

GLOBAL ANOMALIES IN CANONICAL GRAVITY

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Abstract

In this note we study the structure of diffeomorphism anomalies in $3 + 1$ canonical gravity coupled to a chiral massless fermion. We find that when the spatial manifold Σ is S^3 or a Lens space $L(p, q)$, the first homotopy group of the related diffeomorphism group can be nontrivial and hence the question of global anomalies becomes relevant. Here we show that for gravity coupled to $SU(2)$ chiral fermions, assuming the strong form of the Hatcher conjecture, $SU(2)$ -induced diffeomorphism anomalies do not occur whenever Σ is S^3 or a Lens space.

1 Introduction

The existence of a global gauge anomaly in a 4-dimensional Euclidean path-integral picture relies on the non-triviality of the group of large gauge transformations $\pi_0(\mathcal{G}^4)$. The $SU(2)$ anomaly can be attributed, first, to the non-triviality of $\pi_0(\mathcal{G}^4) = \pi_4(SU(2)) = \mathbb{Z}_2$, and then, to the existence of an odd number of zero modes for

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an appropriately constructed 5-dimensional Dirac operator [1]. In the Hamiltonian approach studied in [2], the $SU(2)$ anomaly comes from a non-trivial Berry phase picked up by the Dirac vacuum when it traverses a non-contractible loop, $\gamma \subset [\gamma] \in \pi_1(\mathcal{G}^3) = \mathbb{Z}_2$. This phase provides a global obstruction to implementing Gauss' Law, since \mathcal{G}^3 no longer has a well-defined action on the vacuum [3]. We note therefore that any gauge theory for which $\pi_1(\mathcal{G}^3)$ is non-trivial has the potential to be anomalous.

In the case of gravity, the 3-dimensional group of gauge transformations \mathcal{G}^3 is replaced by the diffeomorphism group $\text{Diff}(\Sigma)$ of the spatial 3-manifold Σ . Unlike gauge transformations, diffeomorphisms are not fibre-preserving automorphisms of the frame-bundle over Σ and the relation $\pi_i(\mathcal{G}^n) = \pi_{n+i}(G)$ between the homotopy groups of the gauge group G and the group of gauge transformations in n dimensions \mathcal{G}^n , is not valid. Since $\text{Diff}(\Sigma)$ is not known, even up-to homotopy for a generic Σ , it is not even possible to ask whether such a theory has the *potential* to be anomalous, let alone establishing one way or the other the existence of such an anomaly. Our attention to this problem was therefore first excited by noticing that work had been done in precisely this direction in [4] for Σ , a spherical space.

Using Hatcher's conjecture for spherical spaces ³ and the results of [4], it can easily be seen that $\pi_1(\text{Diff}(\Sigma))$ is non-trivial for Lens spaces. Hence we see that the *possibility* of a global anomaly does exist in 3 + 1 quantum gravity coupled to a massless chiral fermion, whenever the spatial manifold is a Lens space. We combine this result with the methods of [1, 2] to examine whether such anomalies exist or not. We are constrained for technical reasons to a theory coupled to an $SU(2)$ gauge field. Our work shows that such theories in fact do not possess $SU(2)$ gauge induced diffeomorphism anomalies.

The importance of chiral fermions in a fundamental theory is obvious and the existence of an anomaly is a serious indication that the theory is inconsistent with the known physical world. If indeed a spatial topology Σ exists for which the associated theory is anomalous, one can conclude that this topology must be "forbidden". In

³This conjecture states that $\text{Diff}(\Sigma)$ is of the same homotopy type as the isometry group of Σ [5].

other words it would provide a “selection rule” for spatial topology at such energies. It is in this spirit that we wish to pose our question. Of course these considerations would become inconsequential at energies at which the “frozen” topology picture breaks down, but presumably it is still valid in an effective sense. The presence of the “external” $SU(2)$ field as we mentioned earlier, is only a technical crutch at this point and its physical importance is not clear. We hope to extend our analysis to the pure gravity case in the near future.

2 Global Anomalies and 3 + 1 Gravity

In the standard canonical approach to quantum gravity, spacetime has the topology $\Sigma \times \mathbb{R}$ where the topology of space is “frozen”. If \mathcal{R} denotes the space of all Riemannian metrics on Σ , and Diff the group of all diffeomorphisms of Σ , then the true configuration space for pure gravity is the space of 3-geometries, \mathcal{R}/Diff . Thus Diff acts as the group of gauge transformations for such a theory. If Σ is closed, Diff contains non-pointed diffeomorphisms that do not act freely on \mathcal{R} . Thus \mathcal{R}/Diff does not have a manifold structure, but is an orbifold. Orbifold-quantization may be dealt with by removing the isolated singular points of \mathcal{R}/Diff and using appropriate self-adjoint extensions of the quantum operators (see for example [6] and references therein). On the other hand, if Σ is thought of as a one-point compactification of a manifold that is asymptotically like \mathbb{R}^3 then the relevant group is the group of “frame” fixing diffeomorphisms, $\text{Diff}_F \subset \text{Diff}$ which does act freely on \mathcal{R} so that \mathcal{R}/Diff is a manifold. In what follows we only consider the former case with Σ closed. Problems associated with orbifold quantisation do not affect our present analysis, however, since we are concerned with a more basic question: does there exist a well defined action of Diff on \mathcal{R} at all, free or otherwise.

