

LETTER TO THE EDITOR

Limitations on the operational definition of spacetime events and quantum gravity

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Abstract. Using simple arguments from general relativity and quantum theory we show that it is not possible to devise experiments (or operational procedures) which will measure the position of a particle to an accuracy better than the Planck length ($G\hbar/c^3 \approx 10^{-33}$ cm). It is also impossible to synchronise clocks to a precision better than Planck time. The implications of the result are discussed.

Classical general relativity treats spacetime as a four-dimensional differential manifold. Coordinate labels and clock readings can be introduced into this manifold using rigid rods and clocks. In physically reasonable spacetimes, it is also possible to synchronise the clocks by light signals. It is the existence of these operational procedures which allows one to attribute a unique quadruple of coordinates x^i to any event of the spacetime manifold (Misner *et al* 1973).

There is no reason to expect such operational procedures to be valid when quantum theory is introduced. Indeed, it can be shown, using a simple model of quantum gravity, that the Planck length L_p ($\sim 10^{-33}$ cm) acts as a lower bound on all proper lengths measured in any spacetime manifold (Padmanabhan 1985a, b). This 'zero point length' originates due to the quantum fluctuations of the gravitational field. If x and y are two events on a spacelike hypersurface Σ , then the mean square value of the geodesic distance between x and y (averaged over the metric fluctuations) can be shown to be

$$\langle \sigma^2(\mathbf{x}, \mathbf{y}) \rangle \approx \sigma_0^2(\mathbf{x}, \mathbf{y}) + L_p^2 \quad (1)$$

when x and y are sufficiently close. (Here $\sigma_0(\mathbf{x}, \mathbf{y})$ is the classical value of the proper distance between x and y .) This result suggests that no experiment can be devised which will measure distances to a precision better than L_p .

We shall show in this letter that such is indeed the case. We shall consider simple thought experiments and demonstrate how quantum mechanics and gravity conspire together to prevent observations at scales lower than L_p . The crucial factor which is introduced into the discussion by gravity is the following: an amount of energy E cannot be confined to a region smaller than $(2GE/c^2)$ without the formation of an event horizon. This principle, combined with the standard results of quantum mechanics, will help us to demonstrate our claim.

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It must be noted that the present letter does not use any explicit model for quantum gravity (in contrast with the previous works in which (1) was derived). Elementary considerations based on general relativity and quantum theory will be all that we require. This makes the result self-contained and compelling.

The result also illustrates a hierarchy of limitations on measurability. In classical physics all observables are measurable with arbitrary accuracy. In combining non-relativistic physics with quantum theory we lose the simultaneous measurability of two non-commuting variables. Special relativity added to quantum theory prevents localisation of massive particles to regions smaller than the Compton wavelength. As we shall argue below, combining gravity with quantum theory prevents even the measurement of a single event with arbitrary accuracy.

The limitations on the determination of the coordinates of an event, considered here, has interesting parallels with the classic work of DeWitt on the quantum measurability of the gravitational field (DeWitt 1962a, b). Our results are consistent with his conclusions and complements that analysis.

The plan of the letter is as follows. First we consider the measurement of position. We discuss two particular examples: that of a particle bound to a region R and that of a 'Heisenberg microscope'. Next we consider limitations on the synchronisation of clocks by light signals and, finally, we summarise the letter and offer some further speculations.

Several operational procedures usually used for introducing coordinates on a manifold require the concept of a 'bound particle'. By 'bound particle' we mean a particle which—by the use of suitable forces and potentials—is confined within a region of size R . We now ask: can R be made arbitrarily small?

It is easy to see that R is bounded from below by L_p . Consider a particle of mass m bound in a region of size R . The uncertainty principle implies that the energy E of such a particle should be

$$E = (|\mathbf{p}|^2 + m^2)^{1/2} \geq |\mathbf{p}| \geq 1/R. \quad (2)$$

This energy will produce a gravitational field around it. Assuming that the distribution is spherically symmetric (at least, on the average) the gravitational field will be described by a Schwarzschild metric with the metric component:

$$g_{00}(r) = (1 - 2GE/r). \quad (3)$$

Since we do not want the particle to be surrounded by an event horizon we must have $g_{00}(r=R) > 0$, or

$$R > 2GE. \quad (4)$$

Combining with (2) we obtain

$$R > 2GE \geq 2G/R \quad (5)$$

so that

$$R > G^{1/2} = L_p. \quad (6)$$

Thus, if particles kept at various locations are used as coordinate markers, then such a coordinate frame can never achieve an accuracy better than L_p . Neither can rigid rods have lengths specified to an accuracy greater than L_p . The simple argument outlined in (2)–(6) above takes care of a large number of practical situations.

