CMB Non Gaussianity from Cosmic Magnetic Fields

T R Seshadri

Department of Physics and Astrophysics, University of Delhi

Indo-UK meeting on
Confronting particle-cosmology with Planck and LHC
August 12, 2011, IUCAA, Pune

Based on
TRS and **Kandaswamy Subramanian :** Physical Review Letters **103** 081303 (2009)
Pranjal Trivedi, TRS and **Kandaswamy Subramanian :** Physical Review **D 82** 123006 (2010)

Introduction

Several features of CMBR are good probes of a variety of physical processes in the Universe. Anisotropy, Polarization (Two types), temperature-polarization correlation, spectral distortion, Non-Gaussianity

Question:
Can statistics of CMB anisotropy be used as a probe to study the Cosmic Magnetic Fields?

Aim of the talk:
The Non-gausianity can be a useful signal to detect/put bounds on cosmic magnetic fields.

Plan of the talk

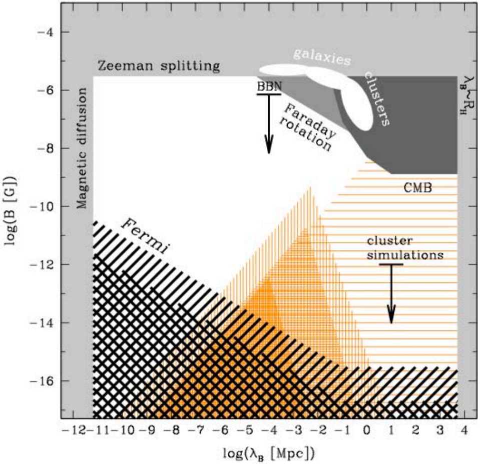
- Observational evidence of Magnetic Fields
- Special relevance of CMB for CMF - Role of Non-Gaussianity
- Nature of magnetic fileds considered
- CMB Bispectrum from magnetic filed energy density
- Reduced bispectrum for equilateral case and Isosceles case
- CMB Bispectrum from passive modes of Scalar anisotropic stress of magnetic filed
- Reduced bispectrum for squeezed collinear case and contribution from s-independent terms.
- Conclusions

Study of Cosmic Magnetic Field - Motivation

- μGauss $\vec{B}$ observed in galaxies: both coherent & stochastic
- $\vec{B}$ growth via either dynamo amplification or flux freezing $\longrightarrow$ a seed $\vec{B}$ field is required
- These seed fields may be of primordial origin
- Evidence for equally strong $\vec{B}$ in high redshift ($z \sim 2$) galaxies [Bernet et al. 08, Kronberg et al. 08]
  - Enough time for dynamo to act?
- Recent FERMI/*LAT* observations of γ-ray halos around AGN
  - ***Detection*** of intergalactic $\vec{B} \approx 10^{-15} G$ [Ando & Kusenko 10]
  - ***Lower*** limit: $\vec{B} \geq 3 \times 10^{-16}$ G on intergalactic $\vec{B}$ [Neronov & Vovk, *Science* 10]

No compelling mechanism yet for origin of strong primordial $\vec{B}$ fields [e.g. Martin & Yokoyama 08]

Constraints on Cosmic Magnetic Fields



[Neronov & Vovk, *Science* 10]

Why is CMB-Nongaussianty of special significance for studying Cosmic Magnetic Fields

Inflationary models:
Small fluctuations in the field (and hence, linear order)

$\downarrow$

Gaussian statistics for Fluctuation

$\downarrow$

Gaussian statistics for CMB Temperature Anisotropy

CMB Non-gaussianity only from higher order effects

From Magnetic Fields:
Magnetic Stresses inherently quadratic in $\vec{B}$ field

$\downarrow$

Even for Gaussianity $\vec{B}$ field
Magnetic stresses non-gaussian

$\downarrow$

Non-Gausianity in $\vec{B}$ field induced CMB anisotropy

CMB Non-gaussianity even from lowest order orders

Measures of Non-Gaussianity

- Bispectrum $\leftrightarrow$ 3-point correlation function
- Trispectrum $\leftrightarrow$ 4-point correlation function

Here we estimate the bispectrum of the CMBR temperature anisotropy statistics

Nature of the Magnetic Field Considered here

1. Magnetic Field: Stochastic. Statistically homogeneous and isotropic.
2. Assumed to be a Gaussian Random Field. Statistical properties specified completely by 2-point correlation function.
3. Magnetic field induces velocity fields via lorentz force
   On scales $> L_G$ (galactic scales) velocities small enough that the magnetic fields do not change.