The importance of Diff having a well defined action on \mathcal{R} becomes clear when we consider the momentum constraint (Gauss law constraint). Implementing this constraint on the physical states by the Dirac procedure makes them invariant under Diff_0 , the identity connected subgroup of Diff . However, if Diff_0 does not have a well defined action on \mathcal{R} then it cannot have a well-defined induced action on the states

$\psi : \mathcal{R} \rightarrow \mathcal{E}$ and this provides an obstruction to implementing the constraint [3]. In the case of pure gravity, such a problem does not exist since there is a well defined action of Diff_0 on \mathcal{R} [7]. However, when gravity is coupled to an odd number of chiral fermion fields, one needs to re-examine this question.

Using the ideas presented in [2], one can carry out the quantization of gravity coupled to chiral fermions in two stages. First, one quantizes the fermion field in a background metric, and second, the gravitational field. At the end of the first step, for every generic $q \in \mathcal{R}$, there exists a well-defined Dirac Fock space, \mathfrak{H}_q . The Fock bundle \mathcal{H} over \mathcal{R} can then be constructed from these \mathfrak{H}_q . The Fock vacuum \mathfrak{H}_q^0 (defined as the state in Fock space for which all the negative energy states are filled) will not be well defined at all points of \mathcal{R} , because of the existence of zero energy eigenvalues. It is the existence of such (non-generic) points that allows one to check for anomalies.

To complete the quantization, the momentum constraint needs to be implemented and Diff_0 must have a well defined action on \mathcal{H}^0 , the vacuum bundle over \mathcal{R} . Now, the spectrum of the Dirac Hamiltonian $\hat{\mathbb{H}}_q$ at any generic point q in \mathcal{R} is the same at all q' related to q by a diffeomorphism. However, $\mathfrak{H}_{q'}^0$ need not be the same as \mathfrak{H}_q^0 for all such q' . In particular, if Diff is not simply connected, the Fock bundle may “twist” along a non-trivial element of $\pi_1(\text{Diff})$. Since $\hat{\mathbb{H}}$ is a real operator, one can always choose a real Hilbert space on \mathcal{R} (c.f. [8] for meaning of real operator) and hence the phase reduces to ± 1 . The existence of a twist implies that Diff_0 does not have a well defined action on \mathcal{H}^0 which in turn points to a global obstruction in implementing the momentum constraint.

Clearly, if Diff is simply connected, by continuity arguments, such a twisting is not in principle possible, and one need not worry about the possibility of an anomaly. Thus, the first question we need to ask is whether Diff is simply connected or not. Below we see that the first homotopy group for both the full diffeomorphism group and the frame fixing subgroup is nontrivial for several Lens spaces.

Table 1: $\pi_1(\text{Diff}(L(p, q)))$

$L(p, q)$	Topology of $Isom$	$\pi_1(\text{Diff})$
S^3	$\mathbb{Z}_2 \times S^3 \times \mathbb{RP}^3$	\mathbb{Z}_2
\mathbb{RP}^3	$\mathbb{Z}_2 \times \mathbb{RP}^3 \times \mathbb{RP}^3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$q^2 \equiv 1 \pmod{p}; q \not\equiv \pm 1 \pmod{p}$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times S^1 \times S^1$	$\mathbb{Z} \oplus \mathbb{Z}$
$q^2 \equiv -1 \pmod{p}; p \neq 2$	$\mathbb{Z}_4 \times S^1 \times S^1$	$\mathbb{Z} \oplus \mathbb{Z}$
$q \equiv \pm 1 \pmod{p}; p \neq 2$	$\mathbb{Z}_2 \times S^1 \times \mathbb{RP}^3$	$\mathbb{Z} \oplus \mathbb{Z}_2$
remaining cases	$\mathbb{Z}_2 \times S^1 \times S^1$	$\mathbb{Z} \oplus \mathbb{Z}$

2.1 $\pi_1(\text{Diff})$ for Lens Spaces

The Hatcher conjecture, $\pi_n(Isom) \simeq \pi_n(\text{Diff})$, holds for both S^3 and \mathbb{RP}^3 [5, 9, 10]. For all other Lens spaces, a weaker form of the conjecture holds, i.e., $\pi_0(Isom) \simeq \pi_0(\text{Diff})$ and the number of generators of $\pi_1(Isom)$ is equal to the number of generators of $\pi_1(\text{Diff})$ [11–13]. Assuming the strong form of the conjecture for all Lens spaces $L(p, q)$, we simply read off $\pi_1(\text{Diff})$ from [4] where the topology of $Isom(L(p, q))$ has been found in all cases (see Table 1). We see that $\pi_1(\text{Diff})$ is indeed non-trivial for all $L(p, q)$, including S^3 , and hence these spaces are candidates for anomalous theories.