It is conceivable that coordinate marking can be achieved without actually using bound particles. One can use streams of free particles and use (possibly) their intersections to mark the coordinates. We shall now show that it is not possible to measure (with arbitrary accuracy) the position of the free particle either.

The position of a free particle can be measured by an arrangement usually called the 'Heisenberg microscope'. A photon (or any suitable high-energy particle) is scattered off the free particle (the position of which we are trying to measure) into a microscope and observed. Other methods can, of course, be devised but this is as good as any (Heisenberg 1925).

The uncertainty originates for the following reason. To improve the accuracy of observation one has to use a very small wavelength (high-frequency) photon. Such a photon *produces* a gravitational field which acts on the particle during the time of scattering and disturbs its position. It turns out that this disturbance cannot be minimised arbitrarily without sacrificing the resolution.

Consider a photon of frequency ν (i.e. the frequency is ν when the photon is far away) which scatters off a particle of mass m (the position of which we are trying to measure). We choose the coordinates such that after the scattering the photon travels roughly along the y axis and is detected by a microscope. The microscope cannot measure the direction of the scattering photon with arbitrary accuracy because of aperture limitations. Let us suppose that the direction of the photon was spread over an angle ε around the y axis. Then, because of the finite resolution of the microscope, the (transverse) position of the particle is known only to an accuracy

$$\Delta x \geq 1/\nu \sin \varepsilon. \quad (7)$$

Let us suppose that the interaction between the photon and the particle took place in a region of size r , over a timescale $r/c = r$ (since $c = 1$). When the photon is in the vicinity of the particle (after the scattering), its momentum will be

$$p \approx \nu + Gm\nu/r. \quad (8)$$

The direction of this momentum is unknown by the angle ε giving an uncertainty in the transverse component of the momentum to be

$$\Delta p_x \approx p \sin \varepsilon \approx (\nu + Gm\nu/r) \sin \varepsilon. \quad (9)$$

By momentum conservation, the x component of the momentum of *the particle* will be uncertain by the same amount, leading to an uncertainty in the velocity of the particle:

$$\Delta v_x = \frac{\Delta p_x}{m} \approx \left(\frac{\nu}{m} + \frac{G\nu}{r} \right) \sin \varepsilon. \quad (10)$$

This uncertainty in the velocity persists (at least) during the time when the photon is in the vicinity of the particle leading to an uncertainty in the position:

$$\Delta x \geq r \Delta v_x \approx \left(\frac{\nu r}{m} + G\nu \right) \sin \varepsilon \geq G\nu \sin \varepsilon. \quad (11)$$

Combining (11) and (7) we immediately obtain

$$(\Delta x)^2 \geq \frac{1}{\nu \sin \varepsilon} G\nu \sin \varepsilon = G = L_p^2. \quad (12)$$

The result (12) can be obtained in other ways as well. The photon exerts a gravitational acceleration ($G\nu/r^2$) on the particle for a time interval r , leading to a

displacement $(G\nu/r^2)r^2 \sim G\nu$ in the direction of the scattered photon (loosely speaking, the photon 'drags' the particle along with it). The transverse component of this displacement is $G\nu \sin \epsilon$.

A much more sophisticated way of deriving (12) would involve following the trajectory of the particle in the metric produced by a photon (which can be obtained as a limiting case of a relativistic particle). This rigorous analysis reproduces the same result after complicated algebra.

The limits in (11) and (12) are special to gravity and depend on the principle of equivalence in a rather subtle way. If the gravitational and inertial masses, m_g and m_i , were unequal (or if we were using 'electromagnetism' rather than gravity in our interactions) then the lower bound would have been $L_p^2(m_g/m_i)$. By using particles with $m_g \ll m_i$ we could have achieved arbitrarily high accuracy in position measurement. This is why non-gravitational interactions do not lead to any result similar to (12).

Now we consider the measurement of time. A test clock is usually synchronised with respect to a standard clock by passing light signals back and forth between them (see, e.g., Landau and Lifshitz 1975, p 84). If the position of one of the clocks is uncertain (due to the considerations presented above) then we will automatically introduce a corresponding error in the synchronisation.