$$\vec{B}(\vec{x}, t) = \frac{\vec{b}_0(\vec{x})}{a^2(t)}$$

Statistical specification of the Magnetic Field

Field: Non helical, Gaussian and spectrum specified by
$\langle b_i(\vec{k}) b_j^*(\vec{q}) \rangle = (2\pi)^3 \delta(\vec{k} - \vec{q}) P_{ij}(\vec{k}) M(k)$
$\rightarrow$ Completely determined by $M(k)$

P_{ij} is the projection operator that ensures $\vec{\nabla} \cdot \vec{b}_0 = 0$

$\langle \vec{b}_0 \cdot \vec{b}_0 \rangle = 2 \int \frac{dk}{k} \Delta_b^2(k)$ with $\Delta_b^2 = k^3 M(k)/2\pi^2$

Form of $M(k)$:
$M(k) \propto A k^n$ with a cutoff at
Alfen wave damping scale

Fixing A: In terms of variance, B_0,
of Magnetic Field at $k_G = 1h\text{Mpc}^{-1}$

$$\Rightarrow \Delta_b^2(k) = \frac{B_0^2}{2}(n+3)\left(\frac{k}{k_g}\right)^{n+3}$$

$\Delta T/T$ from Magnetic Field Energy Density

$\Omega_B = | b_0^2(\vec{x}) |^2 /(8\pi\rho_0)$ is the fractional contribution of magnetic fields towards energy density

In Fourier space,

$$\vec{\Omega}_B(\vec{k}) = \frac{1}{(2\pi)^3}\int d^3s \;\; b_i(\vec{k}+\vec{s})b_i^*(\vec{s})/(8\pi\rho_0)$$

$$\frac{\Delta T(\hat{n})}{T} \sim 0.03 \; \Omega_B(\vec{x}_0 - \hat{n}D^*)$$

[Giovannini 2007]

$\hat{n} \longrightarrow$ direction of observation
$D^* \longrightarrow$ angular diameter distance to SLS.
$\vec{x}_0 \longrightarrow$ position vector of the observer.

3-point correlation function

$$\frac{\Delta T(\hat{n})}{T} = \sum_{lm} a_{lm} Y_{lm}(\hat{n})$$

Bispectrum $\leftrightarrow \langle \frac{\Delta T(\hat{n}_1)}{T} \frac{\Delta T(\hat{n}_2)}{T} \frac{\Delta T(\hat{n}_3)}{T} \rangle$

$$B^{m_1 m_2 m_3}_{l_1 l_2 l_3} = \langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle$$

$$= \mathcal{R}^3 \int \left[\prod_{i=1}^{3} (-i)^{l_i} \frac{d^3 k_i}{2\pi^2} j_{l_i}(k_i D^*) Y^*_{l_i m_i}(\hat{k}_i) \right] \zeta_{123}$$

$$\zeta_{123} = < \hat{\Omega}_B(\vec{k}_1) \hat{\Omega}_B(\vec{k}_2) \hat{\Omega}_B(\vec{k}_2) > .$$

3-point correlation function continued .....

Recall: $\vec{\Omega}_B(\vec{k}) = \frac{1}{(2\pi)^3} \int d^3s \;\; b_i(\vec{k}+\vec{s}) b_i^*(\vec{s})/(8\pi\rho_0)$

$\zeta_{123} = \psi_{123}\delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)$ with

$$\psi_{123} = \frac{1}{(4\pi\rho_0)^3} \int d^3s \, M(|\,\vec{k}_1 + \vec{s}\,|) M(s) M(|\,\vec{s} - \vec{k}_3\,|) F$$

$F = \alpha^2 + \beta^2 + \gamma^2 - \alpha\beta\gamma$
$\alpha = (\widehat{s} \cdot \widehat{s + k_1}) \;\; ; \;\; \beta = (\widehat{s} \cdot \widehat{s - k_3}) \;\; ; \;\; \gamma = (\widehat{s + k_1} \cdot \widehat{s - k_3})$

Calculation of the Bispectrum

Main term to be evaluated:

$$\psi_{123} = \frac{1}{(4\pi\rho_0)^3} \int d^3s\, M(|\,\vec{k}_1 + \vec{s}\,|)M(s)M(|\,\vec{s} - \vec{k}_3\,|)F$$

1. $l_1 = l_2 = l_3$: equilateral case
2. $l_2 = l_3 \gg l_1$: local isosceles case

In the equilateral case the presence of $j_{li}(k_i D^*)$ picks out configuration with $k_1 \sim k_2 \sim k_3$

Local Isosceles case picks out those with $k_2 \sim k_3 \gg k_1$

Range of n in $M(k) = Ak^n$: $-3 < n < -3/2$
Value chosen close to -3

Estimation of ψ_{123}

$$\psi_{123} = \frac{1}{(4\pi\rho_0)^3} \int d^3s \, M(|\, \vec{k}_1 + \vec{s} \,|) M(s) M(|\, \vec{s} - \vec{k}_3 \,|) F$$

$F = F(\vec{s}, \vec{k_1}, \vec{k_3})$

Equilateral case: Integral over s expressed as sum over integrals in the range, $0 < s < k_1$ and $s > k_1$.