When the spatial manifold Σ is asymptotically flat and of the form $\mathbb{R}^3 \# L(p, q)$, the relevant group is $\text{Diff}_F(L(p, q))$, the frame fixing diffeomorphism group of the one-point compactification, $\bar{\Sigma} = L(p, q)$. We can reduce the following exact sequence from [4],

$$1 \rightarrow \pi_1(\text{Diff}_F) \rightarrow \pi_1(\text{Diff}^+) \rightarrow \pi_1(\Sigma) \times \mathbb{Z}_2 \rightarrow \pi_0(\text{Diff}_F) \xrightarrow{a} \pi_0(\text{Diff}^+) \rightarrow 1, \quad (1)$$

(where Diff^+ is the group of orientation preserving diffeomorphisms) to

$$1 \rightarrow \pi_1(\text{Diff}_F) \rightarrow \pi_1(\text{Diff}^+) \rightarrow \mathbb{Z}_p \oplus \mathbb{Z}_2 \rightarrow 1. \quad (2)$$

To do this, we have used the isomorphism $\pi_0(\text{Diff}_F) \simeq \pi_0(\text{Diff}^+)$ for Lens spaces [4], which implies that a is an isomorphism (since it is already surjective), and $\pi_1(\Sigma) = \mathbb{Z}_p$. From (2), $\pi_1(\text{Diff}^+)/\pi_1(\text{Diff}_F) = \mathbb{Z}_p \oplus \mathbb{Z}_2$. Diff^+ and Diff differ only

Table 2: $\pi_1(\text{Diff}^+(L(p, q)))$

$L(p, q)$	$\pi_1(\text{Diff}^+)/\pi_1(\text{Diff}_F) = \mathbb{Z}_p \oplus \mathbb{Z}_2$	$\pi_1(\text{Diff}_F)$
S^3	$\mathbb{Z}_2/\pi_1(\text{Diff}_F) = \mathbb{Z}_2$	1
\mathbb{RP}^3	$(\mathbb{Z}_2 \oplus \mathbb{Z}_2)/\pi_1(\text{Diff}_F) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$	1
$q^2 \equiv 1 \pmod{p}; q \not\equiv \pm 1 \pmod{p}$	$(\mathbb{Z} \oplus \mathbb{Z})/\pi_1(\text{Diff}_F) = \mathbb{Z}_p \oplus \mathbb{Z}_2$	$p\mathbb{Z} \oplus 2\mathbb{Z}$
$q^2 \equiv -1 \pmod{p}; p \neq 2$	$(\mathbb{Z} \oplus \mathbb{Z})/\pi_1(\text{Diff}_F) = \mathbb{Z}_p \oplus \mathbb{Z}_2$	$p\mathbb{Z} \oplus 2\mathbb{Z}$
$q \equiv \pm 1 \pmod{p}; p \neq 2$	$(\mathbb{Z} \oplus \mathbb{Z}_2)/\pi_1(\text{Diff}_F) = \mathbb{Z}_p \oplus \mathbb{Z}_2$	$p\mathbb{Z}$
remaining cases	$(\mathbb{Z} \oplus \mathbb{Z})/\pi_1(\text{Diff}_F) = \mathbb{Z}_p \oplus \mathbb{Z}_2$	$p\mathbb{Z} \oplus 2\mathbb{Z}$

by orientation reversing diffeomorphisms which are not identity connected, so that their first homotopy groups are isomorphic. Using the right most column of Table 1 we list $\pi_1(\text{Diff}_F)$ in Table 2.

Again, we see that $\pi_1(\text{Diff}^+)$ is non-trivial for all $L(p, q)$ except \mathbb{RP}^3 and S^3 , and hence are candidates for anomalous theories.

2.2 Determining the Berry Phase

Berry has shown that for a family of real Hamiltonians $\{\widehat{\mathbb{H}}_q\}$, the phase picked up by \mathfrak{H}_q^0 around a non-trivial loop $\gamma \subset [\gamma] \in \pi_1(\text{Diff}(\Sigma))$ is equal to $(-1)^k$, where k is the number of points in a disc \mathbb{D} , bounded by γ , at which $\widehat{\mathbb{H}}_q$ is degenerate [14, 15]. In our case, $\gamma \subset \text{Diff}$ parameterizes a family of real Dirac Hamiltonians and $\mathbb{D} \subset \mathcal{R}$. In the $SU(2)$ case, the number of degeneracies k was found by using the Atiyah-Singer mod-2 index theorem for an appropriately constructed real and antisymmetric 5-dimensional Dirac operator [2, 8]. The index theorem states that for such an operator, the number of zero modes is a mod-2 invariant which was found by Witten to be odd in the $SU(2)$ case [1]. In the Hamiltonian framework [2], the adiabatic approximation was used to show that the kernel of the 5-dimensional Dirac operator is related to the spectral flow of an associated 4-dimensional operator, whose kernel is in turn related to the spectral flow of $\widehat{\mathbb{H}}_q$ in \mathbb{D} and thence to the anomaly.

