relations between the surface and bulk terms in the action functionals describing several theories of gravity, and the surface term in the action is closely related to the entropy of horizons in all these theories. Our aim is to elucidate this further.

Let us begin by reviewing some known facts and interpreting them in a manner useful for our discussion, starting from the Einstein-Hilbert action. It is well known that the Einstein-Hilbert action can be separated into a bulk term and a surface term. (Several facts related to this decom-

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position were known fairly early in the literature and, in particular, to Einstein himself [10]; for a modern textbook description, see chapter 6 of Ref. [11].) The bulk term (the ‘‘Gamma-Gamma’’ term) depends on the metric and its first derivatives and is quadratic in the latter; the surface term arises from integrating a total divergence, and it contains both the normal and tangential derivatives of the metric on the boundary. Because of the dependence of the surface term on the normal derivatives of the metric, the action principle cannot be formulated in the usual manner. In general, there are two ways of handling this issue:

(a) One can add an extra term [12] to the Einstein-Hilbert action such that the variation of this term precisely cancels the unwanted terms arising in the variation of the original surface term.

(b) One can simply ignore the surface term in the Einstein-Hilbert action and vary the bulk term, keeping the metric fixed at the boundary; even though the bulk term is not generally covariant, the resulting field equations are indeed covariant.

In either approach, it is *only* the variations of the bulk $\Gamma\Gamma$ term that contribute to the field equations. That is, the field equations (and their solutions) do *not* depend in any way on the surface term. It is therefore a mystery—in the conventional approach—that the surface term, which is ignored while obtaining the field equations, can be used to determine the entropy of the horizons that arise in the theory.

The solution to this mystery was first pointed out in [9], where it was emphasized that the bulk and surface terms in the Einstein-Hilbert action are connected by a peculiar relation:

$$L_{\text{sur}} = -\partial_i \left(g_{ab} \frac{\partial L_{\text{bulk}}}{\partial (\partial_i g_{ab})} \right), \quad (1)$$

which allows the information about either one to be extracted from the other. It was also shown that one can obtain the bulk action from the surface term if one adopts the thermodynamic perspective of gravity. Later on, these ideas were generalized to a wide class of models [13] including the Lanczos-Lovelock [14] models. The relationship between the bulk and boundary terms in the action was termed ‘‘holographic’’ because the information about the bulk action functional (which we can vary to obtain the dynamical equations) is encoded in the boundary action functional. In this paper, we shall continue to use the terminology holographic action with this understanding.

The holographic nature of the action fits very well with the thermodynamic approach to gravity and can be thought of as the hidden signal in the action functionals, indicating that the description of gravity is an emergent one. In fact, one can provide very general arguments to suggest that the action functional describing *any* theory of gravity that obeys the principle of equivalence and the principle of general covariance will have a bulk term and a boundary term related holographically (see, e.g., [7].) If this is the

case, one might like to explore this connection further and see what insights it can provide. In particular, we would like to address the following concrete questions:

(a) Of the two terms—bulk and boundary—the boundary term has a clear interpretation as being related to the entropy of horizons. But the physical interpretation of the bulk term is unclear and one would like to have a thermodynamic interpretation for the same.

(b) Is there something special in the particular decomposition of the Einstein-Hilbert action such that it admits a holographic relationship? Or can the holographic relationship arise in other contexts when we decompose the Einstein-Hilbert action into a bulk term and a surface term in a different manner?

It turns out that the answers to these two questions are closely related. We will show that there is an alternative way of decomposing the action functionals in the case of static geometries which gives a simple thermodynamic interpretation for both bulk and boundary terms as energy and entropy. More importantly, this decomposition is also holographic so that one can extract the information about the energy from the entropy and vice versa.

The paper is organized as follows. In Sec. II, we briefly review the previous work done as regards the holography of the action. We also set up the notations that we use in the rest of the paper. In Sec. III, we look at the decomposition of the Einstein-Hilbert action into a new pair of surface and bulk terms, which is different from the usual splitting. We show that this new pair also obeys a holographic relationship. We will study this decomposition from a thermodynamic point of view and give meaning to the holographic relationship as playing the same role as the relationship between two thermodynamic potentials—viz., the entropy and the energy of a thermodynamic system. In Sec. IV, we generalize the results of Sec. III to the Lanczos-Lovelock models of gravity. Further, we prove a general result which gives a holographic relationship between any decomposition of the Lanczos-Lovelock action into an arbitrary bulk term and a surface term, provided the surface term is homogeneous in its dynamical degrees of freedom. The conclusions are discussed in Sec. V.

The metric signature is $(-, +, +, \dots, +)$, and all the fundamental constants such as G , \hbar , and c have been set to unity. Latin indices run from 0–3, whereas Greek indices run from 1–3.

II. HOLOGRAPHY OF THE GRAVITATIONAL ACTION

We begin by expressing the Einstein-Hilbert Lagrangian $L_{\text{EH}} = R$ in a manner which will be convenient for our further discussions and generalization to Lanczos-Lovelock models. We write

$$L_{\text{EH}} \equiv Q_a{}^{bcd} R^a{}_{bcd} = Q_{cd}{}^{ab} R^ab{}_{cd} \equiv \delta_{ab}{}^{cd} R^ab{}_{cd} = R, \quad (2)$$

where

$$Q_{cd}^{ab} = \delta_{ab}^{cd} = \frac{1}{2}(\delta_a^c \delta_b^d - \delta_a^d \delta_b^c). \quad (3)$$

Here δ_{ab}^{cd} is the alternating (“determinant”) tensor. The tensor Q^{abcd} is the only fourth rank tensor that can be constructed from the metric (alone) that has

- (i) all the symmetries of the curvature tensor and
- (ii) zero divergence on all indices, $\nabla_a Q^{abcd} = 0$, etc.

The total action can be written as a sum of the Einstein-Hilbert action and the matter action A_m so that

$$A_{\text{total}} = \int_{\mathcal{V}} d^D x \sqrt{-g} L_{\text{EH}} + \int_{\mathcal{V}} d^D x \sqrt{-g} L_{\text{matter}}. \quad (4)$$

The variation of the metric in this action, after ignoring the surface terms, leads to the Einstein field equations

$$G_{ab} = Q_a{}^{cde} R_{bcde} - \frac{1}{2} g_{ab} R = R_{ab} - \frac{1}{2} g_{ab} R = \frac{1}{2} T_{ab}, \quad (5)$$

where T_{ab} is the stress-energy tensor obtained from the variation of the matter part of the total action.

