

## CHAPTER 2.6

# QUANTUM CONFORMAL COSMOLOGY

**Jayant V. Narlikar**

Inter-University Centre for Astronomy and Astrophysics  
Ganeshkhind, PUNE-411007, India

Developments in general relativity, astrophysics and quantum theory:  
A jubilee volume in honour of Nathan Rosen  
Eds. F. Cooperstock, L.P. Horwitz and J. Rosen

Annals of the Israel Physical Society, V. 9  
© Israel Physical Society 1990

# QUANTUM CONFORMAL COSMOLOGY

Jayant V. Narlikar

Inter-University Centre for Astronomy and Astrophysics  
Ganeshkhind, PUNE-411007

## ABSTRACT

This article discusses an approach to quantum gravity via path integrals. The more conventional approaches look upon general relativity as a field theory. It is argued here that relativity in fact seeks to replace the standard notion of force fields by the concept of non-Euclidean geometry. Any quantization program should take due note of this fundamental aspect. Here it is shown that considerable insight can be gained into the complexities of quantum gravity by a more limited but exact approach that quantizes the conformal degree of freedom only. It is shown, for example, that the likelihood of the universe having originated from a spacetime singularity is vanishingly small. Other interesting consequences of the approach are briefly described.

## 1. INTRODUCTION

Despite numerous efforts, a fully covariant and self consistent theory of quantum gravity continues to be elusive. That such a theory has to exist is not denied by anybody. Like all other basic interactions of physics, gravity must have a quantum description. It may well be that the difficulties encountered today in providing such a description spring from a lack of understanding of gravity at a micro-level. Even assuming that at the macro-level Einstein's general theory of relativity provides an adequate description of the phenomenon of gravity, there are several question marks that still remain, e.g.:

(i) All experimental tests of general relativity relate to the weak field limit of the theory, which essentially test it in a "static" form. Phenomena, like gravitational radiation, which bring out the "dynamic" character of the theory still remain untested. Experimentally, therefore, we are no better off in relation to gravity than our predecessors studying electro-dynamics two centuries ago. So the question is: Do we have a clear idea of what a dynamical

theory of gravity should look like?

(ii) It may be argued that cosmological observations like the expansion of the universe, the microwave background and abundances of light nuclei provide ample demonstration of the validity of general relativistic models of dynamic nature. To those who have actually worked in the field of extragalactic observations the reality may be not so clearcut. There are phenomena that question the universal applicability of the cosmological interpretation of redshifts as arising from the expansion of the universe.<sup>1</sup> There are problems concerning the relic interpretation of the microwave background arising from its spectrum<sup>2</sup> and isotropy<sup>3</sup> and alternative interpretations cannot be ruled out. Likewise, the problem of light nuclei cannot be considered as solved by the standard big bang interpretation.<sup>4-6</sup>

(iii) Do we have experimental/observational evidence on the behavior of strong gravitational fields? An honest answer to this question is "no," despite the popularity of the black hole with high energy astrophysicists.

(iv) In classical general relativity the notion of gravity as a force was done away with. Instead, all the effects of gravity are described in that theory through the curvature of space and time. It is therefore against the spirit of the theory to treat the metric or the Ricci tensor as fields in the ordinary way. This difference is clear when one contrasts general relativity with electrodynamics. In the latter theory the electromagnetic forces are described by fields in a flat spacetime background. In the former theory there is no such description of gravity as a force describable by fields dynamically uncoupled from the ambient spacetime.

This is perhaps the reason why most approaches towards quantization of gravity based on treating gravity as a field theory in general relativity run into self created quagmires of intricate formalism with no emergence of a physically interpretable result. Attempts to graft techniques that have worked elsewhere in particle physics onto general relativity cannot be expected to succeed if the point (iv) above is ignored.

It is possible to proceed with the quantization program by divesting oneself of the notion that general relativity is a field theory like any other theory in particle physics. Instead, it is to be recognized for what it was originally conceived to be, a dynamical theory of spacetime and matter with the geometry of the former having a direct coupling to the stress-energy of the latter. Can one make progress with quantization on this basis alone?

This paper deals with the above question and ends with a possibly affirmative answer. We begin by describing a general technique based on path integrals. We will then specialize to the so called conformal quantization. Finally, we will apply the result to cosmology.

