

Gravoelectromagnetic approach to the gravitational Faraday rotation in stationary spacetimes

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Abstract

Using the $1 + 3$ formulation of stationary spacetimes we show, in the context of gravoelectromagnetism, that the plane of the polarization of light rays passing close to a black hole undergoes a rotation. We show that this rotation has the same integral form as the usual Faraday effect, i.e. it is proportional to the integral of the component of the gravomagnetic field along the propagation path. We apply this integral formula to calculate the Faraday rotation induced by the Kerr and NUT spaces using the quasi-Maxwell form of the vacuum Einstein equations.

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1 Introduction

It is a well known fact that the plane of the polarization of light rays passing through plasma in the presence of a magnetic field undergo a rotation which

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is called Faraday rotation (Faraday effect) [5]. One can show that a plane polarized wave is rotated through an angle $\Delta\theta$ given by

$$\Delta\theta = \frac{2\pi e^3}{m^2 c^2 \omega^2} \int_a^b n B_{\parallel} dl \quad (1)$$

where B_{\parallel} is the component of the magnetic field along the line of sight. It is also a well known consequence of the general relativity that light rays passing a massive object are bent towards it. Several authors have considered the gravitational effect on the polarization of light rays by analogy with the Faraday effect [2,4,9]. In particular they have considered the propagation of electromagnetic waves in the Kerr spacetime. In [4] the authors have used the Walker-Penrose constant to calculate this effect for a Kerr black hole. They have shown that in the weak field limit the rotation angle of the plane of the polarization is proportional to the line-of-sight component of the black hole's angular momentum at the third order. In what follows we will use the Landau-Lifshitz 1+3 splitting of stationary spacetimes and show that the gravitational Faraday rotation has the same integral form as the usual Faraday effect if one replaces the magnetic field with the gravomagnetic field of the spacetime under consideration. Having found this integral form, one can use the quasi-Maxwell form of the vacuum Einstein equations to calculate the effect much more easily. In particular we show that the gravitational Faraday rotation in NUT space is zero, a result which needs a lot of calculation if one uses the approach based on the Walker-Penrose constant.

2 1+3 formulation of stationary spacetimes (projection formalism)

Suppose that \mathcal{M} is the 4-dimensional manifold of a stationary spacetime with metric g_{ab} ¹ and $p \in \mathcal{M}$, then one can show that there is a 3-dimensional manifold Σ_3 defined invariantly by the smooth map [3]

$$\Psi(p) : \mathcal{M} \rightarrow \Sigma_3$$

where $\Psi(p)$ denotes the orbit of the timelike Killing vector ξ_t passing through p . The 3-space Σ_3 along with the 3-dimensional metric $\gamma_{\alpha\beta}$ (defined on Σ_3) is called the factor space $\frac{(\mathcal{M},g)}{G_1}$, where G_1 is the 1-dimensional group of

¹Note that the Roman indices run from 0 to 3 and Greek indices from 1 to 3.

transformations generated by ξ_t . One can use $\gamma_{\alpha\beta}$ to define differential operators on Σ_3 in the same way that g_{ab} defines differential operators on \mathcal{M} . For example the covariant derivative of a 3-vector \mathbf{A} is defined as follows

$$A_{;\beta}^{\alpha} = \partial_{\beta}A^{\alpha} + \lambda_{\gamma\beta}^{\alpha}A^{\gamma}$$

$$\alpha, \beta = 1, 2, 3$$

$$A_{\alpha;\beta} = \partial_{\beta}A_{\alpha} - \lambda_{\alpha\beta}^{\gamma}A_{\gamma}$$

where $\lambda_{\gamma\beta}^{\alpha}$ is the 3-dimensional Christoffel symbol constructed from the components of $\gamma_{\alpha\beta}$ in the following way

$$\lambda_{\mu\nu}^{\sigma} = \frac{1}{2}\gamma^{\sigma\eta}(\partial_{\nu}\gamma_{\eta\mu} + \partial_{\mu}\gamma_{\eta\nu} - \partial_{\eta}\gamma_{\mu\nu})$$