We will now state a feature of the Einstein-Hilbert action relevant to our discussion, viz., its decomposition into a bulk term and a surface term. It can be shown [13] that when the Lagrangian has the form $Q_a{}^{bcd} R^a{}_{bcd}$, with Q^{abcd} obeying the two properties (i) and (ii) mentioned in the last paragraph, there is a natural decomposition of the Lagrangian into bulk and surface terms which are holographically related. (The actual form of Q^{abcd} is irrelevant as long as it has the symmetries of the curvature tensor and is divergence-free.). In the case of the Einstein-Hilbert action, $\sqrt{-g} L_{\text{EH}}$ can be written as a sum $L_{\text{bulk}} + L_{\text{sur}}$, where L_{bulk} is quadratic in the first derivatives of the metric and L_{sur} is a total derivative which leads to a surface term in the action:

$$\begin{aligned} \sqrt{-g} L_{\text{EH}} &= 2\partial_c [\sqrt{-g} Q_a{}^{bcd} \Gamma_{bd}^a] + 2\sqrt{-g} Q_a{}^{bcd} \Gamma_{dk}^a \Gamma_{bc}^k \\ &\equiv L_{\text{sur}} + L_{\text{bulk}} \end{aligned} \quad (6)$$

with

$$L_{\text{bulk}} = 2\sqrt{-g} Q_a{}^{bcd} \Gamma_{dk}^a \Gamma_{bc}^k, \quad (7)$$

$$L_{\text{sur}} = 2\partial_c [\sqrt{-g} Q_a{}^{bcd} \Gamma_{bd}^a]. \quad (8)$$

It is well known that one can obtain the Einstein equations, Eq. (5), by varying only L_{bulk} , keeping g_{ab} fixed at the boundary (see Appendix B 2 for a brief demonstration). What is more remarkable is the fact that there exists a simple relation between L_{bulk} and L_{sur} allowing L_{sur} to be determined completely by L_{bulk} [9,15]. It is given as

$$L_{\text{sur}} = -\frac{1}{[(D/2) - 1]} \partial_i \left(g_{ab} \frac{\partial L_{\text{bulk}}}{\partial (\partial_i g_{ab})} \right), \quad (9)$$

which is a generalization of Eq. (1) to D dimensions. As discussed in Sec. I, we call such a relation holographic.

All the above results generalize to a class of actions known as the Lanczos-Lovelock action [14], which is a

generalization of the Einstein-Hilbert action. The Lanczos-Lovelock Lagrangian is constructed as a special product of m curvature tensors R_{cd}^{ab} given by

$$L_m = \delta_{2468\dots 2k}^{1357\dots 2k-1} R_{13}^{24} R_{57}^{68} \dots R_{2k-3}^{2k-2} R_{2k-1}^{2k}; \quad k = 2m, \quad (10)$$

where $k = 2m$ is an even number. For $m = 1$, the L_m reduces to the Einstein-Hilbert Lagrangian in the form given in Eq. (2). The L_m is clearly a homogeneous function of degree m in the curvature tensor R_{cd}^{ab} so that it can also be expressed in the form

$$L = \frac{1}{m} \left(\frac{\partial L}{\partial R^a{}_{bcd}} \right) R^a{}_{bcd} \equiv \frac{1}{m} P_a{}^{bcd} R^a{}_{bcd} = Q_a{}^{bcd} R^a{}_{bcd}, \quad (11)$$

where we have defined $P_a{}^{bcd} \equiv (\partial L / \partial R^a{}_{bcd})$ so that $P^{abcd} = m Q^{abcd}$. Hence Q^{abcd} inherits all the symmetries of the curvature tensor. It can also be directly verified that for these Lagrangians,

$$\nabla_c P^{ijcd} = m \nabla_c Q^{ijcd} = 0. \quad (12)$$

Because of the symmetries, Q^{abcd} is divergence-free in *all* indices. So the Q^{abcd} satisfies the two conditions (i) and (ii) mentioned at the beginning of this section.

The total action is obtained from adding the Lanczos-Lovelock Lagrangian to the matter Lagrangian and integrating over a D -dimensional region \mathcal{V} . The variation of this action, ignoring the boundary terms on $\partial \mathcal{V}$, leads to the following field equations:

$$\mathcal{G}_{ab} \equiv P_a{}^{cde} R_{bcde} - \frac{1}{2} g_{ab} L \equiv \mathcal{R}_{ab} - \frac{1}{2} g_{ab} L = \frac{1}{2} T_{ab}, \quad (13)$$

which, of course, reduce to Eq. (5) for $m = 1$. The notation with calligraphic font is motivated by the fact that $\mathcal{G}_{ab} \rightarrow G_{ab}$ and $\mathcal{R}_{ab} \rightarrow R_{ab}$ in Einstein’s theory.

Since Q^{abcd} also satisfies the same two properties (i) and (ii), L_m can be again separated [13] into bulk and surface terms as

$$\begin{aligned} \sqrt{-g} L_m &= 2\partial_c [\sqrt{-g} Q_a{}^{bcd} \Gamma_{bd}^a] + 2\sqrt{-g} Q_a{}^{bcd} \Gamma_{dk}^a \Gamma_{bc}^k \\ &\equiv L_{\text{sur}} + L_{\text{bulk}}. \end{aligned} \quad (14)$$

One crucial difference between Einstein gravity and the more general Lanczos-Lovelock models is the following: In Einstein gravity $L_{\text{bulk}} \simeq Q \Gamma \Gamma$ is quadratic in the first derivatives of the metric and does not involve second derivatives of the metric. This is because in this case, Q^{abcd} depends only on the metric. In the case of ($m > 1$) Lanczos-Lovelock models, Q^{abcd} will have a nontrivial dependence on the curvature tensor and hence on the second derivatives of the metric. Therefore, L_{bulk} now depends on both first derivatives of the metric as well as second derivatives, and hence, while varying the action (based on either L_m or L_{bulk}) we need to keep both the metric *and* its normal derivatives fixed at the boundary to

get the field equations. It can be shown that (see Appendix B 2) under these variations both L_m and L_{bulk} lead to the same field equations in Eq. (13). (In principle, one can add counterterms to the general Lanczos-Lovelock action to make it well defined [16], but the nature of these counterterms, in general, is quite complicated. Fortunately, this is irrelevant to our discussion.)

Furthermore, as in the case of the Einstein-Hilbert Lagrangian, the L_{sur} of the Lanczos-Lovelock Lagrangian can be obtained from the L_{bulk} as [13]

$$[D/2 - m]L_{\text{sur}} = -\partial_i \left[g_{ab} \frac{\delta L_{\text{bulk}}}{\delta(\partial_i g_{ab})} + \partial_j g_{ab} \frac{\partial L_{\text{bulk}}}{\partial(\partial_i \partial_j g_{ab})} \right], \quad (15)$$

where differentiation by δ is the Euler derivative. This is a natural generalization of the holographic relation of Eq. (9) and reduces to Eq. (9) for $m = 1$. We also mention that the surface term A_{sur} evaluated on the horizon gives one-quarter of the area of the boundary when the boundary is a horizon [15] in Einstein-Hilbert gravity, while it is proportional to the corresponding Wald entropy of the horizon [17] in the case of Lanczos-Lovelock gravity [13], with the proportionality constant being $1/m$. In the absence of the holographic relation between the surface and bulk terms, this fact is difficult to understand because the field equations are independent of the boundary term. The holographic relationship in Eqs. (9) and (15) explains how the two terms are interrelated, thereby offering a possible reason why the surface term might have a physical meaning on shell. We shall now probe these aspects further.

III. AN ALTERNATIVE LOOK AT THE EINSTEIN-HILBERT ACTION

We are interested in providing a thermodynamic interpretation for the action functionals in the theories of gravity, taking a clue from the fact that the surface term is related to the horizon entropy. Since the notion of temperature and thermal equilibrium is well defined in static situations, we shall consider the class of all *static* spacetime metrics to elucidate the thermodynamic relationships.