## 2. THE PATH INTEGRAL APPROACH

The classical theory of general relativity is described by the Hilbert action

$$S = - \frac{c^3}{16\pi G} \int_V R \sqrt{-g} d^4v - \int_V \mathfrak{S}_m \sqrt{-g} d^4v = S_g + S_m, \quad (1)$$

with the gravitational equations given by

$$\frac{\delta S}{\delta g_{ik}} = 0. \quad (2)$$

In the above formulae, we have used  $g_{ik}$  for the metric tensor corresponding to the coordinates  $x^i$  ( $i=0,1,2,3$ ;  $i=0$  timelike and  $i=1,2,3$  spacelike), and  $g = \det \|g_{ik}\|$ . The signature is  $(+, -, -, -)$ ;  $R$  = curvature scalar and  $\mathfrak{S}_m$  = scalar Lagrangian density for any matter present in the spacetime volume  $v$ .  $S_g$ , the first term of (1), contains the dynamics of the geometry of spacetime.

Following the work of Isenberg and Wheeler,<sup>7</sup> we may formulate the geometro-dynamic problem as follows. Let  $(\Sigma)$  denote a sequence of spacelike hypersurfaces on which the above geometry of spacetime induces a sequence of 3-geometries  $G^3$ . To initialize the problem we need to specify the conformal part of  $G^3$  (of this particular set) on a particular  $\Sigma$  along with the trace  $K$  of the extrinsic curvature tensor  $K_{\mu\lambda}$ . The equations (2) then determine the 3+1 geometry across the entire spacetime.

This classical problem must have a quantum counterpart. Formally we may state it with the help of the Feynman path integral formulation. Thus, instead of everything being specified on one initial surface  $\Sigma_i$ , we specify partial data on  $\Sigma_i$  as well as a final hypersurface  $\Sigma_f$ , and ask for the probability amplitude that the system evolves from the data specified on  $\Sigma_i$  to the data specified on  $\Sigma_f$ :

$$K[\Sigma_f; \Sigma_i] = \sum_{\Gamma} \exp(iS/\hbar), \quad (3)$$

where  $\Gamma$  is a typical "path" of evolution of the system.

If we ignore the matter terms for the present and concentrate on geometry alone, the above sum is over the sequences of 3-geometries from  $\Sigma_i$  to  $\Sigma_f$ . This may be formally written as the path integral

$$K[\Sigma_f; \Sigma_i] = \int \exp(iS[\Gamma]/\hbar) D\Gamma. \quad (4)$$

This statement is the quantum geometro-dynamic version of the classical equations (2).

From this technically compact statement it is not easy to extract any practical information. Indeed, attempts to do so bring in several difficulties. What is the measure of the 3-geometries that goes into the above path integral? What meaning can be attached to the "wave functionals" of spacetime geometry on which the propagator supposedly acts? How

are causal relationships between matter-matter interactions preserved when the underlying geometries undergo quantum transitions? What are the essential degrees of freedom in the geometry that are to be quantized?

Notice that these difficulties remain even after we have taken care not to bring in "fields in background spacetimes" which have the additional difficulties mentioned earlier. A full theory of quantum gravity based on classical general relativity must answer all these questions.

### 3. CONFORMAL QUANTIZATION

These problems can be resolved if we lower our ambitions and begin with a limited scope for quantization. Consider the typical conformal transformation

$$g_{ik} = (1 + \phi)^2 g_{ik}, \quad (5)$$

where  $\phi$  denotes the "conformal fluctuation" from the original metric. If we write

$$g_{ik} = (-g)^{1/4} h_{ik}, \quad (6)$$

then, under the transformation (5), we have

$$(-\tilde{g})^{1/4} = (1 + \phi)^2 (-g)^{1/4} \quad (7)$$

and

$$\tilde{h}_{ik} = h_{ik}. \quad (8)$$

Thus we may look upon  $h_{ik}$  as the part of the total metric that remains unaltered under conformal transformations. Of the six implicit degrees of freedom present in  $g_{ik}$ , five rest in the  $h_{ik}$  while the remaining one is the conformal degree of freedom (since we are talking about the space of continuous functions of spacetimes, these degrees of freedom are respectively  $\infty^5$  and  $\infty^1$  in number, where  $\infty$  is the uncountable infinity of such functions).