There is, however, another way of visualising this uncertainty. To transmit light beams of finite duration, we need to superpose light waves of different frequencies. Suppose the light beams were of mean frequency ν and width $\Delta\nu$ (or equivalently the photon energy ν was known only with an accuracy of $\Delta\nu$). If the time of (emission or) return of the photon is uncertain by an amount Δt , then the reading T of the test clock is also uncertain by $\Delta T \geq \Delta t$. But from the time-energy uncertainty principle, we must have

$$\Delta T \approx \Delta t \geq 1/\Delta\nu > 1/\nu \quad (13)$$

since $\Delta\nu \ll \nu$. If the clock and the photon were interacting strongly in a region of size R for a time $t \approx R$, then the clock would have been in the gravitational field of the photon during this time and would have picked up a gravitational time delay

$$T_{\text{grav}} \approx \tau \sqrt{g_{00}} \approx R(1 - 2G\nu/R)^{1/2}. \quad (14)$$

This time delay can be easily accommodated into the calculation if ν is known with arbitrary accuracy. However, since ν is uncertain by a factor $\Delta\nu$, T_{grav} is uncertain by the amount

$$\Delta T \approx (1 - 2G\nu/R)^{-1/2} G\Delta\nu \geq G\Delta\nu. \quad (15)$$

Combining (13) and (15) we obtain

$$(\Delta T)^2 \geq G = L_p^2. \quad (16)$$

The same result can be re-expressed in the operator language of quantum mechanics if the gravitational field is weak. Consider any physical system of size R described by a Hamiltonian operator \hat{H} . This system may be called a 'clock' if there exists a dynamical operator \hat{T} with the commutation relation

$$[\hat{T}, \hat{H}] = i. \quad (17)$$

(If (17) is satisfied, then $i\hat{T} = [\hat{T}, \hat{H}] = i$ allowing \hat{T} to measure the *local proper time*.) This operator \hat{T} , for example, could be the position of a pointer on a dial. The time

measured by \hat{T} will differ from the coordinate time (or the time measured by 'clocks at infinity') by a factor

$$F = \sqrt{g_{00}} \approx 1 - G\langle\hat{H}\rangle/R. \quad (18)$$

Let us define an operator \hat{F} by

$$\hat{F} \equiv 1 - G\hat{H}/R \quad (19)$$

so that

$$[\hat{T}, \hat{F}] = -iG/R. \quad (20)$$

The time required to make the clock synchronised is at least of the order of R . During this interval the proper time will change by at least

$$\tau = R\langle\hat{F}\rangle \quad (21)$$

to which we can associate an operator $\hat{\tau}$:

$$\hat{\tau} \equiv R\hat{F} = R(1 - G\hat{H}/R). \quad (22)$$

From (20) and (22) we obtain

$$[\hat{T}, \hat{T} + \hat{\tau}] = -iG. \quad (23)$$

Note that $\langle\hat{T}\rangle$ would have been the expected reading of the clock at the beginning of measurement and $\langle\hat{T} + \hat{\tau}\rangle$ would have been the expected reading of the clock at the end of the measurement. The commutation relation (23) implies that

$$\Delta T \Delta(T + \tau) \geq G = L_p^2. \quad (24)$$

Thus the product of dispersions in the initial and final readings of the clock will be of the order of L_p^2 . Since synchronisation require exact knowledge of both initial and final readings, it follows immediately that clocks cannot be synchronised to arbitrary accuracy.

The arguments on position measurement and those here can be polished and improved in many ways. However, the essential physics should be clear from the above analysis. The combination of the uncertainty principle and gravity prevents spacetime events from being well defined.

In these arguments we have treated gravity as a classical field. This attitude is comparable to the derivation of the uncertainty principle in the pre-quantum theory days. Using simple ideas of wave-particle duality one can analyse various thought experiments and demonstrate the validity of the uncertainty principle. Similarly, simple application of quantum theory and classical gravity helps one to study various thought experiments and show that scales below L_p are not measurable. The burden of quantum theory was to provide a theoretical framework from which these results followed automatically. Similarly, the burden of quantum gravity will be to produce a theoretical formalism in which these results follow naturally. One such model involves quantisation of the conformal degree of freedom of gravity (see, e.g., Narlikar and Padmanabhan 1985). Various other theoretical structures are possible, some of which are discussed below.