Our first task is to introduce an alternative decomposition of the Einstein-Hilbert action into a surface term and a bulk term, in any static spacetime. Using the time-time component of the Einstein tensor G_b^a in Eq. (5), the Einstein-Hilbert Lagrangian can be expressed as

$$L_{\text{EH}} = -2G_0^0 + 2R_0^0. \quad (16)$$

In any static spacetime, the R_0^0 components, in particular R_0^0 , can be expressed as a divergence term [15]

$$R_0^0 = \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} g^{0k} \Gamma_{0k}^\alpha). \quad (17)$$

This is most easily seen from noting that any static space-

time has a timelike Killing vector which, for a natural choice of the time coordinate, has the components $\xi^a = (1, \mathbf{0})$. The standard identity satisfied by the Killing vector now gives

$$R_j^i \xi^j = R_0^0 = \nabla_b \nabla^a \xi^b = \frac{1}{\sqrt{-g}} \partial_b (\sqrt{-g} \nabla^a \xi^b), \quad (18)$$

where the last relation follows from the fact that $\nabla^a \xi^b$ is an antisymmetric tensor. Equation (17) now follows directly after noticing that all quantities are time independent. Hence, we see that the Einstein-Hilbert Lagrangian for static spacetimes can be expressed as a sum of a bulk term and a surface term in the form

$$L_{\text{EH}} = R = -2G_0^0 + 2 \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} g^{0k} \Gamma_{0k}^\alpha). \quad (19)$$

Since G_0^0 contains second derivatives of the metric, it is obvious that this decomposition is different from the standard decomposition in Eq. (6). We shall now describe several interesting features of this decomposition.

To begin with, it provides yet another variational principle for obtaining the field equations in the static case. As proved in Appendix B 3, the variation of $-2G_0^0$, keeping the static metric fixed on the boundary, leads to the usual field equations, viz., Einstein's equations.

Second, the action functional, as well as the two terms in it, has a direct thermodynamic interpretation which is clear in the Euclidean sector obtained by replacing t with it . In any static spacetime with a suitable gauge, one has the result $-2G_0^0 = -16\pi \mathcal{H}_{\text{ADM}}$. We can now express the Euclidean Einstein-Hilbert action as an integral over $R = 2(-8\pi \mathcal{H}_{\text{ADM}} + R_0^0)$. Since the spacetime is static, the integrand is independent of time and we limit the time integration to a finite range $(0, \beta)$ to get a finite result in the Euclidean sector. Converting the volume integral of R_0^0 over three-space to a surface integral over the two-dimensional boundary, we can write [15] the Euclidean Einstein-Hilbert action in any static spacetime as

$$A_E = -\beta \int N \sqrt{h} d^3x \mathcal{H}_{\text{ADM}} + \frac{\beta}{8\pi} \int d^2x \sqrt{\sigma} N n_\alpha (g^{0k} \Gamma_{0k}^\alpha) \equiv -\beta E + S, \quad (20)$$

where $N = \sqrt{g_{00}^E}$ is the lapse function, h is the determinant of the spatial metric, and σ is the determinant of the two-metric on the surface. [We have also put in a factor of $1/16\pi$ such that $L_{\text{EH}} = (1/16\pi)R$.] In a class of static metrics with a horizon and associated temperature, the time interval has natural periodicity in β , which can be identified with the inverse temperature. Once this identification is made, the βN factor in Eq. (20) is exactly what is needed to give the local Tolman temperature $T_{\text{loc}} = \beta_{\text{loc}}^{-1} \equiv (\beta N)^{-1} = T/\sqrt{-g_{00}}$. So we are actually integrating $\beta_{\text{loc}} \mathcal{H}_{\text{ADM}}$ over all space, as one should, and we take the resulting quantity to be βE . (One can think of E as the

thermally averaged energy, obtained with a weightage factor which is the local inverse temperature.) When the two-surface is a horizon, the integral over R_0^0 gives one-quarter of the area of the horizon, which is the expression for entropy [15]. (We will show in the next section that R_0^0 is in fact equal to βQ , where Q is the Noether charge as used by Wald [17] to define entropy.) This allows identification of the two terms with energy and entropy; together the Euclidean Einstein action can be interpreted as giving the (negative of) free energy of space time.

With this thermodynamic interpretation of the action, one can interpret extremizing the integral over $-2G_0^0$ while keeping the surface term fixed at the boundary as extremizing the bulk energy of the static spacetime while keeping the entropy fixed. From previous work we know that the resulting field equations can be interpreted as the thermodynamic identity $dE = TdS - PdV$ on the horizon. In the usual thermodynamic systems, if we know the entropy functional $S(E, V)$, we can obtain the other thermodynamic variables like (T, P) , etc. of the system. Alternatively, one can invert the form of $S(E, V)$ to get energy $E(S, V)$ in terms of entropy. Similarly, in the case of the Einstein-Hilbert action, we can consider the extremization of $-2G_0^0$ as equivalent to obtaining the thermodynamic relation $dE = TdS - PdV$ (which is the same as field equations of the theory) from an energy functional. Further, if gravity is truly an emergent, thermodynamic phenomenon of an underlying microscopic theory, then one should be able to invert the energy functional $E(S, V)$ of gravity to obtain the $S(E, V)$ functional. This motivates us to ask the following question: Can we obtain the surface term $2R_0^0$ directly from the bulk term $-2G_0^0$?

Holography again answers this question. Even in the new decomposition we have introduced, the two terms continue to be related holographically. We can show by direct computation that $L_{\text{sur}} = 2\sqrt{-g}R_0^0$ and $L_{\text{bulk}} = -2\sqrt{-g}G_0^0$ are related by

$$L_{\text{sur}} = -\frac{1}{[(D/2) - 1]} \partial_i \left(g_{ab} \frac{\delta L_{\text{bulk}}}{\delta(\partial_i g_{ab})} + \partial_j g_{ab} \frac{\partial L_{\text{bulk}}}{\partial(\partial_i \partial_j g_{ab})} \right) \quad (21)$$

(see Appendix A 1 for the proof). One should note that Eq. (21) is the general expression of the holographic relation for the Einstein-Hilbert action. It reduces to the form in Eq. (9) in the standard decomposition because $L_{\text{bulk}} = 2\sqrt{-g}Q_a{}^{bcd}\Gamma_{dk}^a\Gamma_{bc}^k$ is independent of the second derivatives of the metric.

In this analysis, we have given a physical meaning to the holographic relationship in the gravitational action. It is seen as playing the same role as the relationship between two thermodynamic potentials, viz., the entropy and the energy of a normal thermodynamic system. These results also suggest that the dynamics of spacetime can be encoded in an entropy functional at the boundary of the

spacetime, and one could obtain the field equations from this entropy functional. Thus one could, in principle, formulate a theory of gravity completely by specifying only these entropy functionals at the boundary. Some recent attempts in this direction have been made in Ref. [18].

IV. GENERALIZATION TO LANCZOS-LOVELOCK THEORY

It has been repeatedly noticed in the literature that the thermodynamic aspects transcend Einstein's theory and occur in any reasonable theory of gravity that obeys the principle of equivalence and general covariance. If we further demand that the field equations not be of a degree higher than two in the metric, one is led to the Lanczos-Lovelock models. Just like several other thermodynamic features, the results obtained above are also applicable to Lanczos-Lovelock models. We shall now describe this generalization in a manner similar to the discussion in the previous section.