Unfortunately, in the case of gravity, a similarly constructed 5-dimensional Dirac

operator is not real and hence the number of its zero modes is *not* a mod-2 invariant. One might look for other ways to calculate the number of degeneracies of $\widehat{\mathbb{H}}_q$ in the disc, but here we side step the issue a little by posing a slightly different question.

Namely, we consider gravity coupled to an “external” $SU(2)$ gauge field and ask whether this theory possesses an $SU(2)$ -induced diffeomorphism anomaly. The existence of the external $SU(2)$ gauge field ensures that the associated 5-dimensional Dirac operator will be real. If a diffeomorphism anomaly exists then it is “induced” by the $SU(2)$ field. It is likely that even for non-trivial spatial topologies, a pure $SU(2)$ anomaly exists. Our concern in this paper however, will not be to determine whether the full theory is anomalous or not, but to ascertain, rather, the role played by the diffeomorphisms.

Let us consider the trivial $SU(2)$ bundle $SU(2) \times \Sigma$ over Σ . The configuration space for this theory is the Cartesian product, $\mathfrak{A} \times \mathcal{R}$, where \mathfrak{A} is the space of all 3-dimensional $SU(2)$ connections, and the full group of gauge transformations is $\mathcal{G}^3 \times \text{Diff}$. Note that since the $SU(2)$ bundle is trivial, connection, $A^0 \equiv 0 \in \mathfrak{A}$. Although the action of $\text{Diff} \in \mathcal{G}^3 \times \text{Diff}$ leaves A^0 fixed, this is not true for a generic $A \in \mathfrak{A}$.

Since we are only interested in the *diffeomorphism* anomaly, the non-contractible loop we consider lies only in $\text{Diff} \subset \mathcal{G}^3 \times \text{Diff}$. Let $\mathbb{D} \subset \mathcal{R}$ be the disc suspended by γ and let $(A(\alpha, \beta), q(\alpha, \beta))$ be the 2-parameter family of fields that make up \mathbb{D} where $\alpha \in [0, 1]$ and $\beta \in [0, 2\pi]$ (Figure 1). Let $(A(0, 0), q(0, 0))$ be the basepoint and $(A(1, \cdot), q(1, \cdot))$ the fields along γ (where the notation (α, \cdot) means that α is fixed and β is allowed to vary.) Since $\gamma \subset \text{Diff} \subset \mathcal{G}^3 \times \text{Diff}$, all points on it are purely gauge related to each other and the spectrum of $\widehat{\mathbb{H}}_q$ is invariant along γ . Note that $\widehat{\mathbb{H}}_q$ is generically non-degenerate so that starting from a non-degenerate basepoint, the spectrum along γ will be non-degenerate.

From the loop of 3-metrics $q(\alpha, \cdot) \subset \mathcal{R}$, we construct the Riemmanian 4-metric on $\Sigma \times S^1$,

$$g(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & q(\alpha, \cdot) \end{pmatrix}. \quad (3)$$

which is periodic in β , and from the loop of 3-dimensional gauge potentials $A(\alpha, \cdot)$,

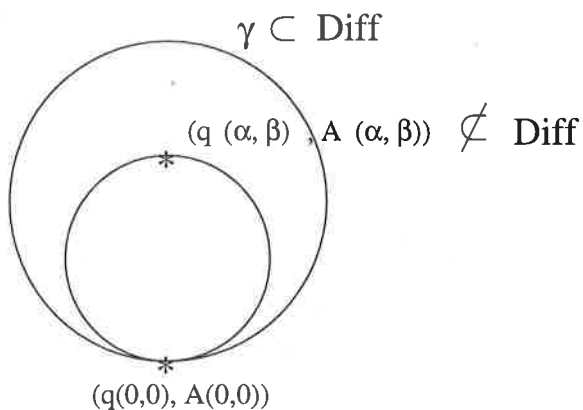


Figure 1: Loops in *Riem*

we construct the 4-dimensional gauge potential,

$$\mathcal{A}(\alpha) = \begin{pmatrix} 0 \\ A(\alpha, \cdot) \end{pmatrix}, \quad (4)$$

in the $\mathcal{A}_0 \equiv 0$ gauge. For the purpose of brevity, we will henceforth omit mentioning the $SU(2)$ gauge field, whenever its role is obvious.

Varying α , we get a 1-parameter family of 4-dimensional fields interpolating between $(\mathcal{A}(0), g(0))$ and $(\mathcal{A}(1), g(1))$, where

$$g(0) = \begin{pmatrix} 1 & 0 \\ 0 & q(0,0) \end{pmatrix} \quad \& \quad g(1) = \begin{pmatrix} 1 & 0 \\ 0 & q(1, \cdot) \end{pmatrix}. \quad (5)$$

and $\mathcal{A}(0)$ and $\mathcal{A}(1)$ are similarly fixed ⁴.

