

# Global monopole as dual-vacuum solution in Kaluza-Klein spacetime

Naresh Dadhich\*

*Inter-University Centre for Astronomy & Astrophysics,  
Post Bag 4, Ganeshkhind, Pune - 411 007, India.*

L.K. Patel

*Department of Mathematics,  
Gujarat University, Ahmedabad 380 009, India.*

R. Tikekar

*Department of Mathematics,  
Sardar Patel University,  
Vallabh Vidyanagar 388 120, India.*

## Abstract

By application of the duality transformation, which implies interchange of active and passive electric parts of the Riemann curvature (equivalent to interchange of Ricci and Einstein tensors) it is shown that the global monopole solution in the Kaluza-Klein spacetime is dual to the corresponding vacuum solution. Further we also obtain solution dual to flat space which would in general describe a massive global monopole in 4-dimensional Euclidean space and would have massless limit analogous to the 4-dimensional dual-flat solution.

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\*E-mail : [nkd@iucaa.ernet.in](mailto:nkd@iucaa.ernet.in)

One of us has proposed a procedure [1-5] to construct spacetimes dual to the vacuum solutions in 4-dimensions. The procedure involves resolution of the Riemann curvature in analogy with the electromagnetics into electric and magnetic parts, defining a duality transformation between the two kinds of electric parts, and then applying it to the "effective" vacuum equation to get the dual-vacuum equation which would yield dual solution. By the effective equation, we would mean an equation that is less restrictive than the vacuum equation yet gives the same vacuum solution. The effective equation and its dual can be written in terms of energy density measured by a static observer and null and timelike convergence density [6]. The vacuum solution would follow from the condition of vanishing of energy and null convergence density while its dual from vanishing of timelike and null convergence density. Under the duality transformation energy density goes to timelike convergence density while null convergence density goes to itself. The effective equation would thus have a direct physical meaning.

It turns out that the dual solution to the Schwarzschild black hole describes a black hole with global monopole charge [1,7]. That is under the duality transformation black hole acquires a global monopole charge. This procedure works for the Reissner-Nordström and the NUT solutions [5,8]. In the 2+1 gravity, it leads to new classes of black hole spacetimes [9].

In this paper we wish to show that the same procedure would work for the 5-dimensional Kaluza-Klein spacetime, and the global monopole solution obtained by Banerjee et al [10] could be obtained as solution to the dual equation. In the Kaluza-Klein spacetime, analogue of the Schwarzschild solution was obtained by Gross and Perry [11] and Banerjee et al [10] have obtained a solution of a global monopole which amounts to putting a global monopole charge on the vacuum solution. In this paper we wish to show that the global monopole solution [10] is dual to the vacuum solution [11].

The Riemann curvature can be resolved similar to the electromagnetic field into electric and magnetic parts relative to a unit timelike vector. The electromagnetic parts would be given by second rank 3-tensors orthogonal to the resolving vector and they are defined as follows [1]:

$$E_{ac} = R_{abcd}u^b u^d, \tilde{E}_{ac} = *R *_{abcd} u^b u^d \quad (1)$$

$$H_{ac} = *R_{abcd}u^b u^d, \tilde{H}_{ac} = R *_{abcd} u^b u^d \quad (2)$$

where a star denotes the hodge dual in 5-dimensions and  $*R_{abcd}$  indicates the left dual (on first 2-form) and  $R*_{abcd}$  the right dual (on second 2-form). We shall call  $E_{ab}$  and  $\tilde{E}_{ab}$  as the active and passive electric parts. The former is anchored to non-gravitational energy distribution while the latter to the gravitational field energy. By the duality transformation we shall mean interchange of active and passive electric parts with magnetic parts remaining undisturbed. The electrogravity duality transformation is thus defined by [1],

$$E_{ab} \longleftrightarrow \tilde{E}_{ab}. \quad (3)$$

The contraction of Riemann tensor is the Ricci tensor and that of its double dual (when all but one indices are properly contracted [12]) is the Einstein tensor. This means that the above duality transformation will take Ricci to Einstein and vice-versa. Henceforth we shall directly work with the Ricci and the Einstein tensors rather than the electromagnetic components, manipulation of which would be quite complicated in 5-dimensions. Since the vacuum equation is invariant under the interchange of Einstein and Ricci tensors, it would be invariant under the electrogravity duality transformation. Thus we will have to make use of the effective equation which is duality-variant to obtain dual solution.