Using the time-time component of Eq. (13), the m th order Lanczos-Lovelock Lagrangian can be expressed as

$$L_m = -2G_0^0 + 2P^{0cde}R_{0cde}. \quad (22)$$

In any stationary spacetime, the $P^{acde}R_{0cde}$ components, in particular $P^{0cde}R_{0cde}$, can be expressed as a divergence term,

$$P^{0cde}R_{0cde} = \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} P_a{}^{ba0} \Gamma_{0b}^a). \quad (23)$$

This is again most easily seen from the identity for the Killing vector $\xi^a = (1, \mathbf{0})$,

$$\begin{aligned} P^{0cdb}R_{acdb}\xi^a &= P^{0cdb}\nabla_c\nabla_d\xi_b \\ &= \frac{1}{\sqrt{-g}} \partial_c (\sqrt{-g} P^{0cdb} \nabla_d \xi_b), \end{aligned} \quad (24)$$

where the last relation follows from the fact that $\nabla_c P^{acdb} = 0$ and P^{acdb} is antisymmetric in its first two indices. Equation (23) now follows directly after noticing that all quantities are time independent. Hence, the Lanczos-Lovelock Lagrangian for static spacetimes can be expressed as a sum of a bulk term and a surface term,

$$L_m = -2G_0^0 + 2\frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} P_a{}^{ba0} \Gamma_{0b}^a), \quad (25)$$

which is a direct generalization of the result in Eq. (19) for Einstein's theory.

The $L_{\text{sur}} = 2\sqrt{-g}P^{0cdb}R_{0cde}$ in Eq. (22) has a nice physical interpretation. It is actually the Noether charge density (i.e., the time component of the Noether current) which arises from the diffeomorphism invariance of the theory. To see this, we rederive Eq. (25) using the definition of the Noether charge along the lines of Ref. [2]. Consider the variation of a Lanczos-Lovelock Lagrangian, which can be expressed in the form (see Appendix B 1)

$$\delta(L\sqrt{-g}) = \sqrt{-g}(\mathcal{G}_{ab}\delta g^{ab} + \nabla_a \delta v^a), \quad (26)$$

where

$$\mathcal{G}_{ab} = P_a{}^{cde}R_{bcde} - \frac{1}{2}g_{ab}L, \quad (27)$$

$$\delta v^a = 2P^{cbad}(\nabla_b \delta g_{dc}), \quad (28)$$

and

$$P_a{}^{bcd} = \frac{\partial L}{\partial R^a{}_{bcd}}. \quad (29)$$

When the variations in δg^{ab} arise due to the diffeomorphism $x^a \rightarrow x^a + q^a$, then we have $\delta g^{ab} = \nabla^a q^b + \nabla^b q^a$ and $\delta(L\sqrt{-g}) = -\sqrt{-g}\nabla_a(Lq^a)$. Substituting these in Eq. (26) and using the Bianchi identity $\nabla_a \mathcal{G}^{ab} = 0$, we obtain a conservation law $\nabla_a J^a = 0$, for the Noether current,

$$J^a = Lq^a + \delta_q v^a + 2\mathcal{G}^{ab}q_b, \quad (30)$$

where $\delta_q v^a$ is the variation of the surface term when the variation in the metric δg^{ab} is due to the diffeomorphism. Since J^a is divergenceless, it is convenient to write J^a as $J^a = \nabla_b J^{ab}$, where J^{ab} is an antisymmetric tensor. For Lanczos-Lovelock Lagrangians, knowing the variation of the surface term $\delta_q v^a$, one can obtain the explicit form of J^{ab} to be [2,19]

$$J^{ab} = 2P^{abcd}\nabla_c q_d. \quad (31)$$

As is well known, the expressions of J^a , J^{ab} , etc. are not unique; in what follows we shall use the expressions quoted above.

In the case of static spacetimes with a Killing vector ξ^b , it is natural to consider the Noether current corresponding to $q^a = \xi^a$. When ξ^a satisfies the Killing equations at an event \mathcal{P} , the variation $\delta_\xi v^a$ vanishes at event \mathcal{P} and we get

$$J^a = L\xi^a + 2\mathcal{G}^{ab}\xi_b. \quad (32)$$

In particular, when ξ is a timelike Killing vector given as $\xi = (1, \mathbf{0})$ for static spacetimes, we get

$$J^0 = L + 2\mathcal{G}_0^0 \quad (33)$$

and Eq. (31) becomes

$$J^{0c} = 2P_a{}^{bc0}\Gamma_{0b}^a. \quad (34)$$

Hence, the Lagrangian for static spacetime can be written as a sum of a bulk term and a surface term,

$$L = -2\mathcal{G}_0^0 + \nabla_a J^{0a}, \quad (35)$$

which is the same as Eq. (25) and allows the identification of the divergence term as the time component of the Noether current. (Of course, since Einstein's theory is a special case of Lanczos-Lovelock models, this interpretation of the surface term holds for Einstein gravity as well.) We shall now show that the results obtained in the last

section for Einstein's theory continue to hold in the present case.

To begin with, let us consider the thermodynamic interpretation of the Euclideanized action [2]. The conserved Noether current for the displacement $x^a \rightarrow x^a + \xi^a$ is given by Eq. (32). We will work in the Euclidean sector and integrate this expression over a constant- t hypersurface with the measure $d\Sigma_a = \delta_a^0 N \sqrt{h} d^{D-1}x$, where $g_{00}^E = N^2$ and h is the determinant of the spatial metric. Multiplying by the period β of the imaginary time, we get

$$\begin{aligned} \beta \int J^a d\Sigma_a &= \beta \int 2\mathcal{G}_b^a \xi^b d\Sigma_a + \beta \int L \xi^a d\Sigma_a \\ &= \int (\beta N) 2\mathcal{G}_b^a u_a u^b \sqrt{h} d^{D-1}x \\ &\quad + \int_0^\beta dt_E \int L \sqrt{g} d^{D-1}x, \end{aligned} \quad (36)$$

where we have introduced the four-velocity $u^a = \xi^a/N = N^{-1}\delta_0^a$ of observers moving along the orbits of ξ^a , and the relation $d\Sigma_a = u_a \sqrt{h} d^{D-1}x$. The term involving the Lagrangian gives the Euclidean action for the theory. In the term involving $2\mathcal{G}_{ab}$ we note that $\beta N \equiv \beta_{\text{loc}}$ corresponds to the correct redshifted local temperature. Hence, taking a cue from our procedure for the Einstein-Hilbert action, here too we define the (thermally averaged) energy E as

$$\begin{aligned} \int (\beta N) 2\mathcal{G}_b^a u_a u^b \sqrt{h} d^{D-1}x &= \int \beta_{\text{loc}} 2\mathcal{G}_b^a u_a u^b \sqrt{h} d^{D-1}x \\ &\equiv \beta E. \end{aligned} \quad (37)$$

We thus get

$$A_E = \beta \int J^a d\Sigma_a - \beta E. \quad (38)$$

The first term involving the Noether charge is just the Wald entropy, which continues to hold true in the Euclidean sector. Therefore, we find that

$$A_E = S - \beta E = -\beta F, \quad (39)$$

where F is the free energy. Thus we have a thermodynamic interpretation for the Lanczos-Lovelock action in the case of static spacetimes.