Since the spatial metrics $q(1, \cdot)$ are all diffeomorphism related to one another and to $q(0,0)$ by a sequence of “small” diffeomorphisms ⁵, this sequence “extends” to a single 4-dimensional diffeomorphism on $\Sigma \times S^1$. This is because any “small” diffeomorphism can be continuously deformed to the identity diffeomorphism; in fact because Diff_0 is a Lie group and thus has a manifold structure, this deformation can be made smooth. These deformations thus take $\Sigma \times S^1$ smoothly into itself so that

⁴ Note that $g(0)$ is “static” since it has no β dependence.

⁵ The loop γ lies in the identity connected component of Diff .

$g(0)$ and $g(1)$ are related by a 4-dimensional diffeomorphism. A 4-dimensional Dirac operator defined on $g(0)$ would therefore have the same spectrum as one defined on $g(1)$.

From the 1-parameter family $(\mathcal{A}(\alpha), g(\alpha))$, one can similarly construct a single 5-dimensional Riemannian metric G on $\Sigma \times S^1 \times I$,

$$G = \begin{pmatrix} 1 & 0 \\ 0 & g(\cdot) \end{pmatrix} \quad (6)$$

and a 5-dimensional gauge potential \mathbb{A} (again in the $\mathbb{A}_0 \equiv 0$ gauge.)

Now, consider a single loop of 3-metrics $q(\alpha, \cdot)$. As β varies from 0 to 2π , the spectrum of $\widehat{\mathbb{H}}_q$ might undergo a spectral flow, although it must return to itself. Since the energy spectrum of $\widehat{\mathbb{H}}_q$ consists of pairs $(E, -E)$, where E is real, a spectral flow leads to a cross-over point β_i at which $\widehat{\mathbb{H}}_{q(\beta_i)}$ is degenerate. Thus, the loop $q(\alpha, \cdot)$, will contain as many degenerate points β_i as its spectral flows.

In order to find the number of these spectral flows, consider the Euclidean Dirac operator $i\mathcal{D}^4$ on $\Sigma \times \mathbb{R}$ with the Riemannian 4-metric $g(\alpha)$ (3) and connection $\mathcal{A}(\alpha)$ (4), whose induced spatial metric on Σ varies adiabatically with “time” t along γ (as $t \rightarrow \pm\infty$, the induced metric on Σ goes to $q(0, 0)$.) Assuming the adiabatic approximation [1, 2], the zero mode equation $i\mathcal{D}^4\psi(t, x) = 0$ reduces to

$$\partial_t\psi(t, x) \approx -\gamma^4\gamma^i D_i\psi(t, x), \quad (7)$$

where x labels the spatial coordinates. Since $[\partial_t, \gamma^4\gamma^i D_i] \approx 0$, these modes can be separated into $\psi(t, x) = G(t)\chi(x)$ where $\chi(x)$ satisfies the eigenvalue equation, $\gamma^4\gamma^i D_i\chi(x) = E(t)\chi(x)$. Thus,

$$G(t) = G(0)e^{-\int E(t')dt'}. \quad (8)$$

Now, the spinor ψ on $\Sigma \times \mathbb{R}$ will not in general be normalisable. However, when $G(t)$ is normalisable, as $t \rightarrow -\infty$, $E(t)$ is negative and as $t \rightarrow +\infty$, $E(t)$ is positive. Thus, although the spectrum of $\gamma^4\gamma^i D_i$ is the same as $t \rightarrow \pm\infty$, there will be a spectral flow for every normalisable zero mode of $i\mathcal{D}^4$. The spectrum of $\gamma^4\gamma^i D_i$ coincides with that of $\widehat{\mathbb{H}}_q(t) = \gamma^0\gamma^i D_i$, so that a spectral flow for $\gamma^4\gamma^i D_i$ is a spectral flow for $\widehat{\mathbb{H}}_q(t)$.

In restricting to normalisable zero modes of $i\mathcal{D}^4$, one is effectively compactifying $\Sigma \times \mathbb{R}$ to $\Sigma \times S^1$, since $\psi \rightarrow 0$ as $t \rightarrow \pm\infty$, and Σ is “essentially” closed (i.e., it is not a one-point compactification). Since the fields on $\Sigma \times \mathbb{R}$ are periodic, the trivial $SU(2)$ bundle on $\Sigma \times \mathbb{R}$ is compactified to a trivial bundle over $\Sigma \times S^1$. The associated Dirac operators are then identical except for the existence of non-normalisable modes in the latter case. Thus, the number of spectral flows of $\widehat{\mathbb{H}}$ along the loop $q(\alpha, \cdot) =$ the number of zero modes of the associated $i\mathcal{D}^4$ on $\Sigma \times S^1$. However, there is no obvious method of finding the number of zero modes of $i\mathcal{D}^4$ at this stage.