Approximation used :
$s \ll k_1$ and $s \gg k_1$, respectively.

Local Isosceles Case case: Integral over s expressed as sum over integrals in the range, $0 < s < k_1, \quad k_1 < s < k_3,$ and $\quad s > k_3$.

Approximation used :
$s \ll k_1 \ll k_3 \quad k_1 \ll s \ll k_3 \quad s \gg k_3$, respectively.

Bispectrum $B^{m_1 m_2 m_3}_{l_1 l_2 l_3} = \mathcal{G}^{l_1 l_2 l_3}_{m_1 m_2 m_3} b_{l_1 l_2 l_3}$
$\mathcal{G}^{l_1 l_2 l_3}_{m_1 m_2 m_3} = \int d\Omega Y_{l_1 m_1} Y_{l_2 m_2} Y_{l_3 m_3}$ is the Gaunt Factor.

$b_{l_1 l_2 l_3} \longrightarrow$ Reduced Bispectrum

$$b_{l_1 l_2 l_3} = \frac{\pi}{2} \left(\frac{\mathcal{R}}{\pi^2} \right)^3 V_A^6 \left[\int \frac{dk_3}{k_3} j^2_{l_3} (k_3 D^*) \left(\frac{k_3}{k_G} \right)^{n+3} \right]$$
$$\times \left[\int \frac{dk_1}{k_1} j^2_{l_1} (k_1 D^*) \left(\frac{k_1}{k_G} \right)^{2(n+3)} \right] C(n)$$

Equilateral Case: $C(n) = (\frac{4}{3})^4 \frac{\pi^7}{2} \frac{(n+3)^2 (7-n)}{(|n+1|)}$

Local Isosceles Case: $C(n) = (\frac{16}{3})^3 \pi^7 \frac{(n+3)^2}{|2n+3|}$

Sachs Wolf contribution in the two Limits: Equilateral Case and Isosceles Case

Equilateral Case

$$l_1(l_1+1)l_3(l_3+1)b_{l_1 l_2 l_3} \sim 2.3 \times 10^{-23} \left(\frac{n+3}{0.2}\right)^2 \left(\frac{B_{-9}}{3}\right)^6$$

Isosceles Case

$$l_1(l_1+1)l_3(l_3+1)b_{l_1 l_2 l_3} \sim 1.5 \times 10^{-22} \left(\frac{n+3}{0.2}\right)^2 \left(\frac{B_{-9}}{3}\right)^6$$

with $B_{-9} \equiv (B_0/10^{-9}\mathrm{Gauss})$**.**

Comparison with Standard Inflationary Models

$l_1(l_1+1)l_3(l_3+1)b_{l_1 l_2 l_3} \sim 4 \times 10^{-18} f_{NL}$
$\longrightarrow$ reduced bispectrum from nonlinear terms in gravitational potential characterized by f_{NL}

for $f_{NL} = 1$ Magnetically induced bispectrum is 10^4 times smaller.

for $f_{NL} \sim 100$ [WMAP5, Komatsu et. al. 2009]
$l_1(l_1+1)l_2(l_2+1)l_3(l_3+1)b_{l_1 l_2 l_3} < 4 \times 10^{-16} \rightarrow B_0 < 35$ nG.

Here only SW contribution considered.
Stronger limits expected from ISW.

Stock taking

- CMB Non-Gaussianity expected to probe stochastic magnetic fields better than the power spectrum.
  Reason: Magnetically induced signals are fundamentally non Gaussian even to the lowest order. Non-Gaussianity in inflationary models - a higher order effect.
- For $B_0 \sim 3\text{nG}$, $l_1(l_1+1)l_2(l_2+1)l_3(l_3+1)b_{l_1 l_2 l_3} \sim 10^{-22}$ for a scale invariant magnetic field spectrum.
- This is a new probe of primordial magnetic fields. But only scalar modes included.
- Present limits on bispectrum $\rightarrow$upper limits on $B_0 \sim 35\text{nG}$. Limits expected to improve significantly when vector and tensor modes also to be included.