The next question to ask, as in the case of the Einstein-Hilbert action, is as follows: Are the bulk terms and surface terms related by holography? The answer again is "yes." One can show by direct calculation—which is somewhat more involved as outlined in Appendix B 2—that $L_{\text{sur}} = 2\sqrt{-g}P^{0cde}R_{0cde}$ is related to $L_{\text{bulk}} = -2\sqrt{-g}\mathcal{G}_0^0$ through the holographic relation

$$\begin{aligned} [D/2 - m]L_{\text{sur}} &= -\partial_i \left[g_{ab} \frac{\delta L_{\text{bulk}}}{\delta(\partial_i g_{ab})} \right. \\ &\quad \left. + \partial_j g_{ab} \frac{\partial L_{\text{bulk}}}{\partial(\partial_i \partial_j g_{ab})} \right]. \end{aligned} \quad (40)$$

This is the same relation as Eq. (15). We thus find that the holographic relation between the surface term and the bulk term is not only valid for the Einstein-Hilbert Lagrangian but also for the Lanczos-Lovelock Lagrangian which shares the basic geometric structure of the Einstein-Hilbert Lagrangian. Further, we see that the relation is true for the Lagrangian written in two different ways [see Eqs. (14) and (25)]. (In Appendix B 2, we have shown that the relation is true for an arbitrary pair of bulk and surface terms provided the surface term is homogeneous in its dynamical degrees of freedom.) This suggests that holography has deep roots in the very nature of the Lanczos-Lovelock Lagrangian itself (and hence applies to the Einstein-Hilbert Lagrangian as a special case).

Our thermodynamic interpretation of the holographic relation also carries over to the Lanczos-Lovelock models in a straightforward manner. We can now consider the bulk and boundary terms of the action as providing the energy and entropy of the system, and the holographic relation as providing the means for obtaining $S(E)$ from $E(S)$ and vice versa. It may, however, be noted that these results do not generalize beyond Lanczos-Lovelock models (to, e.g., arbitrary Lagrangians leading to higher derivative field equations) in a natural manner. This is because our results depend crucially on the fact that in Eq. (11) the tensor Q^{abcd} is divergence-free. This is true for all Lanczos-Lovelock models but not for more general gravitational Lagrangians, e.g., even the $f(R)$ -type Lagrangians [except for the trivial case $f(R) \propto R$]. The fact that Lanczos-Lovelock models are special as regards thermodynamic properties has been noticed in several previous works (e.g., [13]) as well.

For the Lagrangian written in two different ways [as in Eqs. (14) and (25)], let us consider the difference between the two expressions, each of which leads to the entropy when evaluated on the horizon. Using the surface term in Eq. (14), we define

$$S_1 = mL_{\text{sur}}^{(1)} = 2m\partial_\beta[\sqrt{-g}Q_a^{b\beta d}\Gamma_{bd}^a], \quad (41)$$

and using Eq. (25),

$$S_2 = L_{\text{sur}}^{(2)} = 2\partial_\beta(\sqrt{-g}P_a^{b\beta 0}\Gamma_{0b}^a). \quad (42)$$

Both S_1 and S_2 terms give the Wald entropy when integrated over a horizon. This is because their difference

$$S_1 - S_2 = 2m\partial_\beta(\sqrt{-g}Q_a^{b\beta\alpha}\Gamma_{b\alpha}^a) \quad (43)$$

vanishes on a spherically symmetric horizon, as only the Q^{0110} component contributes on the horizon (see Sec. IV.B of Ref. [13]). The direct difference between the two surface terms ΔL_{sur} and the difference between the two corresponding bulk terms ΔL_{bulk} follow the holographic relation, Eq. (40) due to the linearity of the relation.

V. DISCUSSION AND CONCLUSIONS

The conventional approach to gravity begins from the principle of equivalence and the principle of general covariance. These are sufficient to argue that effects of gravity on matter can be given a geometrical interpretation in terms of a nontrivial metric. The (minimal) coupling of gravity to matter can be obtained by demanding the validity of special relativistic laws in local inertial frames and general covariance. Unfortunately, there is no such elegant principle to write down an action functional and obtain the dynamics of gravity. One can construct a large class of generally covariant action functionals, of which the Einstein-Hilbert action is the simplest in four dimensions.

The alternative paradigm considers gravity to be an emergent phenomenon [2,3] and obtains the field equations from thermodynamic considerations (see, e.g., [7]). Since the same field equations are obtained by both procedures, it seems reasonable to believe that the structure of action functionals used in the conventional approach must contain certain clues to the fact that gravity has an emergent description.

This expectation turns out to be true. Unlike all other theories in physics (in particular, non-Abelian gauge theories which are in some sense closest to gravity), the gravitational action—determined by the symmetries of the theory—has a nontrivial surface term. An investigation of the surface term shows that (i) it has a deep connection with the entropy of horizons that arise in certain on-shell solutions of the theory [15] and (ii) it has a holographic relation to the bulk term [9]. What is more, the existence of the surface term and its properties seems to be generic [13] and arises in a wide class of theories more general than Einstein gravity.

In the conventional approach, no intuitive explanation is available for these features, but in the thermodynamic paradigm, one can interpret the bulk term as energy and the surface term as entropy. (In fact, one can argue that [7] the action functional in any reasonable theory of gravity should contain a surface term.) The holographic relationship between the bulk and surface terms then acquires a thermodynamic interpretation and is analogous to the usual Legendre-like transformations in thermodynamics, allowing one to construct the energy functional from the entropy functional and vice versa. Some explicit constructions along these lines have already been done, demonstrating the utility of the holographic relation [18].

The extra feature not present in the conventional thermodynamics is the dimensional reduction that occurs in any holographic relation. Note that the holographic relation itself can be stated as a relation between Lagrangian densities as in Eq. (1), Eq. (9), or Eq. (15), etc. But on integrating the Lagrangian over a region \mathcal{V} to obtain the action, the L_{sur} contributes a surface term in $\partial\mathcal{V}$. Thus the dynamics of gravity, expressed through the field equations in D -dimensional space (“bulk” \mathcal{V}) is equally well en-

coded in a functional expressed in $D - 1$ -dimensional space (“boundary” $\partial\mathcal{V}$). The fundamental reason for this is the existence of horizons in gravitational theories and the need to encode information blocked by the horizons on its surface.

Our analysis has also revealed the relationship between the Noether charge density and the surface term in the action for static geometries. One can, in fact, start from the Noether charge density, construct the entropy functional, and then determine the bulk term of the action through the holographic condition. Such an approach requires some careful considerations of uniqueness, which were sorted out for the case of Einstein gravity in previous works [9,15]. We hope to address corresponding issues in the case of Lanczos-Lovelock models in a future work. Finally, it would be interesting to extend the study to cover nonstatic spacetimes, which might require extension of the ideas to situations away from thermodynamic equilibrium.

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APPENDIX A

1. Proof of Eq. (21) in Einstein’s gravity by direct calculation

We first expand the Euler derivative and write

$$\begin{aligned} & \partial_i \left[g_{ab} \frac{\delta f}{\delta(\partial_i g_{ab})} + \partial_j g_{ab} \frac{\partial f}{\partial(\partial_i \partial_j g_{ab})} \right] \\ &= \partial_i \left[g_{ab} \frac{\partial f}{\partial(\partial_i g_{ab})} - g_{ab} \partial_h \frac{\partial f}{\partial(\partial_h \partial_i g_{ab})} \right. \\ & \quad \left. + \partial_j g_{ab} \frac{\partial f}{\partial(\partial_i \partial_j g_{ab})} \right]. \end{aligned} \quad (\text{A1})$$

