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UNCERTAINTY PRINCIPLE AND THE QUANTUM FLUCTUATIONS OF THE SCHWARZSCHILD LIGHT CONES

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We consider the gravitational field of a point mass and show that the application of the uncertainty principle leads to (i) an uncertainty relation for the metric and its conjugate momentum and (ii) finite fluctuations of the light-cone at the event horizon.

1. Introduction

Why should gravity be quantized? At present there is no compelling observational result which demands the quantization of gravity. However, there exist theoretical reasons for quantizing gravity.

The simplest argument is based on the uncertainty principle, applied to the measurement of the gravitational field.[1] It was shown that the components of the metric tensor cannot be measured to arbitrary accuracy if the measuring apparatus obeys laws of quantum mechanics. The only natural way of incorporating this stochastic nature of the metric components is to quantize gravity. In this approach, the uncertainty in g_{ik} arises from the (ultimate) quantum nature of the measuring apparatus.

It is also possible to approach the issue from a different direction. Gravity is produced by the energy momentum tensor T_{ik} of the matter fields. We do know that these source fields are quantized, and obey well-defined uncertainty relations. Consistency demands that the uncertainty in the source variables must induce an uncertainty in the gravitational field. We shall explore these source-related uncertainties in this paper. (It should be noted that, these source-related uncertainties cannot be used to say anything about quantization of source-free modes, e.g., gravitational wave modes.)

We use the metric of a mass point m as a toy model in this paper. Quantum mechanics demands that the position and velocity of such a mass point obey an uncertainty relation. We shall now work out the consequences of this uncertainty relation for the metric variables.

The necessary preliminary results are obtained in the next section. The quantum fluctuations are evaluated in Sec. 3 and the physical effects are discussed in Sec. 4.

2. Lorentz Transformation of Schwarzschild Metric

In order to perform the analysis suggested in the previous section, we first require the metric due to a point mass moving with uniform *coordinate* velocity. This can be obtained by boosting the Schwarzschild metric (expressed in isotropic coordinates) along, say, the x-axis. (For previous application of this technique see Refs. 2 and 3.)

The Schwarzschild metric can be written in the isotropic coordinates as:

$$ds^2 = \left(\frac{1-A}{1+A}\right)^2 d\bar{t}^2 - (1+A)^4(d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2), \tag{2.1}$$

where $A = m/2\bar{r}$, $\bar{r}^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2$ and m is the mass of the source. Consider an observer moving with respect to the mass with a coordinate velocity v along the x-axis. Let the coordinates in his frame be denoted by (t, x, y, z). Let the initial conditions be given by

$$\bar{x} = x = x_0 \text{ at } \bar{t} = t = 0 \text{ with } x_0 > m/2. \tag{2.2}$$

The Lorentz transformation relating the two coordinate systems are

$$\begin{aligned} \bar{x} - x_0 &= \gamma(x - x_0 + vt), \qquad \bar{y} = y, \\ \bar{t} &= \gamma(t + v(x - x_0)), \qquad \bar{z} = z, \end{aligned} \tag{2.3}$$

where $\gamma = (1 - v^2)^{-1/2}$. In the frame of the uniformly moving observer, the line element is

$$\begin{aligned} ds^2 &= (1+A)^4(dt^2 - dx^2 - dy^2 - dz^2) \\ &\quad - \left\{(1+A)^4 - \left(\frac{1-A}{1+A}\right)^2\right\}(dt - v\,dx)^2\gamma^2. \end{aligned} \tag{2.4}$$

Here $A = m/2r = \dfrac{m}{2\gamma[(x-vt)^2 + (y^2+z^2)/\gamma^2]^{1/2}}$. We can write (2.4) as

$$\begin{aligned} ds^2 &= \left[(1+A)^4 - \gamma^2(1+A)^4 + \left(\frac{1-A}{1+A}\right)^2\gamma^2\right]dt^2 \\ &\quad - \left[(1+A)^4 + (1+A)^4\gamma^2v^2 - \left(\frac{1-A}{1+A}\right)^2\gamma^2v^2\right]dx^2 \\ &\quad + \left[v\gamma^2(1+A)^4 - v\gamma^2\left(\frac{1-A}{1+A}\right)^2\right]dt\,dx - (1+A)^4(dy^2 + dz^2). \end{aligned} \tag{2.5}$$

In what follows we shall only be concerned with the limit of small velocities. Therefore, we expand (2.5) to lowest order in v, obtaining

$$ds^2 = \left(\frac{1-A}{1+A}\right)^2 dt^2 + v\left[\left(\frac{1-A}{1+A}\right)^2 - (1+A)^4\right] dt\,dx - (1+A)^4(dx^2+dy^2+dz^2). \tag{2.6}$$

We follow the analysis of Aichelburg *et al.*[3] and interpret the metric in (2.6) to be due to a point mass m, moving with a *coordinate* velocity v. (In the curved space-time, we have to make the distinction between coordinate velocity and proper velocity. As we shall see, we will only be interested in the coordinate velocity of the source.)