Now Consider Scalar Anisotropic Stress from $\vec{B}$

- Magnetic stress tensor

$$T^i_j(\mathbf{x}) = \frac{1}{4\pi a^4}\left(\frac{1}{2}b_0^2(\mathbf{x})\delta^i_j - b_0^i(\mathbf{x})b_{0j}(\mathbf{x})\right)$$

- in Fourier space

$$S^i_j(\mathbf{k}) = \frac{1}{(2\pi)^3}\int b^i(\mathbf{q})b_j(\mathbf{k}-\mathbf{q})d^3\mathbf{q}$$

$$T^i_j(\mathbf{k}) = \frac{1}{4\pi a^4}\left(\frac{1}{2}S^\alpha_\alpha(\mathbf{k})\delta^i_j - S^i_j(\mathbf{k})\right).$$

- Magnetic perturbations to $T^i_j(\mathbf{k})$

$$T^i_j(\mathbf{k}) = p_\gamma\left(\Delta_B(\mathbf{k})\delta^i_j + \Pi_{Bj}^{\;i}(\mathbf{k})\right)$$

Scalar Anisotropic Stress and the → Passive Mode

- Assume $\vec{B}$ stresses small compared to total ρ, Π of photons + baryons
- linear perturbations
- scalar, vector, tensor evolve independently
- we focus on the scalar part of ${\Pi_B}^i_j$
  as a source of CMB non-Gaussianity

Scalar Anisotropic perturbations $\Pi_B(\mathbf{k})$ given by.

$$\Pi_B(\boldsymbol{k}) = -\frac{3}{2}\left(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij}\right)\Pi_B^{ij}$$

Neutrinos: also develop scalar anisotropic stress after decoupling

- Prior to neutrino decoupling, $\Pi_B(\mathbf{k})$ only source
- After neutrino decoupling, $\Pi_\nu(\mathbf{k})$ also contributes but with an opposite sign: rapid compensation [Lewis 04]

Magnetic anisotropic stress $\Pi_B(\mathbf{k})$ has effect only till neutrino decoupling

Passive Mode Curvature Perturbation

- After neutrino decoupling there are two types of scalar perturbation modes
- **Passive mode** [J. R. Shaw & A. Lewis, PRD 10]

$$\zeta = \zeta\left(\tau_B\right) - \frac{1}{3} R_\gamma \Pi_B \left[\ln\left(\frac{\tau_\nu}{\tau_B}\right) + \left(\frac{5}{8R_\nu} - 1\right)\right].$$

  - # grown logarithmically from $\vec{B}$ generation at τ_B to ν-decoupling at τ_ν
  - # adiabatic-like passive evolution after ν-decoupling
  - # non-Gaussian statistics (unlike primordial adiabatic perturbations)
- For range τ_B corresponding to temperature range from $T_B \approx 10^{14}$ GeV (inflationary) to $T_B \approx 10^3$ GeV (electroweak)
- $\ln\left(\frac{\tau_\nu}{\tau_B}\right) > 10$

$$\boxed{\zeta \simeq -\frac{1}{3} R_\gamma \Pi_B \ln\left(\frac{\tau_\nu}{\tau_B}\right)}$$

Magnetic CMB Anisotropy

- Passive mode curvature perturbation

$$\zeta \simeq -\frac{1}{3} R_\gamma \Pi_B \ln\left(\frac{\tau_\nu}{\tau_B}\right)$$

- Consider CMB anisotropy sourced by magnetic Sachs-Wolfe effect

$$\frac{\Delta T}{T}(\boldsymbol{n}) = \frac{1}{3}\Phi = \frac{1}{5}\zeta \simeq \mathcal{R}_p \Pi_B \qquad \text{where} \qquad \mathcal{R}_p = -\frac{1}{15} R_\gamma \ln\left(\frac{\tau_\nu}{\tau_B}\right)$$

$$\frac{\Delta T}{T}(\boldsymbol{n}) \simeq -(0.04) \ln\left(\frac{\tau_\nu}{\tau_B}\right) \Pi_B$$

[cf. Bonvin & Caprini 10]

- Spherical harmonic expansion

$$\frac{\Delta T(\boldsymbol{n})}{T} = \sum_{lm} a_{lm} Y_{lm}(\boldsymbol{n})$$

$$a_{lm} = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} \mathcal{R}_p \, \Pi_B(\boldsymbol{k}) \, j_l(kD^*) Y_{lm}^*(\hat{\boldsymbol{k}})$$

Bispectrum Calculation

- Bispectrum

$$B_{l_1 l_2 l_3}^{m_1 m_2 m_3} = \langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle = \mathcal{R}_p^3 \int \left[\prod_{i=1}^{3} (-i)^{l_i} \frac{d^3 k_i}{2\pi^2} j_{l_i}(k_i D^*) Y_{l_i m_i}^*(\hat{\boldsymbol{k}}_i) \right] \zeta_{123}$$

where ζ_{123} is defined as $\zeta_{123} = \langle \hat{\Pi}_B(\boldsymbol{k}_1) \hat{\Pi}_B(\boldsymbol{k}_2) \hat{\Pi}_B(\boldsymbol{k}_3) \rangle$