We prove Eq. (21) in two parts. For the first part we find the following three quantities:

$$\begin{aligned} g_{np} \frac{\partial L}{\partial(\partial_m g_{np})} &= 2\sqrt{-g} [Q_a{}^{bac} \Gamma_{bc}^m - 2Q^{nbmd} \Gamma_{ndb}], \\ -g_{np} \partial_s \frac{\partial L}{\partial(\partial_s \partial_m g_{np})} &= -2\sqrt{-g} [Q_a{}^{bac} \Gamma_{bc}^m - Q^{nbmd} \Gamma_{ndb}], \\ \partial_m g_{np} \partial_s \frac{\partial L}{\partial(\partial_s \partial_m g_{np})} &= 2\sqrt{-g} Q^{nbmd} \Gamma_{ndb}, \end{aligned}$$

where $L = L_{\text{EH}} \sqrt{-g}$ and $Q_a{}^{bcd} = \frac{1}{2}(\delta_a^c g^{bd} - \delta_a^d g^{bc})$. Adding these together we get

$$\partial_i \left[g_{ab} \frac{\delta L}{\delta(\partial_i g_{ab})} + \partial_j g_{ab} \frac{\partial L}{\partial(\partial_i \partial_j g_{ab})} \right] = 0. \quad (\text{A2})$$

For the second part, we expand $L_{\text{sur}} = 2\sqrt{-g} R_0^0$ as

$$\begin{aligned} L_{\text{sur}} &= \sqrt{-g} \partial_b g_{a0} [g^{b0} \partial_c g^{ac} + g^{ac} \partial_c g^{b0} - g^{a0} \partial_c g^{bc} \\ & \quad - g^{bc} \partial_c g^{a0}] + \sqrt{-g} [2Q^{abc0} \partial_c \partial_b g_{a0} \\ & \quad + Q^{abc0} g^{pq} \partial_b g_{a0} \partial_c g_{pq}]. \end{aligned}$$

We calculate the following quantities step by step. First, we have

$$\frac{\partial L_{\text{sur}}}{\partial(\partial_h \partial_k g_{ij})} = 2\sqrt{-g} Q^{ikh0} \delta_0^j. \quad (\text{A3})$$

Next,

$$\begin{aligned} g_{ij} \partial_h \left(\frac{\partial L_{\text{sur}}}{\partial(\partial_h \partial_k g_{ij})} \right) &= \sqrt{-g} g^{pq} Q^{ikh0} g_{i0} \partial_h g_{pq} \\ & \quad + \sqrt{-g} [g^{k0} g_{i0} \partial_h g^{ih} - g^{kh} g_{i0} \partial_h g^{i0} \\ & \quad + \partial_0 g^{k0} - \delta_0^0 \partial_h g^{kh}]. \end{aligned} \quad (\text{A4})$$

The third relation we need is

$$\partial_h g_{ij} \frac{\partial L_{\text{sur}}}{\partial(\partial_k \partial_h g_{ij})} = 2\sqrt{-g} Q^{ikh0} \partial_h g_{i0}. \quad (\text{A5})$$

Finally, we also have

$$\begin{aligned} g_{ij} \frac{\partial L_{\text{sur}}}{\partial(\partial_k g_{ij})} &= \sqrt{-g} g^{pq} Q^{ikh0} g_{i0} \partial_h g_{pq} \\ & \quad + \sqrt{-g} [g^{k0} g_{i0} \partial_h g^{ih} - g^{kh} g_{i0} \partial_h g^{i0} \\ & \quad + \partial_0 g^{k0} - \delta_0^0 \partial_h g^{kh}]. \end{aligned}$$

From these we find that

$$g_{ij} \frac{\delta L_{\text{sur}}}{\delta(\partial_k g_{ij})} = 0. \quad (\text{A6})$$

Hence,

$$\begin{aligned} & \partial_i \left[g_{ab} \frac{\delta L_{\text{sur}}}{\delta(\partial_i g_{ab})} + \partial_j g_{ab} \frac{\partial L_{\text{sur}}}{\partial(\partial_i \partial_j g_{ab})} \right] \\ &= \partial_i \left[\partial_j g_{ab} \frac{\partial L_{\text{sur}}}{\partial(\partial_i \partial_j g_{ab})} \right] = \partial_k (\sqrt{-g} 2Q^{abk0} \partial_b g_{a0}) = L_{\text{sur}}. \end{aligned} \quad (\text{A7})$$

Subtracting (A2) from (A7), we get Eq. (21) for $D = 4$.

2. Holography of Lanczos-Lovelock Lagrangian

We derive a general result which shows that there always exists a holographic relationship, as defined by Eq. (15), between any pair of bulk and surface terms of the Lanczos-Lovelock Lagrangian when the surface term is homogeneous in its variables, as described below.

Let the Lanczos-Lovelock Lagrangian $L = \sqrt{-g} L_{(m)}$ in D dimensions be written as a sum of the bulk term L_1 and a total divergence term L_2 , giving $L = L_1 + L_2$. Let the total divergence term L_2 in the Lagrangian be such that it satisfies the following homogeneity condition:

When the surface term L_2 is expanded and written in terms of g_{ab} , $\partial_c g_{ab}$, $\partial_d \partial_c g_{ab}$, a generic term in the expansion will have the form $(g_{**})^x (\partial_* \partial_* g_{**})^k (\partial_* g_{**})^y$ for some indices x, y, k . We assume that all the terms are homogeneous in degree p ; that is,

$$x + y + k = p. \quad (\text{A8})$$

We can then show that the Lanczos-Lovelock Lagrangian $L = \sqrt{-g} L_{(m)}$ in D dimensions, with this decomposition, is holographic in the sense that

$$[D/2 + p]L_2 = -\partial_i \left[g_{ab} \frac{\delta L_1}{\delta(\partial_i g_{ab})} + \partial_j g_{ab} \frac{\partial L_1}{\partial(\partial_i \partial_j g_{ab})} \right], \quad (\text{A9})$$

where L_1 is defined through $L_1 \equiv L - L_2$ and differentiation indicated by δ is the Euler derivative. The result in Eq. (A9) is a generalization of the holographic relation proved in [13] for the Lanczos-Lovelock Lagrangian written in terms of a $Q\Gamma\Gamma$ bulk term and a $\nabla(Q\Gamma)$ surface term and it uses the same technique.

The proof of Eq. (A9) is as follows. Given an L_2 satisfying the homogeneity condition, we first define

$$L_1 = \sqrt{-g} L_{(m)} - L_2 = L - L_2. \quad (\text{A10})$$

(This can be simplified and written in a compact form depending on the surface term chosen; however, for this proof the given form is useful.) Consider the quantity

$$\partial_i \left[g_{ab} \frac{\delta f}{\delta(\partial_i g_{ab})} + \partial_j g_{ab} \frac{\partial f}{\partial(\partial_i \partial_j g_{ab})} \right]. \quad (\text{A11})$$

One can show, after some manipulations, that

$$\begin{aligned} & \partial_i \left[g_{ab} \frac{\delta f}{\delta(\partial_i g_{ab})} + \partial_j g_{ab} \frac{\partial f}{\partial(\partial_i \partial_j g_{ab})} \right] \\ &= \left\{ g_{ab} \frac{\partial f}{\partial g_{ab}} + (\partial_i g_{ab}) \frac{\partial f}{\partial(\partial_i g_{ab})} + (\partial_i \partial_j g_{ab}) \frac{\partial f}{\partial(\partial_i \partial_j g_{ab})} \right\} \\ & \quad - g_{ab} \frac{\delta f}{\delta g_{ab}}. \end{aligned} \quad (\text{A12})$$