The conformal degree of freedom is a physically relevant degree of freedom: it is not a trivial degree that can be transformed away. This can be seen as follows. First, the classical Einstein equations are not conformally invariant. Thus, we cannot generate arbitrary solutions of these equations by creating new metrics through (5). For example, the standard expanding universe is described by the Robertson-Walker spacetime which is conformally flat. Had conformal transformations been unphysical or trivial, we could have transformed this spacetime to the flat spacetime and argued that the latter also describes the real universe (which it clearly does not!).

In terms of conceptual interpretation the transformation (5) provides a good deal of simplification; for, it leaves the light cones globally unchanged. Thus, two spacetime points have the same causal relationship (or lack of it) under the conformal transformation. Conversely, if we choose to limit our geometrical fluctuations to those that preserve the causal structure of spacetime, then the only fluctuations we are allowed are the conformal ones.

Finally, we are interested in investigating how quantum gravity affects the celebrated conclusion of classical gravity, namely that the universe with "normal" physical contents should inevitably have a spacetime singularity. Here, the singularity (e.g., the big bang one) is most readily linked with the vanishing of proper volumes and hence with the conformal degree of freedom.

If we, therefore, limit our geometrical changes to the conformal fluctuations only, the rest of the problems posed at the end of the previous section are also resolved. For, let  $g_{ik}$  denote the solution of the classical Einstein equations. Then, it can be shown that under the transformation (5)

$$S_g \rightarrow \tilde{S}_g = \frac{c^3}{16\pi G} \int_V ((1+\phi)^2 R - 6\phi_1 \phi^1) \sqrt{-g} d^4v. \quad (9)$$

For details see Narlikar and Padmanabhan.<sup>8</sup> In other words,  $\tilde{S}_g$  contains  $\phi$  and its first derivative,  $\phi_1 \equiv \partial\phi/\partial x^1$  only up to the quadratic form. Path integrals of this kind can be evaluated exactly without ambiguity of measures.

Unlike the other (nonconformal) degrees of freedom, the conformal degree appears explicitly in the action and, as seen above, it can be readily handled. When specifying the wavefunctionals on  $\Sigma_i$  and  $\Sigma_f$ , the conformal part of the 3-geometry remains well specified (as in  $g_{ik}$ , the classical geometry). The only unknown variable is  $\phi$ , which has the initial and final values given by  $\phi_{(i)}$  and  $\phi_{(f)}$ .

The evolution of wavefunctionals of geometry under this scheme is therefore given by the standard propagator equation

$$\Psi[\phi_{(f)}] = \int K[\phi_{(f)}, \Sigma_f; \phi_{(i)}, \Sigma_i] \Psi[\phi_{(i)}] D\phi_{(i)}. \quad (10)$$

In the next section we will consider application of this formula to quantum cosmology.

#### 4. THE QUANTUM CONFORMAL PROPAGATOR

The propagator  $K$  can be computed using (9) and an explicit form for  $\mathfrak{S}_h$ . We shall take  $\mathfrak{S}_h$  to describe an arbitrary distribution of non-interacting massive particles minimally coupled to gravity. This assumption is consistent with particle physics at very high energies. For example, at energies exceeding  $\sim 10^{16}$  GeV the particles may be considered asymptotically free. In the framework of the big bang cosmology, the universe prior to the epoch  $t \sim 10^{-37}$  s contained particles at energies exceeding this value and at the so called Planck epoch, the typical particle energy would be  $\sim 10^{19}$  GeV.