- Mode-Coupling Integral

$$\zeta_{123} = \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)\ \psi_{123}$$

$$\psi_{123} = \frac{1}{(4\pi p_\gamma)^3} \int d^3 s M(|\boldsymbol{k}_1 + \boldsymbol{s}|) M(s) M(|\boldsymbol{s} - \boldsymbol{k}_3|) \times (\mathcal{F}_{\Pi_B \Pi_B \Pi_B})$$

Recall: $\Delta_b^2 = k^3 M(k)/2\pi^2$: **Form of** $M(k)$: $M(k) = Ak^n$

Mode-Coupling Integral

$$\psi_{123} = \frac{1}{(4\pi p_\gamma)^3} \int d^3 s M(|\boldsymbol{k}_1 + \boldsymbol{s}|) M(s) M(|\boldsymbol{s} - \boldsymbol{k}_3|) \times (\mathcal{F}_{\Pi_B \Pi_B \Pi_B})$$

$$\mathcal{F}_{\Pi_B \Pi_B \Pi_B} = \sum_{n=0}^{6} \mathcal{F}^n_{\Pi_B \Pi_B \Pi_B}$$

Mode-Coupling Integral

$$\psi_{123} = \frac{1}{(4\pi p_\gamma)^3} \int d^3 s M(|\boldsymbol{k}_1 + \boldsymbol{s}|) M(s) M(|\boldsymbol{s} - \boldsymbol{k}_3|) \times (\mathcal{F}_{\Pi_B \Pi_B \Pi_B})$$

$$\mathcal{F}_{\Pi_B \Pi_B \Pi_B} = \sum_{0}^{6} \mathcal{F}^n_{\Pi_B \Pi_B \Pi_B}$$

$$\mathcal{F}^0_{\Pi_B\Pi_B\Pi_B} = -9,$$

$$\mathcal{F}^1_{\Pi_B\Pi_B\Pi_B} = 0,$$

$$\mathcal{F}^2_{\Pi_B\Pi_B\Pi_B} = (\bar\beta^2 + \bar\gamma^2 + \bar\mu^2 + 9(\theta_{13}^2 + \theta_{23}^2 + \theta_{12}^2) + 3(\alpha_3^2 + \alpha_1^2 + \alpha_2^2 + \beta_3^2 + \beta_1^2 + \beta_2^2 + \gamma_3^2 + \gamma_1^2 + \gamma_2^2)),$$

$$\begin{aligned}\mathcal{F}^3_{\Pi_B\Pi_B\Pi_B} = {} & -3(\bar\mu(\beta_3\gamma_3 + \beta_1\gamma_1 + \beta_2\gamma_2 + \tfrac{1}{3}\bar\beta\,\bar\gamma) + \bar\gamma(\alpha_3\gamma_3 + \alpha_1\gamma_1 + \alpha_2\gamma_2) + \bar\beta(\alpha_3\beta_3 + \alpha_1\beta_1 + \alpha_2\beta_2) \\ & + 3\theta_{13}(\alpha_3\alpha_1 + \beta_3\beta_1 + \gamma_3\gamma_1) + 3\theta_{23}(\alpha_3\alpha_2 + \beta_3\beta_2 + \gamma_3\gamma_2) + 3\theta_{12}(\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2) \\ & + 9\theta_{13}\theta_{23}\theta_{12}),\end{aligned}$$

$$\begin{aligned}\mathcal{F}^4_{\Pi_B\Pi_B\Pi_B} = {} & 3(\bar\gamma\,\bar\mu\,\alpha_3\beta_3 + \bar\beta\,\bar\mu\,\alpha_1\gamma_1 + \bar\beta\,\bar\gamma\,\beta_2\gamma_2 + 3(\bar\mu\theta_{13}\beta_3\gamma_1 + \bar\gamma\theta_{23}\alpha_3\gamma_2 + \bar\beta\theta_{12}\alpha_1\beta_2) \\ & + 3(\alpha_3\beta_3(\alpha_1\beta_1 + \alpha_2\beta_2) + \alpha_1\gamma_1(\alpha_3\gamma_3 + \alpha_2\gamma_2) + \beta_2\gamma_2(\beta_3\gamma_3 + \beta_1\gamma_1)) \\ & + 9(\theta_{13}\theta_{23}\gamma_1\gamma_2 + \theta_{13}\theta_{12}\beta_3\beta_2 + \theta_{23}\theta_{12}\alpha_3\alpha_1)),\end{aligned}$$

$$\begin{aligned}\mathcal{F}^5_{\Pi_B\Pi_B\Pi_B} = {} & -9(\bar\mu\alpha_3\beta_3\alpha_1\gamma_1 + \bar\gamma\alpha_3\beta_3\beta_2\gamma_2 + \bar\beta\alpha_1\gamma_1\beta_2\gamma_2 + 3(\theta_{13}\beta_3\gamma_1\beta_2\gamma_2 + \theta_{23}\alpha_3\alpha_1\gamma_1\gamma_2 \\ & + \theta_{12}\alpha_3\beta_3\alpha_1\beta_2)),\end{aligned}$$

$$\mathcal{F}^6_{\Pi_B\Pi_B\Pi_B} = 27\alpha_3\beta_3\alpha_1\gamma_1\beta_2\gamma_2.$$