Let us consider the spherically symmetric metric in the Kaluza-Klein spacetime

$$ds^2 = A^2 dt^2 - B^2 dr^2 - M^2(d\theta^2 + \sin^2 \theta d\varphi^2) - C^2 d\psi^2 \quad (4)$$

where  $A, B, M, C$  are functions of  $r$  only. We denote the coordinates as  $x^0 = t, \dots, x^4 = \psi$ . For this metric the non-zero Ricci components in the basis frame are given by

$$\begin{aligned} R_{00} &= \frac{1}{B^2} \left[ -\frac{A''}{A} - \frac{A'}{A} \left( \frac{2M'}{M} - \frac{B'}{B} + \frac{C'}{C} \right) \right] \\ R_{11} &= \frac{1}{B^2} \left[ \frac{A''}{A} + \frac{C''}{C} + \frac{2M''}{M} - \frac{B'}{B} \left( \frac{2M'}{M} + \frac{C'}{C} + \frac{A'}{A} \right) \right] \\ R_{22} &= R_{33} = \frac{1}{B^2} \left[ \frac{M''}{M} + \frac{M'}{M} \left( \frac{A'}{A} + \frac{C'}{C} + \frac{M'}{M} - \frac{B'}{B} \right) - \frac{B^2}{M^2} \right] \end{aligned}$$

$$R_{44} = \frac{1}{B^2} \left[ \frac{C''}{C} + \frac{C'}{C} \left( \frac{2M'}{M} + \frac{A'}{A} - \frac{B'}{B} \right) \right] \quad (5)$$

Note that in the Kaluza-Klein spacetime the vacuum solution unlike the Schwarzschild solution of GR is not unique. If all the metric coefficients are assumed to be given by powers of the same function; i.e.  $A = f^a, B = f^b, C = f^c, M = r f^d, f = f(r)$ . Then it turns out that for the vacuum solution it is sufficient to solve

$$R_2^2 = 0, R_0^0 = R_1^1 = R_4^4 \quad (6)$$

rather than all the Ricci components zero. The solution of this set would imply vacuum (all Ricci zero). This is the effective vacuum equation. This gives the vacuum solution,

$$f = \left( 1 - \frac{2m}{r} \right)^{\frac{1}{2(d-b)}}, a + b + c = 0, a(a + b) + d(2b - d) = 0. \quad (7)$$

We thus have the vacuum solution [11] having two free dimensionless parameters.

Under the duality transformation Ricci goes to Einstein and hence the above effective equation (6) would go over to

$$G_2^2 = 0, G_0^0 = G_1^1 = G_4^4 \quad (8)$$

which means

$$R_0^0 = R_1^1 = R_4^4 = 0. \quad (9)$$

This equation would under the same assumption admit the solution

$$f = \left( k - \frac{2m}{r} \right)^{\frac{1}{2(d-b)}} \quad (10)$$

$a, b, c, d$  satisfying the same two relations as in (7). This is the solution obtained by Banerjee et al [10] for a global monopole charge in the Kaluza-Klein spacetime. The parameters,  $m$  is the core mass and  $k = 1 - \eta^2$  is due to the monopole field.

The metric for the dual solution would read as

$$ds^2 = f^{2a} dt^2 - f^{2b} dr^2 - r^2 f^{2d} (d\theta^2 + \sin^2\theta d\varphi^2) - f^{2c} d\psi^2 \quad (11)$$

where

$$f = (1 - \eta^2 - 2m/r)^{\frac{1}{2(d-b)}} \quad (12)$$

and  $a, b, c, d$  are constrained by the same two relations as given in (7) above.

This is the Banerjee et al solution [10] which is the Kaluza-Klein analogue of the Barriola-Vilenkin solution [7] (which follows when  $d = 0, a = 1$ ) for 4-dimensional global monopole charge. The Gross-Perry vacuum solution follows when  $\eta = 0$ . It may be noted that unlike GR, neither the vacuum nor the dual-vacuum solutions are unique.

It generates the stresses

$$T_0^0 = T_1^1 = T_4^4 = \frac{1 - k}{M^2} \quad (13)$$

which disappear to go over to vacuum when  $k = 1$ . The global monopole stresses take to this form asymptotically [10].

Let us show that the effective equation (6) and its dual (8) can be written in terms of energy and timelike and null convergence density. We define the energy density measured by a static observer by  $\rho = T_{ab}u^a u^b, u_a u^a = 1$ , and the timelike and null convergence density respectively by  $\rho_t = (T_{ab} - 1/2Tg_{ab})u^a u^b$  and  $\rho_n = T_{ab}v^a v^b, v_a v^a = 0$ . The convergence density is responsible for focussing of timelike and null congruence as indicated in the Raychaudhuri equation [13].