Since L_2 is a divergence, its Euler derivative identically vanishes,

$$\frac{\delta L_2}{\delta g_{ab}} = 0. \quad (\text{A13})$$

The Euler derivative, say $M^{ab}[L]$, of the Lanczos-Lovelock Lagrangian L satisfies the property that its trace is proportional to the Lagrangian itself,

$$\begin{aligned} g_{ab} \frac{\delta L}{\delta g_{ab}} &= g_{ab} M^{ab}[L] = -\sqrt{-g} g_{ab} \mathcal{G}^{ab}[L] \\ &= [D/2 - m] \sqrt{-g} L_m. \end{aligned} \quad (\text{A14})$$

We will now prove two preliminary results using the

homogeneity condition of L_2 and the natural homogeneity of L . We show that

$$\begin{aligned} & g_{ab} \frac{\partial L}{\partial g_{ab}} + (\partial_i g_{ab}) \frac{\partial L}{\partial(\partial_i g_{ab})} + (\partial_i \partial_j g_{ab}) \frac{\partial L}{\partial(\partial_i \partial_j g_{ab})} \\ &= (D/2 - m)L \end{aligned} \quad (\text{A15})$$

and

$$\begin{aligned} & g_{ab} \frac{\partial L_2}{\partial g_{ab}} + (\partial_i g_{ab}) \frac{\partial L_2}{\partial(\partial_i g_{ab})} + (\partial_i \partial_j g_{ab}) \frac{\partial L_2}{\partial(\partial_i \partial_j g_{ab})} \\ &= (D/2 + p)L_2. \end{aligned} \quad (\text{A16})$$

To prove these relations, consider any generic term $f^{(k)}$ (here k is a label) which arises when f is expanded in terms of g_{ab} , $\partial_c g_{ab}$, $\partial_d \partial_c g_{ab}$, where k is the degree of $\partial_d \partial_c g_{ab}$. Here f is a dummy scalar, and we will later put $f = L$ or $f = L_2$ according to our need. Let x and y be the degree of g_{ab} and $\partial_c g_{ab}$ in the term $f^{(k)}$. Hence, by definition,

$$\begin{aligned} g_{ab} \frac{\partial f^{(k)}}{\partial g_{ab}} &= (D/2 + x)L^{(k)}, & (\partial_i g_{ab}) \frac{\partial f^{(k)}}{\partial(\partial_i g_{ab})} &= yf^{(k)}, \\ (\partial_i \partial_j g_{ab}) \frac{\partial f^{(k)}}{\partial(\partial_i \partial_j g_{ab})} &= kf^{(k)}. \end{aligned} \quad (\text{A17})$$

The $D/2$ factor arises due to $\sqrt{-g}$. Adding the three, we get

$$\begin{aligned} & \left\{ g_{ab} \frac{\partial f^{(k)}}{\partial g_{ab}} + (\partial_i g_{ab}) \frac{\partial f^{(k)}}{\partial(\partial_i g_{ab})} + (\partial_i \partial_j g_{ab}) \frac{\partial f^{(k)}}{\partial(\partial_i \partial_j g_{ab})} \right\} \\ &= (D/2 + (x + y + k))f^{(k)}. \end{aligned} \quad (\text{A18})$$

When $f^{(k)} = L_2^{(k)}$, the homogeneity condition on L_2 tells us $x + y + k = p$, which is independent of the k th term and hence true for any generic term in L_2 . Hence the above expression is valid for L_2 and leads us to Eq. (A16).

The Lanczos-Lovelock Lagrangian $L = \sqrt{-g} L_{(m)}$ also satisfies the homogeneity condition naturally with the degree $p = -m$. To see this, we note that the scalar L is made up of the metric, and its first and second derivatives; hence the upper indices must be equal to the number of lower indices in any generic term of L , giving us the relation $2(-x) = 3y + 4k$. This fixes the number of g_{ab} in terms of $\partial_i g_{ab}$ and $\partial_i \partial_j g_{ab}$. Since the Lanczos-Lovelock Lagrangian L is made up of a product of m curvature tensors $R \sim \partial^2 g + (\partial g)^2$, the $L^{(k)}$ term will have k factors of $\partial^2 g$ and $(m - k)$ factors of $(\partial g)^2$; that is, $y = 2(m - k)$ and hence $(-x) = 3m - k$. This gives $x + y + k = -m$, which is again independent of the k th term and hence leads us to Eq. (A15).

Using Eqs. (A14) and (A15) in Eq. (A12), we get

$$\partial_i \left[g_{ab} \frac{\delta L}{\delta(\partial_i g_{ab})} + \partial_j g_{ab} \frac{\partial L}{\partial(\partial_i \partial_j g_{ab})} \right] = 0. \quad (\text{A19})$$

Using Eqs. (A13) and (A16) in Eq. (A12), we get

$$[D/2 + p]L_2 = \partial_i \left[g_{ab} \frac{\delta L_2}{\delta(\partial_i g_{ab})} + \partial_j g_{ab} \frac{\partial L_2}{\partial(\partial_i \partial_j g_{ab})} \right]. \quad (\text{A20})$$

Writing L_2 in the right-hand side of the above equation as $L_2 = L - L_1$ and using Eq. (A19), we get the holographic relation of Eq. (A9),

$$[D/2 + p]L_2 = -\partial_i \left[g_{ab} \frac{\delta L_1}{\delta(\partial_i g_{ab})} + \partial_j g_{ab} \frac{\partial L_1}{\partial(\partial_i \partial_j g_{ab})} \right]. \quad (\text{A21})$$

One can now check that the surface terms we discussed satisfy the homogeneity condition. (i) Consider the form of L_m where $L_{\text{sur}} = \sqrt{-g} \nabla_a J^{0a}$ in Eq. (25). Expanding $\sqrt{-g} \nabla_a J^{0a}$ in terms of the metric and its derivatives, one can easily see that it satisfies the homogeneity condition with $p = -m$. (ii) $L_{\text{sur}} = 2\partial_c [\sqrt{-g} Q_a{}^{bcd} \Gamma_{bd}^a]$ in Eq. (14) is homogeneous with the degree $p = -m$. Hence the holographic relationship follows.

APPENDIX B

1. Variation of the Lanczos-Lovelock Lagrangian

The variation of the quantity $L\sqrt{-g}$, where L is the Lanczos-Lovelock Lagrangian, can be expressed as

$$\begin{aligned} \delta(L\sqrt{-g}) &= \left(\frac{\partial L\sqrt{-g}}{\partial g^{ab}} \right) \delta g^{ab} + \left(\frac{\partial L\sqrt{-g}}{\partial R_{bcd}^a} \right) \delta R_{bcd}^a \\ &= \left(\frac{\partial L\sqrt{-g}}{\partial g^{ab}} \right) \delta g^{ab} + \sqrt{-g} P_a{}^{bcd} \delta R_{bcd}^a. \end{aligned} \quad (\text{B1})$$

The term $P_a{}^{bcd} \delta R_{bcd}^a$ is generally covariant and hence can be evaluated in the local inertial frame using

$$\begin{aligned} \delta R_{bcd}^a &= \nabla_c (\delta \Gamma_{db}^a) - \nabla_d (\delta \Gamma_{cb}^a) \\ &= \frac{1}{2} \nabla_c [g^{ai} (-\nabla_i \delta g_{db} + \nabla_d \delta g_{bi} + \nabla_b \delta g_{di})] \\ &\quad - \{\text{term with } c \leftrightarrow d\}. \end{aligned} \quad (\text{B2})$$