It has been shown that the metric of a stationary spacetime can be written in the following form [6]

$$ds^2 = h(dx^0 - A_{\alpha}dx^{\alpha})^2 - dl^2 \quad (2)$$

where

$$A_{\alpha} \equiv g_{\alpha} = \frac{-g_{0\alpha}}{g_{00}}, \quad h \equiv g_{00}$$

and

$$dl^2 = \gamma_{\alpha\beta}dx^{\alpha}dx^{\beta} = (-g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}})dx^{\alpha}dx^{\beta}$$

is the spatial distance written in terms of the 3-dimensional metric $\gamma_{\alpha\beta}$ of Σ_3 . Using this formulation for a stationary spacetime one can write the vacuum Einstein equations in the following quasi-Maxwell form [7]

$$\text{div } \mathbf{B}_g = 0 \quad (3)$$

$$\text{Curl } \mathbf{E}_g = 0 \quad (4)$$

$$\text{div } \mathbf{E}_g = - \left(\frac{1}{2}hB_g^2 + E_g^2 \right) \quad (5a)$$

$$\text{Curl } (\sqrt{h}\mathbf{B}_g) = 2\mathbf{E}_g \times \sqrt{h}\mathbf{B}_g \quad (5b)$$

$$P^{\alpha\beta} = E_g^{\alpha;\beta} + \frac{1}{h}(B_g^{\alpha}B_g^{\beta} - B_g^2\gamma^{\alpha\beta}) + E_g^{\alpha}E_g^{\beta} \quad (6)$$

where the gravitational fields are

$$\mathbf{E}_g = -\frac{1}{2} \frac{\nabla h}{h} \quad (7)$$

$$\mathbf{B}_g = \text{Curl } \mathbf{A}. \quad (8)$$

and $P^{\alpha\beta}$ is the 3-dimensional Ricci Tensor constructed from the metric $\gamma^{\alpha\beta}$. Note that all operations in these equations are defined in the 3-dimensional space with metric $\gamma_{\alpha\beta}$. Note that using the timelike Killing vector of the spacetime one can define the above gravoelectromagnetic fields in the following covariant forms

$$E_g^b = -\frac{1}{2} \frac{(\xi^a \xi_a)^{;b}}{|\xi|^2}$$

$$B_g^b = -\frac{1}{2} |\xi| \xi^a \varepsilon_a^{bcd} \left[\left(\frac{\xi_d}{|\xi|^2} \right)_{;c} - \left(\frac{\xi_c}{|\xi|^2} \right)_{;d} \right]$$

where ε_a^{bcd} is the 4-dimensional antisymmetric tensor and " ; " denotes covariant differentiation.

3 Derivation of the gravitational Faraday rotation

We use the analogy with the flat spacetime and take the plane of the polarization of an electromagnetic wave to consist of two 3-vectors \mathbf{k} and \mathbf{f} , the wave vector and the polarization vector respectively. The 4-vectors corresponding to these two 3-vectors have the following relations

$$k^a k_a = 0 \quad f^a f_a = 0 \quad k^a f_a = 1 \quad a = 0, 1, 2, 3 \quad (9)$$

Both of these 4-vectors are parallelly transported along null geodesics [6] i.e.

$$\frac{\partial k^a}{\partial \tau} + \Gamma_{mn}^a k^n k^m = 0$$

$$\frac{\partial f^a}{\partial \tau} + \Gamma_{mn}^a f^n k^m = 0$$

Concentrating on the spatial part of these two equations we have

$$\frac{\partial k^\alpha}{\partial \tau} + \Gamma_{mn}^\alpha k^n k^m = 0 \quad (10)$$

$$\frac{\partial f^\alpha}{\partial \tau} + \Gamma_{mn}^\alpha f^n k^m = 0 \quad (11)$$

Now we try to write these two equations in terms of the 3-dimensional quantities defined on Σ_3 . From equation (10) we have

$$\frac{\partial k^\alpha}{\partial \tau} = -\Gamma_{00}^\alpha (k^0)^2 - 2\Gamma_{0\beta}^\alpha k^0 k^\beta - \Gamma_{\beta\gamma}^\alpha k^\beta k^\gamma$$

to calculate this we need the following components of the christoffel symbol [6]

$$\begin{aligned}\Gamma_{00}^\alpha &= \frac{1}{2}h_{;\alpha}^\alpha \\ \Gamma_{0\beta}^\alpha &= \frac{h}{2}(g_{;\beta}^\alpha - g_\beta^{i\alpha}) - \frac{1}{2}g_\beta h_{;\alpha}^\alpha \\ \Gamma_{\beta\gamma}^\alpha &= \lambda_{\beta\gamma}^\alpha + \frac{h}{2}[g_\beta(g_\gamma^{i\alpha} - g_{;\gamma}^\alpha) + g_\gamma(g_\beta^{i\alpha} - g_{;\beta}^\alpha)] + \frac{1}{2}g_\beta g_\gamma h_{;\alpha}^\alpha\end{aligned}\quad (12)$$