For future reference, we shall also calculate the path of light rays in the (x, t) plane. For a light ray travelling along $y = z = 0$, the condition $ds^2 = 0$ will reduce to

$$g_{00}(dt)^2 + g_{11}(dx)^2 + 2g_{01}\,dx\,dt = 0, \tag{2.7}$$

or

$$\frac{dx}{dt} = -\frac{g_{01}}{g_{11}} \pm \left[\left(\frac{g_{01}}{g_{11}}\right)^2 - \frac{g_{00}}{g_{11}}\right]^{1/2}. \tag{2.8}$$

Here the two solutions refer to the two light rays—one travelling along the positive x-axis and the other along the negative x-axis. Substituting the value of the components of the metric tensor from (2.6) into the expression (2.8), we can express the "coordinate velocity", v_{ph}, of the photon as

$$v_{ph} \equiv \left(\frac{dx}{dt}\right)_{\pm} = v\left[1 - \frac{(1-A)^2}{(1+A)^6}\right] \pm \frac{1-A}{(1+A)^3}. \tag{2.9}$$

3. Fluctuations in the Metric

Classically a particle can have simultaneously well-defined position and velocity. For such a particle all the metric coefficients in (2.6) can simultaneously possess definite values. By measuring the components of g_{ik} (by rods and clocks) one can simultaneously determine the position and velocity of the source particle.

Such a determination is, of course, forbidden if the source is quantum mechanical. If an exact measurement of position is made, then the velocity will be completely uncertain and vice versa. It follows that all the metric coefficients in (2.6) cannot simultaneously possess definite values. Uncertainties in position and velocity will induce uncertainty in the metric coefficients.

Simple reasoning might suggest that the last conclusion is premature. Is it not possible that the position and velocity that appear in g_{ik} in (2.6) are actually expectation values? In other words, can we not interpret Einstein's equation as

$$G_{ik} = R_{ik} - \frac{1}{2} g_{ik} R = 8\pi G \langle \psi | T_{ik} | \psi \rangle . \tag{3.1}$$

If such were the case, uncertainty in the source will not lead to uncertainty in g_{ik}. However, it can be shown that (3.1) *cannot be the correct description of gravity.* There are many ways of obtaining this result; one version of the argument, due to Unruh,[4] is briefly reproduced in the Appendix, for the sake of completeness. There is also "experimental evidence" (albeit somewhat controversial!) against (3.1).[7] Therefore, one cannot replace source parameters by their expectation values in solutions to Einstein's equations. (It should be borne in mind that we are concerned with an *exact* description. Using $\langle T_{ik} \rangle$ as the source can be a very good approximation in many situations, especially in the physics of the early universe.[5])

If the metric g_{ik} for an arbitrary source T_{ik} is known, then the quantum commutation rules for T_{ik} will shed more light on the quantum nature of the problem. The non-linearity of Einstein's equations prevents progress in this direction. Therefore, we shall confine our attention in this paper to evaluating the quantum fluctuations in a particular *solution, viz.* metric of a moving point mass.

We now proceed to calculate the uncertainty relations between the physical components of the metric tensor. As is well known, the induced 3-metric on the space-like hypersurfaces, $\gamma_{\alpha\beta}$, represents the true dynamical degrees of freedom. [Einstein's equations involve the second time derivatives of only the induced 3-metric ($\gamma_{\alpha\beta}$) of the space-like hypersurface. Thus g_{00} and $g_{0\alpha}$ components of g_{ik} are not dynamical degrees of freedom in gravity, and can be "gauged away" by coordinate transformations (see Chap. 21 of Ref. 6).] This 3-metric is given in Ref. 8,

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha} g_{0\beta}}{g_{00}} . \tag{3.2}$$

To first order in v, we have (using Eqs. (3.2) and (2.6))

$$\gamma_{11} = (1 + A)^4 , \tag{3.3}$$

and also

$$\gamma^{11} = -(1 + A)^{-4} . \tag{3.4}$$

The momentum conjugate to γ^{11} is given by[6]

$$\dot{\gamma}^{11} = \frac{2mv}{r^2 (1 + A)^5} . \tag{3.5}$$

The quantum uncertainties in γ_{11} and $\dot{\gamma}^{11}$ can be easily related to the uncertainties in the source position Δx and the source velocity Δv, as

$$\Delta\gamma_{11} = \frac{2m}{\langle r\rangle^2}(1 + A)^3\Delta x,$$

$$\Delta\dot{\gamma}^{11} = \frac{2m}{\langle r\rangle^2}(1 + A)^{-5}\Delta v. \tag{3.6}$$