Let $i\mathcal{D}^5(G, \mathbb{A})$ be a Dirac operator on $\Sigma \times S^1 \times \mathbb{R}$ with the Riemannian 5-metric G (6) whose induced metric on $\Sigma \times S^1$ varies adiabatically with another “time” parameter τ , from $g(0)$ to $g(1)$, defined in (5). Using the same adiabatic approximation technique as above for the zero mode equation $i\mathcal{D}^5(G, \mathbb{A})\Psi(\tau, t, x) = 0$, we get

$$\partial_\tau \Psi(\tau, t, x) = -\gamma^5 \gamma^\mu \mathcal{D}_\mu \Psi(\tau, t, x). \quad (9)$$

Again, the number of normalisable zero modes gives of $i\mathcal{D}^5(G, \mathbb{A}) =$ the number of spectral flows of $\gamma^5 \mathcal{D}^4$, which has the same spectrum as $i\mathcal{D}^4$. The eigenvalues of $i\mathcal{D}^4$ come in pairs, $(\lambda, -\lambda)$ so that a spectral flow leads to a cross-over point α_i at which $i\mathcal{D}^4(\alpha_i)$ has a pair of non-trivial zero modes. Since we are only interested in chiral fermions, only one of these zero modes is relevant to us. Now, a single zero mode of $i\mathcal{D}^4(\alpha_i)$ gives a single spectral flow of $\widehat{\mathbb{H}}(\alpha, \beta)$ along $q(\alpha, \cdot)$ and hence a single degeneracy along this loop at (α_i, β_j) . Thus, the number of normalisable zero modes of $i\mathcal{D}^5(G, \mathbb{A}) =$ number of degeneracies in \mathbb{D} . The logic of this construction should be clear by now.

Again, it would seem that the restriction to normalisable zero modes of $i\mathcal{D}^5(G, \mathbb{A})$ would allow the compactification $\Sigma \times S^1 \times \mathbb{R}$ to $\Sigma \times S^1 \times S^1$. However, the fields on $\Sigma \times S^1 \times \mathbb{R}$ are not periodic. In the pure $SU(2)$ case in [1], for example, a non-trivial \mathbb{Z}_2 gauge transformation between the connections at $t \rightarrow \pm\infty$ gives a non-trivial $SU(2)$ bundle over S^5 under compactification. In our case, however, since we are dealing with only the diffeomorphisms, such a \mathbb{Z}_2 twist in the $SU(2)$ gauge potentials

cannot appear, and the relevant compactification leads to a trivial $SU(2)$ bundle over $\Sigma \times S^1 \times S^1$. The non-periodicity in the metric on the other hand is only the statement that G cannot be deformed continuously to a metric constant in τ .

Now,

$$\mathcal{D}^5(G, \mathbb{A}) = (\gamma^M \otimes 1)(1 \nabla_M^5 \otimes 1 + i1 \otimes \mathbb{A}_M^a T^a), \quad (10)$$

where ∇^5 is the 5-dimensional gravitational connection compatible with G , γ^M is the pseudoreal representation of the generators of the Clifford algebra $\mathcal{C}(0, 5)$ [16], and T^a are a 2-dimensional pseudoreal representation of the generators of the $\mathfrak{su}(2)$ Lie algebra. Since the Clifford algebra has 4 spinor dimensions and the representation of the $\mathfrak{su}(2)$ Lie algebra is 2-dimensional, $\mathcal{D}^5(G, \mathbb{A})$ acts on an 8-dimensional spinor space.

Since $\mathcal{D}^5(G, \mathbb{A})$ is a real antisymmetric operator on a compact manifold, the Atiyah-Singer index theorem [8] tells us that the number of its zero modes are a mod-2 invariant. In other words, given the number n of zero modes for *any* $\mathcal{D}^5(G', \mathbb{A}')$ on $\Sigma \times S^1 \times S^1$, the number of zero modes of $\mathcal{D}^5(G, \mathbb{A}) \bmod 2 = n \bmod 2$, provided (G', \mathbb{A}') can be continuously deformed to (G, \mathbb{A}) . This is precisely what we need to determine the anomaly, since it only depends on the number of degeneracies mod 2 in \mathbb{D} . Thus our problem reduces to finding n for a suitable pair of fields (G', \mathbb{A}') on the trivial $SU(2)$ bundle over $\Sigma \times S^1 \times S^1$.