Bispectrum Configurations

- Parameter space for bispectrum configurations

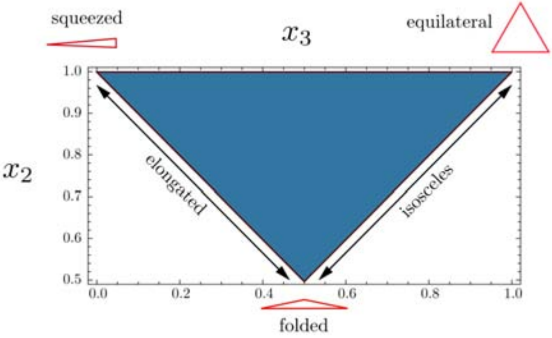


FIG. 3: Shapes of Non-Gaussianity. The shape function $F(k_1, k_2, k_3)$ forms a triangle in Fourier space. The triangles are parametrized by rescaled Fourier modes, $x_2 = k_2/k_1$ and $x_3 = k_3/k_1$. Figure from Ref. [43]

[Yadav & Wandelt 10]

Bispectrum calculation for two specific cases

- **Squeezed collinear case**: $k_2 \ll k_1\ ,\ k_3$ and $\hat{k}_1 = \hat{k}_2 = -\hat{k}_3$
  Only 10 of the 58 terms contribute to the $\mathcal{F}_{\Pi_B \Pi_B \Pi_B}$

$$\begin{aligned}\mathcal{F}_{\Pi_B \Pi_B \Pi_B} \quad = \quad & -8 + \bar{\beta}^2 + 9\left(\mu^2 + 2\gamma^2\right) + 6\mu\bar{\beta}\gamma + 3\bar{\beta}^2\gamma^2 \\ & -9\gamma^2\left(3\mu^2 + \gamma^2\right) - 18\mu\bar{\beta}\gamma^3 + 27\mu^2\gamma^4\end{aligned}$$

- Contribution of the **s-independent terms** in $\mathcal{F}_{\Pi_B \Pi_B \Pi_B}$:
  Only 5 of the 58 term contribute:
  $\mathcal{F}^{I}_{\Pi_B \Pi_B \Pi_B} = -9 + 9\left(\theta_{12}^2 + \theta_{23}^2 + \theta_{13}^2\right) - 27\left(\theta_{13}\theta_{12}\theta_{23}\right)$

The Squeezed colinear case

Recall: $\psi_{123} = \frac{1}{(4\pi p_\gamma)^3} \int d^3 s M(|\boldsymbol{k}_1 + \boldsymbol{s}|) M(s) M(|\boldsymbol{s} - \boldsymbol{k}_3|) \times (\mathcal{F}_{\Pi_B \Pi_B \Pi_B})$

- Split the integral into $0 < s < k_1 \sim k_3$ piece and $s > k_1 \sim k_3$ piece
- Approximate these as $s \ll k_1 \sim k_3$ and $s \gg k_1 \sim k_3$, respectively

Gives: $\psi_{123} = \frac{1}{(4\pi p_\gamma)^3} \mathcal{I} = \left(\frac{3}{4\pi\rho_0}\right)^3 \mathcal{I}$

with

$$\begin{aligned} \mathcal{I} \quad = \quad & \int d^3 s M(|\boldsymbol{k}_1 + \boldsymbol{s}|) M(s) M(|\boldsymbol{s} - \boldsymbol{k}_3|) \\ & \times \left[-8 + \bar{\beta}^2 + 9\left(\mu^2 + 2\gamma^2\right) + 6\mu\bar{\beta}\gamma + 3\bar{\beta}^2\gamma^2 \right. \\ & \left. - 9\gamma^2\left(3\mu^2 + \gamma^2\right) - 18\mu\bar{\beta}\gamma^3 + 27\mu^2\gamma^4 \right] \end{aligned}$$

for $n \longrightarrow -3, \mathcal{I} \simeq 2\pi A^3 \frac{k_1^{2n+3} k_3^n}{n+3} \left[\frac{8}{3}\right]$