[After evaluating $\Delta\dot{\gamma}^{11}$ in terms of Δx and Δv, using (3.5), we have set $v = 0$, because we are interested in the quantum fluctuations around zero mean velocity; thus the $v\Delta x$ term makes no contribution to $\Delta\dot{\gamma}^{11}$.] Taking the product of the uncertainties and using the bound $\Delta x\Delta v \gtrsim m^{-1}\hbar$, we get

$$\Delta\gamma_{11}\Delta\dot{\gamma}^{11} = \left[\frac{2m}{\langle r\rangle^2}\right]^2 \frac{1}{(1 + \langle A\rangle)^2}(\Delta x)(\Delta v)$$

$$\gtrsim \left(\frac{2m}{\langle r\rangle^2}\right)^2 \frac{1}{(1 + \langle A\rangle)^2}\frac{\hbar}{m}. \tag{3.7}$$

The result in (3.7) represents the lower bound on the simultaneous measurement of γ_{11} and $\dot{\gamma}^{11}$. Similar relations exist for other components of $\gamma_{\alpha\beta}$ and $\dot{\gamma}^{\alpha\beta}$. It must be emphasised that (3.7) represents the "source-related" uncertainty of the field. Measurement of $\gamma_{\alpha\beta}$ would require setting up a suitable apparatus which may also lead to an uncertainty due to the essential quantum nature of the detector. The physical origin of these two kinds of uncertainty are, of course, different. For example, the source-related part (represented by (3.7)) depends on the position. The bound in (3.7) becomes weaker at large distances from the source. We note that these fluctuations are everywhere finite and have the maximum values at $r = \frac{1}{2}m$ (for the case of $v = 0$, $r = \frac{1}{2}m$ corresponds to the horizon). The maximum value is given by

$$\Delta\gamma_{11}\Delta\dot{\gamma}^{11}|_{r=(1/2)m} \gtrsim \frac{16L_p^{\,2}}{m^3}, \tag{3.8}$$

where L_p is the Planck length. Note that in the "$G = c = 1$" units m actually stands for (Gm/c^2). Thus the right-hand side actually has the form $(16\hbar/m)(Gm/c^2)^{-2}$, with the dimensions of $(\text{sec})^{-1}$. In our notation $\gamma_{\alpha\beta}$ and $\gamma^{\alpha\beta}$ are dimensionless and hence the left-hand side (3.8) also has the dimension of $(\text{sec})^{-1}$.

4. Fluctuations of the Light Cone

Gravity plays the dual role of "field" and "geometry". Quantum fluctuations of the metric tensor will, in general, lead to fluctuations in the light cone structure of the space-time. We shall now compute the light cone fluctuations in the Schwarzschild metric just outside the horizon.

We have computed in Sec. 2, the trajectory of a null ray travelling along the x-axis. The light cone is defined via the slope of this ray, i.e., by the "coordinate velocity" of the photons (see e.g. Eq. (2.9))

$$v_{ph} \equiv \left(\frac{dx}{dt}\right)_+ = v\left[1 - \frac{(1-A)^2}{(1+A)^6}\right] + \frac{1-A}{(1+A)^3}. \tag{4.1}$$

The range of fluctuations of the light cone is directly related to the uncertainty induced in v_{ph}.

Let the uncertainty in the position and the velocity of the point mass m about a mean velocity $\langle v\rangle = 0$ be Δx and Δv, respectively. This leads to an uncertainty in the value of dx/dt given by

$$\Delta v_{ph} = \Delta v\left[1 - \frac{(1-A)^2}{(1+A)^6}\right] + \Delta A\left[\frac{2(2-A)}{(1+A)^4}\right]. \tag{4.2}$$

Here $\Delta A = -(\Delta r)m/\langle r^2\rangle = -(\Delta x)m/\langle(2r^2)\rangle$, to first order in v, at $t = 0$. From the uncertainty relation between position and momentum in quantum mechanics we know that

$$\Delta x \Delta v \gtrsim \frac{\hbar}{m}. \tag{4.3}$$

In the best possible case $\Delta x = \hbar/m(\Delta v)$. Substituting (4.3) in (4.2), we get

$$\Delta v_{ph} = \Delta v\left[1 - \frac{(1-A)^2}{(1+A)^6}\right] + \frac{\hbar}{m\Delta v}\frac{m}{2r^2}\frac{2(2-A)}{(1+A)^4}. \tag{4.4}$$

The right-hand side achieves a minimum value when the two terms are equal. Therefore the minimum uncertainty in v_{ph} is

$$(\Delta v_{ph})_{\min} = \frac{2}{r}\left[\hbar\left\{1 - \frac{(1-A)^2}{(1+A)^6}\right\}\frac{(2-A)}{(1+A)^4}\right]^{1/2}. \tag{4.5}$$