Since the $SU(2)$ bundle is trivial, an immediate choice for the connection is $\mathbb{A}'_M \equiv 0$. The zero mode equation reduces to,

$$\mathcal{D}^5(G', \mathbb{A}')\Psi = (\nabla^5 \otimes 1)(\Psi_{(1)} \otimes \Psi_{(2)}) = 0, \quad (11)$$

where $\Psi = (\Psi_{(1)} \otimes \Psi_{(2)})$ is a tensor product of a 4 and a 2-dimensional spinor. Thus, for any zero mode Ψ of $\mathcal{D}^5(G', \mathbb{A}')$, $\Psi_{(1)}$ must be a zero mode of ∇^5 , i.e., $\nabla^5 \Psi_{(1)} = 0$. Now, if $\{\Psi_{(1)}^i\}$ is the set of zero modes of ∇^5 then $\Psi = \Psi_{(1)}^i \otimes \Psi_{(2)}$ is a zero mode of $\mathcal{D}^5(G', \mathbb{A}')$ for *any* normalisable 2-spinor $\Psi_{(2)}$. On the other hand, if ∇^5 has *no* zero modes; then $\mathcal{D}^5(G', \mathbb{A}')$ too has no zero modes. Below, we employ suitable metrics on S^3 and the Lens spaces $L(p, q)$ for which ∇^5 has no zero modes. This small but crucial point helps us establish unambiguously that the number of

zero modes of $\mathcal{D}^5(G, \mathbb{A})$ is even, which \Rightarrow there are an even number of degeneracies in \mathbb{D} and therefore no anomaly.

2.3 Zero Modes of the Dirac operator

We now calculate the zero modes of ∇^5 on $S^3 \times S^1 \times S^1$ with the metric $ds^2 = d\tau^2 + dt^2 + d^2\Omega$, where τ and t parameterize the two S^1 's and $d^2\Omega = d^2\psi + \sin^2\psi(d^2\theta + \sin^2\theta d^2\psi)$ is the standard metric on S^3 .

We start with the 4-dimensional space $S^3 \times S^1$ with induced metric $dt^2 + d^2\Omega$. The Dirac operator ∇^4 on this space satisfies,

$$\nabla^4 \psi = (\gamma^4 \partial_t + \gamma^i \nabla_i^3) \psi = i\lambda \psi. \quad (12)$$

where the $\{\gamma^\mu\}$ are the chiral representation of the associated Clifford algebra,

$$\gamma^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \& \quad \gamma^m = \begin{pmatrix} 0 & i\sigma^m \\ -i\sigma^m & 0 \end{pmatrix}, \quad (13)$$

and ∇^3 is the 3-dimensional connection on S^3 . Decomposing ψ into the two 2-dimensional spinors, ϕ_\pm , $\psi = \begin{bmatrix} \phi_+ \\ \phi_- \end{bmatrix}$, we obtain the two coupled first-order differential equations,

$$\mathbf{1} \partial_t \phi_- + i\sigma^m \tilde{e}_m^i \tilde{\nabla}_i^3 \phi_- = i\mathbf{1} \lambda \phi_+ \quad (14)$$

$$\mathbf{1} \partial_t \phi_+ - i\sigma^m \tilde{e}_m^i \tilde{\nabla}_i^3 \phi_+ = i\mathbf{1} \lambda \phi_-, \quad (15)$$

where \tilde{e}_m^i and $\tilde{\nabla}_i^3$ are respectively, the triad and standard connection on S^3 . Or,

$$(\mathbf{1} \partial_t^2 + (\tilde{\nabla}^3)^2) \phi_\pm = -\mathbf{1} \lambda^2 \phi_\pm, \quad (16)$$

where $i\tilde{\nabla}^3 = \sigma^i \tilde{\nabla}_i^3$ is the 3-dimensional Dirac operator on S^3 . Since $[\partial_t^2, (\tilde{\nabla}^3)^2] = 0$, we can decompose $\phi_\pm = N(t)^\pm \chi_{(n\pm)}(\Omega)$, with $\chi_{(n\pm)}(\Omega)$ satisfying $\tilde{\nabla}^3 \chi_{(n\pm)}(\Omega) = i\mu^\pm \chi_{(n\pm)}(\Omega)$ and μ^\pm being real. From reference [17], we find that $\mu^\pm = \pm(n + \frac{3}{2})$, where n is a positive integer. Hence $\tilde{\nabla}^3$ has no zero modes on S^3 with the standard metric $d^2\Omega$. Equation (16) then reduces to

$$\partial_t^2 N(t) = -(\lambda^2 - \mu^2) N(t) = -l^2 N(t). \quad (17)$$

and since ∂_t^2 is an operator on S^1 , its eigenfunctions must therefore satisfy the periodicity condition $l \in \mathbb{Z}$. λ in turn satisfies

$$\lambda^2 = l^2 + \mu^2 = l^2 + \left(n + \frac{3}{2}\right)^2, \quad (18)$$

so that $|\lambda|_{min} = \frac{9}{4}$. Thus ∇^4 has no zero modes on $S^3 \times S^1$ with metric $ds^2 = dt^2 + d^2\Omega$.