Thus $\psi_{123} = \left[\frac{8}{3}\right] \; 2\,(4)^3 \, \frac{\pi^7}{k_G^6} (n+3)^2 \left(\frac{k_1}{k_G}\right)^{2n+3} \left(\frac{k_3}{k_G}\right)^n V_A^6$

Reduced Bispectrum for Squeezed Collinear Case

Recall:

$$B^{m_1 m_2 m_3}_{l_1 l_2 l_3} = \mathcal{G}^{l_1 l_2 l_3}_{m_1 m_2 m_3} \; b_{l_1 l_2 l_3}$$

where

$$b_{l_1 l_2 l_3} = \left(\frac{\mathcal{R}_p}{\pi^2}\right)^3 \int x^2 dx \times \prod_{i=1}^{3} \int k_i^2 dk_i \, j_{l_i}(k_i x) \, j_{l_i}(k_i D^*) \; \psi_{123}$$

$\mathcal{G}^{l_1 l_2 l_3}_{m_1 m_2 m_3} = \int d\Omega \; Y_{l_1 m_1} Y_{l_2 m_2} Y_{l_3 m_3}$ is the Gaunt integral.

For Squeezed Collinear Case:

$$l_1(l_1+1) l_3(l_3+1) b_{l_1 l_2 l_3} = \left[\frac{8}{3}\right] \; \mathcal{R}_p{}^3 \left(\frac{\pi}{2}\right)^2 (4)^3 \, (n+3)^2 V_A^6$$

Gives:

$$l_1(l_1+1) l_3(l_3+1) b_{l_1 l_2 l_3} \approx \left(-1.35 \times 10^{-16}\right) \quad \times \left(\frac{n+3}{0.2}\right)^2 \left(\frac{B_{-9}}{3}\right)^6$$

$\sim 10^6$ times larger than that from the magnetic field density.

Bispectrum Contribution from s-independent terms

$$B^{m_1 m_2 m_3}_{l_1 l_2 l_3} = \langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle = \mathcal{R}_p^3 \int \left[\prod_{i=1}^{3} (-i)^{l_i} \frac{d^3 k_i}{2\pi^2} j_{l_i}(k_i D^*) Y^*_{l_i m_i}(\hat{\boldsymbol{k}}_i) \right] \zeta_{123}$$

$$\zeta_{123} = \langle \hat{\Pi}_B(\boldsymbol{k}_1) \hat{\Pi}_B(\boldsymbol{k}_2) \hat{\Pi}_B(\boldsymbol{k}_3) \rangle = \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)\ \psi_{123}$$

$$\psi_{123} = \frac{1}{(4\pi p_\gamma)^3} \int d^3 s M(|\boldsymbol{k}_1 + \mathbf{s}|) M(s) M(|\mathbf{s} - \boldsymbol{k}_3|) \times (\mathcal{F}_{\Pi_B \Pi_B \Pi_B})$$

$\mathcal{F}^{\rm I}_{\Pi_B \Pi_B \Pi_B} = -9 + 9\left(\theta_{12}^2 + \theta_{23}^2 + \theta_{13}^2\right) - 27\left(\theta_{13}\theta_{12}\theta_{23}\right)$: s-independent part

$\psi_{123} = \frac{m}{(4\pi p_\gamma)^3}\ \mathcal{I} = \left(\frac{3}{4\pi\rho_0}\right)^3\ \mathcal{F}_{\Pi_B \Pi_B \Pi_B} \times \mathcal{I}$

$$\mathcal{I} = \int d^3 s M(|\boldsymbol{k}_1 + \mathbf{s}|) M(s) M(|\mathbf{s} - \boldsymbol{k}_3|)$$

= 2 $\pi A^3 \int_{-1}^{1} d\mu \int_0^\infty ds s^{n+2} \left(s^2 + k_1^2 + 2sk_1\nu\right)^{\frac{n}{2}} \times \left(s^2 + k_3^2 - 2sk_3\mu\right)^{\frac{n}{2}}$

where $\nu = \hat{k}_1 \cdot \hat{s}$ and $\mu = \hat{k}_3 \cdot \hat{s}$.

Value of s-independent part of $\mathcal{F}$ for different configurations

TABLE I. The sum of s-independent terms $m = \mathcal{F}^{\mathrm{I}}_{\Pi_B \Pi_B \Pi_B}$ in four different configurations $(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ for evaluating the Case I bispectrum.