Substituting the above equations into equation (10) we have

$${}^3\nabla_{\mathbf{k}}\mathbf{k} = -k_0(\mathbf{B}_g \times \mathbf{k}) + \frac{k_0^2}{h}\mathbf{E}_g \quad (13)$$

where

$${}^3\nabla_{\mathbf{k}}k^\alpha = \frac{\partial k^\alpha}{\partial \tau} + \lambda_{\beta\gamma}^\alpha k^\beta k^\gamma$$

is the 3-dimensional analogue of the parallel transport of k along itself in Σ_3 , and we have used equations (7) and (8) and the fact that

$$k^0 = \frac{k_0}{h} + \mathbf{g} \cdot \mathbf{k}$$

Now in the same way we write equation (11) in terms of 3-dimensional quantities defined on Σ_3 but before doing so we note the following simplifying fact. From equation (9) one can see that the polarization vector is known up to a constant multiple of the wave vector i.e. both f and $f' = f + \lambda k$ satisfy equation (9). This shows that there is a kind of gauge freedom in choosing f which enables one to put $f_0 = 0$ without loss of generality.² So from equation (11) we have

$$\frac{\partial f^\alpha}{\partial \tau} = -\Gamma_{00}^\alpha k^0 f^0 - \Gamma_{0\beta}^\alpha (f^0 k^\beta + k^0 f^\beta) - \Gamma_{\beta\gamma}^\alpha k^\beta f^\gamma$$

Substituting from equations (12) and putting $f_0 = 0$ one finds

$${}^3\nabla_{\mathbf{k}}\mathbf{f} = -\frac{1}{2}k_0(\mathbf{B}_g \times \mathbf{f}) \quad (14)$$

²This choice corresponds to $\lambda = -\frac{f_0}{k_0}$.

To be able to interpret equations (13) and (14) we need to perform another calculation. Using the facts that $\mathbf{f} \cdot \mathbf{f} = 1$ and $\mathbf{f} \cdot \mathbf{k} = 0$ one can show that ³

$$-\frac{1}{2}k_0(\mathbf{B}_g \times \mathbf{k}) = \frac{k_0^2}{h}(\mathbf{E}_g \cdot \mathbf{f})\mathbf{f} - \frac{1}{2}(\mathbf{B} \cdot \mathbf{f}_g)\mathbf{f} \times \mathbf{k}$$

Using the above equation one can write equations (13) and (14) in the following forms

$${}^3\nabla_{\mathbf{k}}\mathbf{k} = -\frac{1}{2}k_0[\mathbf{B}_g - \frac{1}{2}(\mathbf{B} \cdot \mathbf{f}_g)\mathbf{f} + |\mathbf{k}|(\mathbf{E}_g \cdot \mathbf{n})\mathbf{f}] \times \mathbf{k} + (\mathbf{E}_g \cdot \mathbf{k})\mathbf{k} \quad (15)$$

$${}^3\nabla_{\mathbf{k}}\mathbf{f} = -\frac{1}{2}k_0[\mathbf{B}_g - \frac{1}{2}(\mathbf{f} \cdot \mathbf{B}_g)\mathbf{f} + |\mathbf{k}|(\mathbf{E}_g \cdot \mathbf{n})\mathbf{f}] \times \mathbf{f} \quad (16)$$

where $\mathbf{n} = \mathbf{f} \times \hat{\mathbf{k}}$ is the unit vector normal to the polarization plane⁴. One can write equations (15) and (16) in the following closed forms

$${}^3\nabla_{\mathbf{k}}\mathbf{k} = \mathbf{L} \times \mathbf{k} + (\mathbf{E}_g \cdot \mathbf{k})\mathbf{k} \quad (18a)$$

$${}^3\nabla_{\mathbf{k}}\mathbf{f} = \mathbf{L} \times \mathbf{f} \quad (18b)$$

where

$$\mathbf{L} = -\frac{1}{2}k_0[\mathbf{B}_g - \frac{1}{2}(\mathbf{B} \cdot \mathbf{f}_g)\mathbf{f} + |\mathbf{k}|(\mathbf{E}_g \cdot \mathbf{n})\mathbf{f}] \quad (19)$$

If we had only the second term in the RHS of equation (18a) that would have meant, by comparison with the 4-dimensional definition of the parallel transport, that the 3-dimensional vector \mathbf{k} is parallelly transported along the projection of the null geodesic in Σ_3 space. But the appearance of the first term shows that \mathbf{k} has also been rotated by an angular velocity \mathbf{L} . The same rotation happens to the polarization vector \mathbf{f} as can be seen from equation (18b). Therefore the combination of these two equations leads us to the fact that the polarization plane has rotated by angular velocity \mathbf{L} along the