As explained before, this source-related uncertainty depends on the point of observation. It is of interest to study the light-cone fluctuations at the event horizon, defined by $r = \frac{1}{2}m$, $A = 1$. From (4.5), we see that

$$(\Delta v_{ph})_{\text{horizon}} = \frac{L_p}{m} = (8\pi L_p)T_H, \tag{4.6}$$

where $L_p = (G\hbar/c^3)^{1/2} \sim 10^{-33}$ cm is the Planck length and T_H is the Hawking temperature. *In other words, the root mean square of the fluctuation in the coordinate velocity*

of the photon at the horizon is proportional to the Hawking temperature:

$$(\Delta v_{\text{photon}})_{\text{horizon}} \propto T_H . \tag{4.7}$$

Though this result is extremely suggestive, it is not easy to present a definitive physical interpretation. Since v_{ph} represents the *coordinate* velocity, coordinate transformations will alter its value in a well-defined and calculable way. It is not clear whether our result has any intrinsic coordinate dependence other than the one mentioned above. We emphasize that, (4.6) cannot be arrived at by "dimensional considerations". Since (Δv_{ph}) is dimension-free, any positive function $f(L_p/m)$ with $f(0) = 0$ would have done the job. The simplicity of (4.6) suggests a possible connection between quantum gravity and Hawking effect.

5. Conclusions

We have shown that the application of uncertainty principle of quantum mechanics to a point mass leads to non-trivial effects on the space-time geometry. It is interesting to note that the light-cone fluctuation at the horizon is proportional to its Hawking temperature, although a physical interpretation of this result is not yet available.

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Appendix

In Sec. 3 we pointed out that we cannot replace the source parameters by the expectation values in the Einstein's equation. To demonstrate this, Unruh[4] has considered a gedanken experiment. The experiment consists of two heavy masses connected by a spring. The masses are prevented from coming closer by a rod between them. The rod can be broken by a device triggered by the radioactive decay of an atom. There is also a Geiger counter to record the decay of the atom. When the atom decays, the counter clicks, the device is triggered which in turn breaks the rod and the masses come closer. This whole set-up is enclosed in a box and there is a Cavendish balance outside the box which responds to the position of the masses. In other words, when the masses are separated the balance is in a particular position. And when the masses are close to each other the balance is in a different position. Let us denote by $|1\rangle$ the state of our apparatus when the masses are apart and $|2\rangle$ the state of our apparatus when the masses are together. If our source parameters are to be replaced by the corresponding expectation values, the energy momentum tensor in the Einstein's

equation will be replaced by its expectation value. Initially the two masses will be apart. Hence the state of the system is denoted by $|1\rangle$ and the expectation value of the energy momentum tensor is given by

$$\langle T_{\mu v}\rangle_0 = \langle 1|T_{\mu v}|1\rangle. \tag{A.1}$$

As time passes the state of the system will evolve into a superposition of the two above-mentioned states, i.e., if the state of the system is $|\phi\rangle$, then

$$|\phi\rangle = \alpha|1\rangle + \beta|2\rangle, \tag{A.2}$$

where $|\alpha|^2 = e^{-\lambda t}$ and $|\beta|^2 = 1 - e^{-\lambda t}$. So at a time t, the expectation value of the energy momentum tensor is given by

$$\begin{aligned}\langle T_{\mu v}\rangle_t &= \langle\phi|T_{\mu v}|\phi\rangle \\ &= |\alpha|^2\langle 1|T_{\mu v}|1\rangle + \langle 2|T_{\mu v}|2\rangle|\beta|^2 \\ &\quad + 2\,\mathrm{Re}(\alpha^*\beta\langle 1|T_{\mu v}|2\rangle).\end{aligned} \tag{A.3}$$

If we assume that $|1\rangle$ and $|2\rangle$ are approximately the eigenstates of $T_{\mu v}$, the last term in (A.3) is very small. So we can write

$$\langle T_{\mu v}\rangle_t = e^{-\lambda t}\langle 1|T_{\mu v}|1\rangle + (1 - e^{-\lambda t})\langle 2|T_{\mu v}|2\rangle. \tag{A.4}$$

If in the Einstein's equation, $T_{\mu v}$ is to be replaced by $\langle T_{\mu v}\rangle_t$, then we would find that the position of the Cavendish balance will change *slowly and continuously.*

But this never occurs in practice. The balance always swings *suddenly* when the Geiger counter clicks. But if the balance does move suddenly, replacing the source parameters by their expectation values in Einstein's equation is not valid, then Eq. (3.1) is not a correct description of gravity.

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