For the 5-dimensional operator ∇^5 on $S^3 \times S^1 \times S^1$, the Clifford algebra includes

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (19)$$

Let

$$\nabla^5 \Psi = (\gamma^5 \partial_\tau + \nabla^4) \Psi = ik \Psi, \quad (20)$$

or,

$$(\partial_\tau^2 + (\nabla^4)^2) \Psi = -k^2 \Psi. \quad (21)$$

Since $[\partial_\tau^2, (\nabla^4)^2] = 0$, Ψ can be expressed as $\Psi = R(\tau)\psi(t, \Omega)$, where $(\nabla^4)^2 \psi(t, \Omega) = -\lambda_{n,l}^2 \psi$ and $\lambda_{n,l}^2 = l^2 + \left(n + \frac{3}{2}\right)^2$ (18). Now, $\partial_\tau^2 R(\tau) = -r^2 R(\tau)$, where $r \in \mathbb{Z}$, so that,

$$k^2 = r^2 + \lambda^2 = r^2 + l^2 + \left(n + \frac{3}{2}\right)^2. \quad (22)$$

Hence $k^2_{min} = \frac{9}{4}$ and we can conclude that ∇^5 too has no zero modes for the metric $ds^2 = d\tau^2 + dt^2 + d^2\Omega$, on $S^3 \times S^1 \times S^1$.

Although this calculation has been done for a specific 5-dimensional metric the mod 2 index theorem for $\mathcal{D}^5(G', \mathbb{A}')$ implies that the number of zero modes will only change by 2 if one moves continuously to another point in the space of fields on $S^3 \times S^1 \times S^1$. Thus the number of zero modes of $\mathcal{D}^5(G, \mathbb{A})$ (10) must be even. Putting this information into our previous arguments, we see that there are no diffeomorphism anomalies when the spatial manifold is S^3 .

This can be generalised to the case of the other Lens spaces by changing the 3-dimensional metric $d^2\Omega$ on S^3 to an appropriate one $d^2\Omega_L$ on $L(p, q)$, so that

$ds^2 = d\tau^2 + dt^2 + d^2\Omega_L$ is the metric on $L(p, q) \times S^1 \times S^1$. Repeating the above procedure, we again find the relation (18) where $i\mu$ is now replaced by the eigenvalue of $\tilde{\nabla}^3$ on $L(p, q)$ with metric $d^2\Omega_L$. If $|\mu|_{min} \neq 0$ then from (22) it is immediately obvious there are no zero modes for the associated ∇^5 in this case as well.

Now, heuristically, it can be argued that since $L(p, q)$ is the quotient manifold S^3/\mathbb{Z}_p , its Dirac operator must have a spectrum smaller than the spectrum on S^3 whenever $d^2\Omega_L$ is a “ \mathbb{Z}_p ” quotient of the standard metric $d^2\Omega$ on S^3 , and so doesn’t possess zero modes either.

The spectrum of the Dirac operator ∇^3 has in fact been explicitly calculated in reference [18], using the so-called “Berger” metrics. Fixing the parameter T that appears there to 1 gives us metrics $d^2\Omega_L$ on $L(p, q)$ which are effectively “quotient” metrics of $d^2\Omega$. The results from [18] show that the spectrum of ∇^3 indeed does not possess zero modes. Thus there are no diffeomorphism anomalies for $\Sigma = L(p, q)$ either.

3 Conclusions

We have established that for 3 + 1 quantum gravity, assuming the validity of the strong form of the Hatcher conjecture, $SU(2)$ -induced global diffeomorphism anomalies are absent whenever the spatial slice is diffeomorphic to S^3 or any Lens space $L(p, q)$. In the case of pure gravity, this question is still remains open. We briefly explore the implications of our result.

For a generic 3-manifold, there exists an infinity of inequivalent quantum sectors of 3+1 canonical gravity [19]. This feature is clearly not desirable in a fundamental theory and it is believed that higher energy effects like topology change would be necessary to resolve this problem. If the “frozen” topology theory is viewed as an effective description, however, one might hope that anomalies would provide a selection rule for the allowed spatial topology. Namely, those manifolds which lead to an infinite number of sectors would be “anomalous” (in the sense that the quantum theory on these manifolds possesses non-perturbative global diffeomorphism anomaly of the kind we have investigated.) We notice that the number of quantum

sectors is in fact finite when $\Sigma = L(p, q)$ and thus, in a sense, one doesn't need an anomaly to "rule" out these spatial topologies. This is in keeping with our result, as stated above.

It is generally believed that the full theory of quantum gravity will also be capable of describing processes involving topology change. In the low energy effective description of this full theory, topology change to manifolds that are "anomalous" is expected to be highly suppressed. Our result indicates that all Lens spaces are "well-behaved", and that topology change involving these spaces will not be forbidden.

Note added: After completion of our work we became aware of a paper by Chang and Soo [20] where the question of global anomalies in the self-dual formulation of gravity is discussed using generalised spin structures. The results obtained by these authors have no implications on ours, as they discuss anomalies in a specific formulation of canonical gravity (namely the self-dual formulation) and for the topology of spacetime being S^4 .

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