With this assumption it has been shown by the author<sup>9</sup> that

$$K[\phi_{(f)}, \Sigma_f; \phi_{(i)}, \Sigma_i] = \int \exp\left\{ \frac{c^3}{16\pi G} \int_V (R\phi^2 - 6\phi_1 \phi^1) \sqrt{-g} d^4v \right\} D\phi. \quad (11)$$

The quadratic path integral can be evaluated by using the techniques of Green's functions

for obtaining the solution the wave equation

$$\phi + (1/6) R\phi = 0, \quad (12)$$

for specified values of  $\phi$  on  $\Sigma_1, \Sigma_t$ . For convenience, we choose the time coordinate constant  $t$  over  $\Sigma$ . Thus,  $t = t_1, t_t$  on  $\Sigma_1, \Sigma_t$ , respectively, say. Also, in analogy with the summation convention for discrete tensor indices it is convenient to adopt a similar convention for continuum integration. For example, the relation

$$A(x) = \int \delta(x-y) A(y) dy \quad (13)$$

for the Dirac delta function may be written more compactly as

$$A(x) = \delta(x-y) A(y). \quad (14)$$

The solution of (11) then takes the form

$$K[\phi_{(t)}, t_t; \phi_{(1)}, t_1] = F(t_1, t_t) \exp iQ, \quad (15)$$

where

$$Q \equiv A_{ij}(x_1, x'_1) \phi_{(1)}(x_1) \phi_{(1)}(x'_1) + A_{tt}(x_t, x'_t) \phi_{(t)}(x_t) \phi_{(t)}(x'_t) + 2A_{it}(x_1, x_t) \phi_{(1)}(x_1) \phi_{(t)}(x_t). \quad (16)$$

In the above expression we have taken  $x_1 \equiv x_1^\mu$  to be a typical point on  $\Sigma_1$  and similarly for  $x_t$ . The function  $F(t_1, t_t)$  is a Van Vleck determinant, as shown by DeWitt.<sup>10</sup> The coefficients are related to the Green's functions of (12). In particular,

$$A_{it}(x_1, x_t) = \frac{3}{8\pi} G(x_t, x_1)^{-1}, \quad (17)$$

where  $G(x_t, x_1) \equiv G(x_t, t_t, x_1, t_1)$  is the retarded Green's function.

Suppose we use the propagator  $K$  backwards in time to deduce the initial state of the universe which could have evolved to its present "final" state. Since the present state is very nearly classical, we may approximate it with the wave packet

$$\Psi[\phi_{(t)}(x)] = [2\pi\sigma_t(x)]^{-1/4} \exp \left[ -\left\{ \frac{\phi_{(t)}(x)}{2\sigma_t(x)} \right\}^2 \right], \quad (18)$$

with  $\sigma_t(x)$  denoting the present "uncertainty" of knowledge of  $\phi_{(t)}$ . For  $|\sigma_t| \ll 1$ , we expect  $\phi_{(t)}$  to be very nearly zero.

We now use (10) in the reverse fashion to write

$$\Psi_1[\phi_{(1)}(x_1)] = \int K[\phi_{(t)}(x_t), \phi_{(1)}(x_1)] \Psi_t[\phi_{(t)}(x_t)] D\phi_{(t)}. \quad (19)$$

It can be shown that by a linear transformation of the functions  $\phi_{(t)}$  we can diagonalize  $A_{tt}$ , i.e., take it as

$$A_{tt}(x_t, x'_t) = A(x_t) \delta(x_t - x'_t). \quad (20)$$

Assuming that this diagonalization has been carried out, we can evaluate (19) to find that

$$|\Psi_1|^2 \propto \exp [ -\phi_{(1)}(x_1)^2 / 2\sigma_1^2(x_1) ], \quad (21)$$

where the constant of proportionality is independent of  $x_1$  and

$$\sigma_1^2 = \sigma_t^2 \left\{ A^2 + \frac{1}{16\sigma_t^4} \right\} A_{if}^{-2} . \quad (22)$$

It is interesting to investigate the behavior of (21) and (22) as  $\Sigma_1$  approaches a classical singularity, e.g., the big bang singularity. First, it can be shown that at a curvature singularity  $G(x_t, x_1) \rightarrow \infty$ , so that  $A_{if} \rightarrow 0$ , i.e.,  $\sigma_1 \rightarrow \infty$ . In other words, complete uncertainty prevails regarding the initial conditions if we insist on pushing  $\Sigma_1$  close to the singular epoch. Thus we cannot make any definitive statement about the "origin of the universe" at the singular epoch. Moreover, an evaluation of probabilities based on the wave packet nature of  $\Psi_1$  tells us that the spacetimes which are conformal to the classical singular geometry and which themselves are singular, occur in  $\Psi_1$  with a vanishing measure of probability.