Note that in 5-dimensions,

$$R_{ab} = -8\pi(T_{ab} - 1/3Tg_{ab}). \quad (14)$$

The dual equation is equivalent to  $\rho_t = 0 = \rho_n$  where  $\rho_n$  refers to non-angular coordinate geodesics. Note that  $\rho_n = 0$  will imply  $G_0^0 = G_1^1 = G_4^4$  and  $\rho_t = 0$  will then give  $G_2^2 = 0$ , yielding the dual equation (8). Apply the duality transformation,  $R_{ab} \leftrightarrow G_{ab}$ , to the above relation (14), then  $\rho = 0 = \rho_n$  gives the effective vacuum equation (6).

Similar to dual-vacuum solution obtained above (11), we can also obtain dual-flat solution. In 5-dimensions, flat space is effectively given by the equation

$$H_{[ab]} = 0 = \tilde{E}_{ab}, \quad E_{ab} = \lambda(g_a^1 g_b^1 + g_a^4 g_b^4) \quad (15)$$

where

$$E_b^a = R^{a0}{}_{b0}, \quad \tilde{E}_b^a = -E_b^a + P_b^a + P_0^0 g_b^a - P_0^a g_b^0 - P_b^0 g_0^a \quad (16)$$

where

$$P_b^a = R_b^a - 1/4 R g_b^a$$

The above relation for passive electric part follows from the general expression for the double dual of the Riemann tensor for  $n \geq 5$  dimensions [12]. Here the Riemann curvature is resolved relative to hypersurface orthogonal unit timelike vector.

From the above equation, we shall have  $E_{22} = 0$  which would imply  $A = \text{const.}$  (because  $M$  cannot be constant) which would further imply  $E_{ab} = 0$  leading to flat space because  $\tilde{E}_{ab} = 0$ .

The dual-flat equation would follow by interchange of  $E_{ab}$  and  $\tilde{E}_{ab}$  and would read as

$$H_{[ab]} = 0 = E_{ab}, \quad \tilde{E}_{ab} = \lambda(g_a^1 g_b^1 + g_a^4 g_b^4). \quad (17)$$

Now  $E_{ab} = 0$  gives  $A = \text{const.} = 1$ ,  $\tilde{E}_{22} = 0$  and  $\tilde{E}_1^1 = \tilde{E}_4^4$  would give

$$R^{12}{}_{12} + R^{14}{}_{14} + R^{24}{}_{24} = 0$$

and

$$R^{12}{}_{12} = R^{24}{}_{24}.$$

These two equations solve to give

$$M' = BC \quad (18)$$

and

$$C^2 = k - \frac{2m}{M}. \quad (19)$$

This is the dual-flat solution which is of course non-flat. Here the function  $M$  remains undetermined. It would give rise to the following stresses,

$$T_0^0 = T_1^1 = T_4^4 = \frac{1-k}{M^2} \quad (20)$$

which are exactly the same as the one given above. For  $M = r$ , the dual-flat solution would read as

$$ds^2 = dt^2 - (k - 2m/r)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) - (k - 2m/r)d\psi^2 \quad (21)$$

which follows from the global monopole solution (11-12) when  $a = d = 0, c = -b = 1$ . Note that the dual-flat solution is the 4-dimensional Euclidean global monopole solution. Further when  $m = 0$ , the stress distribution would however remain undisturbed, we shall have zero-mass global monopole, which is the 5-dimensional analogue of the dual-flat solution [1,5]. Thus the dual-flat solution can have  $m$  non-zero as well as zero.

We have thus shown that the global monopole solution [10] is the dual-vacuum solution where the duality is defined by the transformation (3). That is interchange of active and passive electric parts which is equivalent to interchange of the Ricci and Einstein components (or  $\rho \leftrightarrow \rho_t, \rho_n \rightarrow \rho_n$ ) in the effective vacuum equation (6). It is interesting that dual-flat solution in general gives rise to an Euclidean global monopole which has the zero mass limit analogous to the 4-dimensional case. The duality transformation thus as in 4-dimensions puts on a global monopole charge on vacuum or flat spacetime.

Applications of the duality transformation have been considered for non-empty spacetime [14], stringy black holes with dilaton field [15] and 2+1 gravity [9].

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