IUCAA 37/95

String-dust in Einstein and Gödel Universes

L K Patel[1] and Naresh Dadhich[2]
Department of Mathematics and Applied Mathematics
University of Natal
Durban, South Africa.

Abstract

We consider the mixture of perfect fluid and string-dust and obtain string-dust generalizations of the Einstein and Gödel Universes.

PACS Numbers: 04.20Jb, 98.80Dr.

Key words: General Relativity, Einstein Universe, Gödel Universe, rotating Universe, string-dust.

[1]Permanent Address: Department of Mathematics, Gujarat University, Ahmedabad -38009, India.
[2]Permanent Address: IUCAA, Post Bag 4, Pune - 411007, India.
E-mail Address: naresh@incaa.ernet.in

The string-dust distribution is characterised by the stress tensor [1-2],

$$T_{ik} = \lambda(u_i u_k - w_i w_k) \tag{1}$$

where λ is the string energy density and u_i and w_i are the unit time like 4-velocity and the spacelike vectors; $u^i u_i = 1 = -w^i w_i$, $u^i w_i = 0$. For the distribution (1), Einstein's equation implies

$$R_{ik} = -8\pi\lambda(u_i u_k - g_{ik} - w_i w_k) \tag{2}$$

which means the gravitational charge density, $-4\pi\rho_c = R_{ik}u^i u^k$ vanishes for this distribution. That is gravitational charge of the spacetime is zero and that is why one of us [3] would like to look upon (2) as defining generalised empty space. The analogue of the Schwarzschild solution is the general solution of (2) given by

$$ds^2 = (1+2\phi)dt^2 - (1+2\phi)^{-1}dr^2 - r^2(d\alpha^2 + \sin\alpha^2 d\beta^2) \tag{3}$$

where $\nabla^2\phi = 0$, which means $\phi = k - m/r$. It is remarkable to note that $\phi = k \neq 0$ represents a non-flat spacetime. The spacetime is though curved yet free of the Newtonian gravity in the sense that free particles experience no gravitational attraction. With $m \neq 0$, the metric (3) represents a mass point with a string-dust or a generalised empty space surrounding the mass point (i.e. the solution of (2) for a mass point).

The string-dust spacetimes have been considered by several authors [1-7]. Recently we have obtained the string-dust generalisations of the Kerr and NUT solutions[4,5]. That is, the axially symmetric solutions of the generalised empty space equation (2). Vaidya et al [8] have given a unified treatment of the Kerr and NUT metrics by writing

$$\begin{aligned} ds^2 &= 2(du + g\sin\alpha d\beta)dx - M^2(d\alpha^2 + \sin^2\alpha d\beta^2) \\ &- 2L(du + g\sin\alpha d\beta)^2 \end{aligned} \tag{4}$$

where $g = g(\alpha)$ and M, L are functions of u, x, α. For the Kerr $x = t$ while $x = r$ for the NUT solution, and $u = t - r$ always.

Recall that $\phi = k \neq 0$ represented the pure string-dust spacetime which is free of the Newtonian gravity. The analogous situation for (4) will be to consider $L =$ const. Should it analogously describe a string-dust spacetime in the Kerr-NUT symmetry? It turns out that it will give rise to string-dust generalisations of the Einstein and Gödel Universes.

Vaidya has also succeeded in unifying the Einstein and Gödel models by considering $M = M(\alpha)$ and $2L = \epsilon = \pm 1$ in the metric (4). His prescription [9] is,

$$\begin{aligned} ds^2 &= 2(du + g \sin \alpha d\beta)dx - \epsilon(du + g \sin \alpha d\beta)^2 \\ &- Rf(d\alpha^2 + \sin^2 \alpha d\beta^2) \end{aligned} \tag{5}$$

where

$$f = g_\alpha + g \cot \alpha \tag{6}$$

satisfying the equation

$$\frac{1}{2}\left(\frac{f_\alpha}{f}\right)_\alpha + \frac{1}{2}\left(\frac{f_\alpha}{f}\right) \cot \alpha - 1 = -\frac{1}{R}(1 + 3\epsilon)f. \tag{7}$$

R is a constant and $f_\alpha = \partial f / \partial \alpha$. The above equation can be solved to give

$$Rf \sin \alpha = g \exp(-(1 + 3\epsilon)\theta / R) \tag{8}$$

and

$$g \sin \alpha = 1 + \epsilon - \frac{4}{1 + 3\epsilon} \exp(-(1 + 3\epsilon)\theta / 2R) \tag{9}$$

where we have redefined one of the angle coordinates by $d\theta = g d\alpha$.

Now the Einstein Universe follows for $\epsilon = 1$ and $x = t$ while the Gödel will have $\epsilon = -1$ and $x = r$. We wish to consider $2L = \epsilon + k =$ const.,

$M^2 = Rf$ in the metric (4) with f being given by (8). We introduce the orthonormal tetrads,

$$\theta^1 = du + g\sin\alpha d\beta, \qquad \theta^2 = (Rf)^{1/2} d\alpha$$
$$\theta^3 = (Rf)^{1/2}\sin\alpha d\beta, \qquad \theta^4 = dx - L\theta^1$$

and in what follows all the quantities will be referred to the tetrad frame.