Multiplying this expression by $P_a{}^{bcd}$, the middle term $g^{ai} \nabla_d \delta g_{bi}$ will not contribute because of the antisymmetry of P^{ibcd} in i and b . The other two terms will contribute equally. We will get a similar contribution from the term with c and d interchanged. Thus

$$P_a{}^{bcd} \delta R_{bcd}^a = 2\nabla_c [P^{ibcd} \nabla_b (\delta g_{di})] = 2\nabla_c [P_i{}^{bcd} \delta \Gamma_{bd}^i]. \quad (\text{B3})$$

To find $\left(\frac{\partial L\sqrt{-g}}{\partial g^{ab}} \right)$, we write

$$\begin{aligned} \left(\frac{\partial L\sqrt{-g}}{\partial g^{ab}} \right) &= \left(\frac{\partial L}{\partial R_{ij}^{kl}} \frac{\partial R_{ij}^{kl}}{\partial g^{ab}} - \frac{1}{2} g_{ab} L \right) \\ &= \left(P_{kb}^{ij} R_{aij}^k - \frac{1}{2} g_{ab} L \right) \\ &= \left(P_b{}^{kij} R_{akij} - \frac{1}{2} g_{ab} L \right), \end{aligned} \quad (\text{B4})$$

where in arriving at the first inequality we have used the fact that while differentiating $R_{ij}^{kl} = g^{lm} R_{mij}^k$, we should keep R_{mij}^k fixed. Hence we get

$$\begin{aligned} \delta(L\sqrt{-g}) &= (P_b{}^{kij} R_{akij} - \frac{1}{2} g_{ab} L) \delta g_{ab} \\ &\quad + \sqrt{-g} \nabla_j [2P_i{}^{bjd} \delta \Gamma_{bd}^i] \\ &= \sqrt{-g} (\mathcal{G}_{ab} \delta g^{ab} + \nabla_a \delta v^a). \end{aligned} \quad (\text{B5})$$

2. Variation of the $Q\Gamma\Gamma$ term

From Eq. (14), we write

$$\begin{aligned} L_{\text{bulk}} &= 2\sqrt{-g} Q_a{}^{bcd} \Gamma_{dk}^a \Gamma_{bc}^k \\ &= \sqrt{-g} L - 2\partial_c [\sqrt{-g} Q_a{}^{abcd} \Gamma_{bd}^a]. \end{aligned} \quad (\text{B6})$$

Hence, we get

$$\delta L_{\text{bulk}} = \delta(L\sqrt{-g}) - \delta L_{\text{sur}}. \quad (\text{B7})$$

Using the definition of L_{sur} from Eq. (14), we find that

$$\delta L_{\text{sur}} = 2Q_{ak}^{cd} \partial_c [\sqrt{-g} g^{bk} \delta \Gamma_{bd}^a + \Gamma_{bd}^a \delta(\sqrt{-g} g^{bk})]. \quad (\text{B8})$$

Using

$$\begin{aligned} \delta(\sqrt{-g} g^{bk}) &= \sqrt{-g} [\delta_l^b \delta_m^k - \frac{1}{2} g^{bk} g_{lm}] \delta g^{lm} \\ &= \sqrt{-g} B_{lm}^{bk} \delta g^{lm}, \end{aligned} \quad (\text{B9})$$

where the last equality defines B_{lm}^{bk} , we can write the second term in Eq. (B8) as

$$\begin{aligned} 2Q_{ak}^{cd} \partial_c [\Gamma_{bd}^a \delta(\sqrt{-g} g^{bk})] &= 2Q_{ak}^{cd} \partial_c [\sqrt{-g} \Gamma_{bd}^a B_{lm}^{bk} \delta g^{lm}] \\ &\equiv \partial_c [\sqrt{-g} M_{lm}^c \delta g^{lm}], \end{aligned} \quad (\text{B10})$$

where we have defined the three-index nontensorial object

$$M_{lm}^c = 2Q_{ak}^{cd} \Gamma_{bd}^a B_{lm}^{bk} = \Gamma_{lm}^c - \Gamma_{ld}^c \delta_m^d + \frac{g_{lm}}{2g} \partial_b (g^{bc}). \quad (\text{B11})$$

Hence,

$$\delta L_{\text{sur}} = \partial_c [2\sqrt{-g} g^{bk} Q_{ak}^{cd} \delta \Gamma_{bd}^a + \sqrt{-g} M_{lm}^c \delta g^{lm}]. \quad (\text{B12})$$

Using Eqs. (B5) and (B12) in Eq. (B7), we get

$$\begin{aligned} \delta L_{\text{bulk}} &= \sqrt{-g} \mathcal{G}_{ab} \delta g^{ab} + (m-1) \partial_c [2\sqrt{-g} g^{bk} Q_{ak}^{cd} \delta \Gamma_{bd}^a] \\ &\quad - \partial_c [\sqrt{-g} M_{lm}^c \delta g^{lm}]. \end{aligned} \quad (\text{B13})$$

Note that in the Einstein gravity case $m = 1$, the second

term vanishes and we are only required to fix the metric at the boundary. However, in the general case one has to fix the metric as well as the normal derivative of the metric at the boundary.

3. Variation of $-2\sqrt{-g}\mathcal{G}_0^0$

We write

$$\delta(-2\sqrt{-g}\mathcal{G}_0^0) = \delta(L\sqrt{-g}) - \delta(2\sqrt{-g}\mathcal{R}_0^0). \quad (\text{B14})$$

Using the definition of $2\mathcal{R}_0^0$ for static spacetime in Eq. (25),

$$2\sqrt{-g}\mathcal{R}_0^0 = 2m\partial_c(\sqrt{-g}Q_a{}^{bc0}\Gamma_{0b}^a), \quad (\text{B15})$$

we proceed in a manner similar to Appendix B 2. We then find

$$\begin{aligned} \delta(2\sqrt{-g}\mathcal{R}_0^0) &= m\partial_c[2\sqrt{-g}g^{bk}Q_{ak}{}^{c0}\delta\Gamma_{b0}^a \\ &\quad + \sqrt{-g}M^c{}_{lm}\delta g^{lm}], \end{aligned} \quad (\text{B16})$$

where now the three-index nontensorial object is defined as

$$M^c{}_{lm} = 2Q_{ak}{}^{c0}\Gamma_{b0}^a B_{lm}^{bk}.$$

Note that this is different from the second equality in Eq. (B11). From these and Eq. (B5), we get

$$\begin{aligned} \delta(-2\sqrt{-g}\mathcal{G}_0^0) &= \sqrt{-g}\mathcal{G}_{ab}\delta g^{ab} \\ &\quad + m\partial_c[2\sqrt{-g}g^{bk}Q_{ak}{}^{c0}\delta\Gamma_{b\alpha}^a] \\ &\quad - m\partial_c[\sqrt{-g}M^c{}_{lm}\delta g^{lm}]. \end{aligned} \quad (\text{B17})$$

Unlike the situation in Appendix B 2, the second term in the above equation does not vanish trivially even for the Einstein gravity which corresponds to $m = 1$. However, note that the entire analysis is relevant only in the context of static spacetimes, and hence we restrict our variations to metrics which are static; that is, we choose a coordinate system in which the Killing vector has the components $\xi^a = (1, \mathbf{0})$ and consider variations of the form $g^{ab}(\mathbf{x}) \rightarrow g^{ab}(\mathbf{x}) + \delta g^{ab}(\mathbf{x})$. Then only spatial derivatives survive on the boundaries, and the variation is well defined, leading to the static Einstein equations.

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