Configuration	$(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$	$(\theta_{12}, \theta_{23}, \theta_{13})$	m
Local	$k_1 \sim k_3$	$(0, 0, -1)$	0
Isosceles	$k_2 \ll k_1, k_3$		
Equilateral	$k_1 \sim k_2 \sim k_3$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	-5.625
Ssqueezed	$k_1 \sim k_3$	$(1, -1, -1)$	-8 [a]
Collinear	$k_2 \ll k_1, k_3$		
	$\hat{\boldsymbol{k}}_1 = \hat{\boldsymbol{k}}_2 = -\hat{\boldsymbol{k}}_3$		
Midpoint	$k_1 \sim k_2 \sim \frac{k_3}{2}$	$(1, -1, -1)$	-9
Collinear	$\hat{\boldsymbol{k}}_1 = \hat{\boldsymbol{k}}_2 = -\hat{\boldsymbol{k}}_3$		

[a]For the squeezed collinear configuration case, $\mathcal{F}^{\mathrm{I}}_{\Pi_B \Pi_B \Pi_B}$ picks up another term $\bar{\mu}^2 \sim 1$.

Equilateral and squeezed collinear configurations: s integral split into two sub-ranges $0 < s < k_1 \sim k_3$ and $s > k_1 \sim k_3$

Approximate the integrands by assuming $s \ll k_1 \sim k_3$ and $s \gg k_1 \sim k_3$ for the respective sub-ranges.

Midpoint collinear configuration: s integral split into two sub-ranges $0 < s < k_1$ and $s > 2\, k_1 \sim k_3$

A very small contribution from the sub-range $k_1 < s < 2\, k_1 \sim k_3$ is neglected.

Approximate the integrands by assuming $s \ll k_1$ and $s \gg 2\, k_1 \sim k_3$ for the respective sub-ranges.

Focus on $n \to -3$.

$$\psi_{123} = (4)^4\, m\, \frac{\pi^7}{k_G^6}(n+3)^2 \left(\frac{k_1}{k_G}\right)^{2n+3} \left(\frac{k_3}{k_G}\right)^n V_A^6.$$

where V_A, the Alfvén velocity in the radiation dominated era is

$$V_A = \frac{B_0}{(16\pi\rho_0/3)^{1/2}} \approx 3.8 \times 10^{-4} B_{-9},$$

Reduced bispectrum

For s-independent contribution from $\mathcal{F}$

$$l_1(l_1+1)l_3(l_3+1)b_{l_1 l_2 l_3} \approx$$
$$\left(10^{-16}\right)\left(\frac{n+3}{0.2}\right)^2\left(\frac{B_{-9}}{3}\right)^3 \times \left\{\begin{array}{ll} 0 & \text{local isosceles} \\ 5.7 & \text{equilateral} \\ 8.1 & \text{squeezed collinear} \\ 9.2 & \text{midpoint collinear} \end{array}\right\} \tag{1}$$

For comparison, recall the case of squeezed collinear

$$l_1(l_1+1)l_3(l_3+1)b_{l_1 l_2 l_3} \approx$$
$$\left(10^{-16}\right)\left(\frac{n+3}{0.2}\right)^2\left(\frac{B_{-9}}{3}\right)^3 \times \{-1.35\} \tag{2}$$

CMB Bispectrum **Results** for Magnetic Passive Mode

- $l_1(l_1+1)l_3(l_3+1)b_{l_1 l_2 l_3} \approx 10^{-16}$ or $\times 10^6$ stronger than that arising from density

  for n_B = -2.8, 3 nG field, $\tau_B \approx 10^{14}$ GeV

- Squeezed Collinear full evaluation: $l_1(l_1+1)l_3(l_3+1)b_{l_1 l_2 l_3} \approx -1.4 \times 10^{-16}$

  using WMAP7 $-10 < f_{NL}$ get upper limit $B_0 < 2nG$

- Contribution from General configuration approximate evaluation: $l_1(l_1+1)l_3(l_3+1)b_{l_1 l_2 l_3} \approx 6 - 9 \times 10^{-16}$
- using WMAP7 $f_{NL} < 74$ get upper limit $B_0 < 3nG$

- Inflationary bispectrum with $f_{NL} \sim 1$ is $l_1(l_1+1)l_3(l_3+1)b_{l_1 l_2 l_3} \approx 10^{-18}$

- CAVEAT: only Sachs-Wolfe
- CAVEAT: τ_B dependence: But little change $B_0 < 2 - 4nG$

Conclusions

- Cosmological magnetic fields are an interesting possibility: CMB non-Gaussianity a unique probe of them
- 10 times stronger B_0 upper limit of 2 nG from bispectrum (passive vs compensated scalar mode)
- B_0 limit from single bispectrum mode (passive scalar) already competetive with total constraint from all magnetic modes in CMB power spectrum $B_0 < 2$ nG [Paoletti 10, Yamazaki 10]
- First Magnetic CMB bispectrum calculation > inflationary bispectrum (greater by $\times 100$ for $f_{NL} \sim 1$ and $B_0 \sim 3$ nG)
- Magnetic CMB non-Gaussianity: important, needs to be fully understood cf. recovering inflationary non-Gaussianity from CMB data