³We have also used the relation

$$\frac{1}{2}k_0(\mathbf{B}_g \times \mathbf{f}) \cdot \mathbf{k} = -\frac{k_0^2}{h}\mathbf{E}_g \cdot \mathbf{f}$$

which can be found by considering the fact that

$${}^3\nabla(\mathbf{k} \cdot \mathbf{f}) = ({}^3\nabla \mathbf{k}) \cdot \mathbf{f} + \mathbf{k} \cdot ({}^3\nabla \mathbf{f}) = 0.$$

⁴The same relations were also found by Fayos and Llosa [2] where they have not used the unit vector \mathbf{n} that we have used here.

projected null geodesic. What we are interested in is the angle of rotation around the tangent vector $\hat{\mathbf{k}}$ along the path between the source and the observer. So we have

$$\Omega = \int_{sou.}^{obs.} \mathbf{L} \cdot \hat{\mathbf{k}} \, d\tau$$

Substituting for \mathbf{L} from (19) we have

$$\Omega = -\frac{1}{2} \int_{sou.}^{obs.} k_0 \mathbf{B}_g \cdot \hat{\mathbf{k}} \, d\tau \quad (19a)$$

where we used the fact that $\mathbf{f} \cdot \mathbf{k} = 0$, which follows from equation (9) and the gauge freedom which allows the choice $f_0 = 0$. Now combining the following two equations

$$\begin{aligned} k_0 &= g_{0a} k^a = h(k^0 - g_\alpha k^\alpha) \\ k^a k_a &= 0 \equiv h(k^0 - g_\alpha k^\alpha)^2 - \gamma_{\alpha\beta} k^\alpha k^\beta = 0 \end{aligned}$$

we have

$$\frac{k_0^2}{h} - \gamma_{\alpha\beta} k^\alpha k^\beta = 0$$

or equivalently in terms of $k^\alpha = \frac{dx^\alpha}{d\tau}$

$$\frac{k_0^2}{h} = \left(\frac{dl}{d\tau} \right)^2 \quad (19b)$$

finally upon substitution of (19b) in (19a) and putting $\hat{\mathbf{k}} dl = d\mathbf{l}$ we find

$$\Omega = -\frac{1}{2} \int_{sou.}^{obs.} \sqrt{h} \mathbf{B}_g \cdot d\mathbf{l} \quad (20)$$

which has the same integral form as equation (1) i.e. the gravitational Faraday rotation is proportional to the integral of the component of the gravomagnetic field along the propagation path. But their main difference is the fact that the gravitational Faraday rotation, given by (20), is a purely geometrical effect while the usual Faraday effect, equation (1), depends on the frequency of the light ray. In the next two sections we will apply this formula to the cases of NUT and Kerr black holes.

4 Gravitational Faraday rotation in NUT space

There is no gravitational Faraday rotation induced by NUT space and the reason is as follows. Take a closed path \mathcal{C} around the NUT hole which consists of two paths (see figure 1). Path 1, a null geodesic which passes close to the black hole and path 2 so far away that the effect of the gravitational field (including the Faraday rotation) on the light rays is negligible (another reason that on path 2 there is no gravitational Faraday rotation is the fact that $\mathbf{B}_g \rightarrow 0$ as $r \rightarrow \infty$). Now using the Stokes theorem, one can write equation (20) in the following form

$$\Omega = -\frac{1}{2} \oint_{\mathcal{C}} (\sqrt{h} \mathbf{B}_g) \cdot d\mathbf{l} = -\frac{1}{2} \int_s \nabla \times (\sqrt{h} \mathbf{B}_g) \cdot d\mathbf{S}$$

and using equation (5b) we have

$$-\frac{1}{2} \int_1 (\sqrt{h} \mathbf{B}_g) \cdot d\mathbf{l} - \frac{1}{2} \int_2 (\sqrt{h} \mathbf{B}_g) \cdot d\mathbf{l} = - \int_s (\mathbf{E}_g \times \sqrt{h} \mathbf{B}_g) \cdot d\mathbf{S} \quad (21)$$

The second term in the LHS of the above equation is zero by construction. On the other hand for the NUT space we have⁵

$$\mathbf{E}_g = -\frac{1}{2} \partial_r (\ln f(r)) \hat{\mathbf{r}}$$

and

$$\mathbf{B}_g = \frac{2lf(r)^{\frac{1}{2}}}{r^2} \hat{\mathbf{r}}$$

Which together show that the RHS of equation (21) is also zero and therefore

$$-\frac{1}{2} \int_1 (\sqrt{h} \mathbf{B}_g) \cdot d\mathbf{l} = 0 \quad (22)$$

i.e. there is no Faraday effect on the light rays passing a NUT black hole close by. One can show that the same result can be obtained for the NUT space using the approach based on the Walker-Penrose constant. In this case one needs to take into account the simplifying fact that all the geodesics in NUT space including the null ones lie on spatial cones [7].