The non-zero R_{ik} are given by

$$R_{11} = L^2 R44, \qquad R_{14} = \frac{2L}{R^2}, \qquad R_{44} = -\frac{2}{R^2}$$

$$R_{22} = R_{33} = -\frac{1}{R^2}(1 + 3\epsilon - 4L). \tag{10}$$

For string-dust perfect fluid we add to the stress tensor (1) the perfect fluid stresses and write

$$R_{ik} = -8\pi\left[\lambda(u_i u_k - g_{ik} - w_i w_k) + (\rho + p)u_i u_k - \frac{1}{2}(\rho - p)g_{ik}\right] + \wedge g_{ik} \tag{11}$$

where $\wedge$ is the cosmological constant. Let us write

$$u_i = \left(\frac{1}{2\mu}, 0, 0, \mu\right), \qquad w_i = \left(-\frac{1}{2\mu}, 0, 0, \mu\right) \tag{12}$$

where μ is the parameter to be determined.

Equation (11) implies

$$R_{44} = -8\pi(p + \rho)\mu^2, \qquad R_{14} = \wedge - 8\pi p$$
$$R_{22} = -\wedge - 4\pi(\rho - p + 2\lambda) = R_{33}, \qquad \frac{1}{\mu^2} = \pm 2L \tag{13}$$

and then (10) will give

$$8\pi\rho = -\wedge + \frac{2L}{R^2}(1 + 2\epsilon) \tag{14}$$

$$
\begin{aligned}
8\pi p &= \wedge - \frac{2L}{R^2} && (15)\\
8\pi\lambda &= \frac{1}{R^2}[1+3\epsilon - 2L(3+\epsilon)] && (16)
\end{aligned}
$$

where $2L = \epsilon + k,\ \epsilon = \pm 1$.

It may be noted that with string-dust, equation (7) is not required but it turns out that regular behaviour of f and the metric does not permit any other choice for f. Hence we have continued with the same f as in the Einstein and Gödel cases. The string-dust generalisations of the Einstein and Gödel Universes are explicitly given as follows.

Einstein Universe with string-dust: $\epsilon = 1$ and we have from (14-16)

$$
\begin{aligned}
8\pi\rho &= -\wedge + \frac{3}{R^2}(1+k)\\
8\pi p &= \wedge - \frac{1}{R^2}(1+k)\\
8\pi\lambda &= -\frac{4k}{R^2}.
\end{aligned}
$$

The metric is given by (4) with $2L = 1+k,\ M^2 = Rf$ and

$$
g\sin\alpha = 2 - e^{2\theta/R}, \qquad Rf\sin\alpha = g\, e^{-4\theta/R}.
$$

When $k=0$, it reduces to the Einstein Universe. The positivity of ρ, p and λ prescribes $k \le 0$ and $1 \ge (1+k)/\wedge R^2 \ge 1/3$. We can however set $p=0$ for $1+k = \wedge R^2$.

Gödel Universe with string-dust: $\epsilon = -1$ and we get

$$
\begin{aligned}
8\pi\rho &= -\wedge + \frac{1-k}{R^2}\\
8\pi p &= \wedge + \frac{1-k}{R^2}\\
8\pi\lambda &= -\frac{2k}{R^2}
\end{aligned}
$$

and in the metric (4), $2L = -1 + k$, $M^2 = Rf$ and

$$g \sin\alpha = 2e^{\theta/R}, \qquad Rf \sin\alpha = ge^{2\theta/R}.$$

Here we should have $1 - k \geq \wedge R^2$, $k \leq 0$ and $k = 0$ gives the Gödel Universe. We can set $p = 0$ for $1 - k = - \wedge R^2$ or can have $\rho = p$ for $\wedge = 0$.

It is rather interesting to note that the Einstein string density is double of the Gödel string density. In the Einstein case $k = -1$ will lead to $\rho + p = 0$ or $\rho = p = \wedge = 0$ and $8\pi\lambda = 4/R^2$, while for the Gödel $k = 1$, $\rho + p = 0$ or $\rho = p = \wedge = 0$ but $8\pi\lambda = -2/R^2 < 0$. Thus we can have pure string-dust in both the cases but its density is negative in the Gödel case.

Acknowledgement: We thank Professors Sunil Maharaj and Peter Leach for the hospitality at the University of Natal that facilitated this work.

References

[1] P.S. Letelier (1979). Phys. Rev. **D20**, 1274.

[2] J. Stachel (1980) Phys. Rev. **D21**, 2171.

[3] N. Dadhich (1995) How empty must empty space be? (To be published.)

[4] L.K. Patel, N.Dadhich and A. Beesham (1995) $gr - gc$ 9507031. (To be published.)

[5] L.K. Patel, N. Dadhich and K.S. Govinder (1995). (To be published.)

[6] D.R. Matravers (1988) Ge. Rel. Grav. **20**, 279.

[7] J.M. Nevin (1991) Gen. Rel. Grav. **23**, 253.

[8] P.C. Vaidya, L.K. Patel and P.V. Bhatt (1976) Gen.Rel. Grav. **7**, 701.

[9] P.C. Vaidya (1978) Gen. Rel. Grav. **9**, 801.