In the usual formulations of physics, it is always assumed that the coordinates x^i have precise operational meaning. One *possible* way of relaxing this assumption would be as follows.

Consider any particular event E in the spacetime. An observer O is trying to attribute certain coordinates to the event E . It is clear from the discussion above that he will not be able to attribute a unique quadruple x^i to the event E . Every attempt to associate coordinates with E will give a different value $X_1^i, X_2^i, \dots, X_N^i$. Taking each attempt to be independent, we can associate a probability distribution for these values. Suppose we denote by x^i the mean of the distribution. For the best possible measurements, the dispersion of this distribution will be of the order of L_p^2 . It follows from the central limit theorem that one can approximate the distribution to be a Gaussian. In other words, we can associate with *each* event E and an *observer* O , a Gaussian distribution function $G(X, x)$

$$G(X, x) = (2\pi L_p^2)^2 \exp\left(-\frac{1}{2L_p^2}[(T-t)^2 + |\mathbf{X} - \mathbf{x}|^2]\right). \quad (25)$$

This distribution gives the probability that an observer O will associate the coordinates (T, \mathbf{X}) with the event E . As the limit $L_p^2 \rightarrow 0$ is taken, the event E obtains a unique set of coordinates $\langle T \rangle = t, \langle \mathbf{X} \rangle = \mathbf{x}$.

Several points should be stressed about $G(X^i, x^i)$. To begin with, G gives the probability that an event E (labelled classically by x^i) will be attributed a set of coordinates X^i . No provision is made regarding the nature of the coordinates X^i and x^i ; in particular there is no need for them to be inertial, orthogonal coordinates, etc. Of course, one may have to restrict the range and change the normalisation of G in (25) suitably if generalised coordinates are used.

Secondly, X^i in (25) should be looked upon as a set of four real numbers (results of measurement) and x^i should be interpreted simply as the mean values $\langle X^i \rangle$. They are just real numbers obtained as a result of measurements. Conventional coordinate transformations between observers should relate the mean values $\langle X^i \rangle$. Since *no* observer can measure the positions (or set up a coordinate system) with an accuracy better than L_p^2 , the distribution in (25) is valid for all observers with a suitable transformation of the mean $\langle X^i \rangle$.

(In other words, (25) is trivially Lorentz invariant. One should *not* think of X^i and x^i to be events in the same physical manifold and demand a minus sign in the argument of the exponential function. Equation (25) does—and should—have $[(T-t)^2 + |\mathbf{X} - \mathbf{x}|^2]$ rather than $[(T-t)^2 - |\mathbf{X} - \mathbf{x}|^2]$. This is in no way contradictory with the normal ideas of Lorentz invariance which should only be applied to $\langle T \rangle = t$ and $\langle \mathbf{X} \rangle = \mathbf{x}$.)

Finally, a formalism of physical laws incorporating a random element *may* make field theory finite. Consider, for example, a scalar field $\varphi(t, \mathbf{x})$. To specify it uniquely would require (i) a measurement of the value of φ and (ii) exact knowledge of the event (t, \mathbf{x}) at which the field is measured. Conventional field theory imposes limitations on achieving (i); we shall ignore these limitations for a moment. Our discussion above shows that *even* (ii) cannot be achieved. Therefore, we are forced to deal with the expectation values

$$\bar{\varphi}(t, \mathbf{x}) \equiv \int d^4X G(X, x) \varphi(X). \quad (26)$$

In (26), $\varphi(X)$ should be treated as a specific function of X^i determined by some rule: for example, it could be demanded that $\varphi(X)$ satisfies the Klein-Gordon equation. As a result of the averaging in (26), the high-frequency modes will be eliminated from $\bar{\varphi}(x)$. For example, a Fourier mode $\exp(i\mathbf{k} \cdot \mathbf{X})$ in φ will be replaced by $\exp(-L_p^2 k^2 + i\mathbf{k} \cdot \mathbf{x})$ in $\bar{\varphi}$.

The major problem with the above arguments is that the prescriptions in both (25) and (26) are rather *ad hoc*. Possibly a more general mathematical framework is needed to eliminate this, but it is very likely that any framework which incorporates a fundamental limitation on measurements will lead to results similar to that described above.

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