## 5. FURTHER WORK

(i) The avoidance of the spacetime singularity in cosmology has been proved in a more general context by Joshi and Narlikar.<sup>12</sup> They have extended the scope of what is meant by a singularity to the less restrictive definition of the existence of an incomplete nonspacelike curve and have shown that even universes with such a singularity can occur with a probability of measure zero. In deriving this result they have used for  $\Psi(\phi)$  forms that are more general than wave packets.

(ii) Joshi and Joshi have, in a series of papers, recently derived many of the path integral results from the more familiar operator formalism. In particular, they have reconfirmed that the quantum uncertainty diverges at the classical singularity.<sup>13</sup>

(iii) Padmanabhan<sup>14</sup> has used conformal quantization to provide an interesting finite cut off for the renormalization program of quantum electrodynamics. His argument is basically as follows. Consider two events at  $(t, x)$  and  $(t, y)$ . What is the proper distance between them at time  $t$ ? Given the spacetime metric, we may calculate it as  $R = |x - y|$ , say. However, if in the quantum gravity scenario we are not sure of the metric within a conformal fluctuation,  $R$  itself will have built in uncertainty. We may express this as

$$R^2 = (1 + \phi)^2 R_0^2 , \quad (23)$$

say. At the high energy limit ("ultraviolet divergence") we expect  $R \rightarrow 0$ . At this stage we find that

$$\langle R^2 \rangle = R_0^2 (1 + \sigma_\phi^2) , \quad (24)$$

where  $R_0^2 \sigma_\phi^2 \simeq L_p^2$  is the lower bound on  $\langle R^2 \rangle$ . Here  $L_p$  is the Planck length. Thus quantum fluctuations tend to prevent the high energy infinities in renormalizable theories.

To summarize, therefore, the path integral approach limited to the conformal degree of freedom yields meaningful insight into how quantum inputs can alter conclusions derived on the basis of a classical theory of gravity. Although quantization of the conformal degree of

freedom is perhaps the most relevant one for the issue of spacetime singularity, further work is needed on the more difficult problem of quantizing the nonconformal degrees of freedom.

#### A TRIBUTE

This article is written as a tribute to Professor Nathan Rosen, who has contributed semi-nally to our understanding of several fundamental problems of spacetime, gravitation and quantum phenomena.

#### REFERENCES

1. J.V. Narlikar, Sp. Sci. Rev. Nos. 3 and 4, 50 (1989).
2. T. Matsumoto, S. Hayakama, H. Matsuo, H. Murakami, S. Sato, A.E. Lange and P.L. Richards, *Astrophys. Jour.* 329, 567 (1988).
3. J.V. Narlikar, *Introduction to Cosmology*, Jones and Bartlett, Boston (1983).
4. F. Hoyle and N.C. Wickramasinghe, in *Proceedings of the 22nd ESLAB Symposium on Infrared Spectroscopy in Astronomy ESA SP-290*, to be published (1989).
5. N.C. Rana, *Phys. Rev. Lett.* 48, 209 (1982).
6. H. Reeves, in *Proceedings of the Alpbach Summer School on Space Research and Fundamental Physics* (1987).
7. J. Isenberg and J.A. Wheeler, in *Relativity, Quanta and Cosmology*, M. Pantaleo and F. de Finis, eds., Johnson, New York (1979), p. 267.
8. J.V. Narlikar and T. Padmanabhan, *Gravity, Gauge Theories and Quantum Cosmology*, Reidel, Dordrecht (1986).
9. J.V. Narlikar, *Foundations of Physics* 11, 473 (1984).
10. C. DeWitt, *Ann. Phys.* 97, 307 (1976).
11. J.V. Narlikar, *Foundations of Physics* 14, 443 (1981).
12. P.S. Joshi and J.V. Narlikar, *Int. Jour. Mod. Phys. A* 1, 243 (1986).
13. For a collected account of the work by P.S. Joshi and S.S. Joshi, see the Ph.D. thesis of S.S. Joshi entitled *Quantum Effects Near the Space Time Singularity* at Bhavnagar University, Bhavnagar (1989).
14. T. Padmanabhan, *Ann. Phys.* 165, 38 (1985).