⁵We have used the following form of the NUT metric

$$ds^2 = f(r)(dt - 2l \cos \theta d\phi)^2 - f(r)^{-1} dr^2 - (r^2 + l^2)(d\theta^2 + \sin^2 \theta d\phi^2),$$

where

$$f(r) = 1 - \frac{2(mr + l^2)}{(r^2 + l^2)}$$

5 Gravitational Faraday rotation in Kerr metric

Faraday effect in Kerr metric has already been studied and it has been shown that despite previous claims [9], when light ray passes through the vacuum region outside rotating matter its polarization plane rotates [4]. In this section we will consider Two different cases:

- 1- When the orbit lies in the equatorial plane i.e. for $\theta = 0$.
- 2- A more general orbit which intersects the equatorial plane and is symmetric about it.

5.1 Orbits in the equatorial plane

In this case using the definitions of \mathbf{E}_g and \mathbf{B}_g and equation (21) one can see that the gravitational analogue of Poynting vector defined by $\mathbf{E}_g \times \sqrt{h}\mathbf{B}_g$ has only one component along the ϕ direction and therefore is normal to the plane of the orbit, which in turn leads to the fact that in this special case there is no gravitaional faraday effect on light rays.

5.2 A symmetric orbit about the equatorial plane

In this case we need to find the orbit and we will see that one just needs to find the orbit in the zeroth order in a/r and m/r (i.e. straight line approximation) which is done in appendix A.

Writing the Kerr metric in form (2) in Boyer-Lindquist coordinates one can see that

$$\mathbf{A} = A_\phi = \frac{2amr\sin^2\theta}{2mr - \rho^2}$$

from which we have

$$B_g^r = \frac{2amr\sin 2\theta[2mr - r^2 - a^2]}{\sqrt{\gamma}(2mr - r^2 - a^2\cos^2\theta)}$$

and

$$B_g^\theta = \frac{2ams\sin^2\theta(a^2\cos^2\theta - r^2)}{\sqrt{\gamma}(2mr - r^2 - a^2\cos^2\theta)^2}$$

where

$$\gamma = \det\gamma_{\alpha\beta} \quad \text{and} \quad \rho^2 = r^2 + a^2\cos^2\theta$$

Using the definition of the gravoelectric field given in (7) we have

$$E_g^r = \frac{\Delta m(a^2\cos^2\theta - r^2)}{\rho^4(\rho^2 - 2mr)}$$

and

$$E_g^\theta = \frac{rma^2 \sin 2\theta}{\rho^4(\rho^2 - 2mr)}$$

where $\Delta = r^2 + a^2 - 2mr$. Substituting the above fields in equation (21) and putting $\mu = \cos\theta$ we have

$$\Omega = - \int_s (\mathbf{E}_g \times \sqrt{h} \mathbf{B}_g)_\phi d\mathbf{S}^\phi = 2am^2 \int_{-\mu_o}^{\mu_o} \int_{r_{orb.}(\mu)}^{\infty} \frac{dr d\mu}{(r^2 + a^2\mu^2 - 2mr)^2} \quad (23)$$

where $r_{orb.}(\mu)$ is the equation (of the projection) of the orbit in the (r, θ) plane. To find the lowest order Faraday effect we calculate the above integral neglecting the a^2/r^2 and m/r terms in which case we have

$$\Omega = 2am^2 \int_{-\mu_o}^{\mu_o} \int_{r_{orb.}(\mu)}^{\infty} \frac{1}{r^4} dr d\mu = -\frac{4}{3} am^2 \int_0^{\mu_o} \frac{1}{r_{orb.}^3} d\mu$$

Using the (r, θ) equation of the orbit given in appendix A one can calculate the above integral which gives

$$\Omega = -\frac{4}{3} am^2 \int_{\mu_o=\sqrt{\eta}/r_{min}}^0 \left(1 - \frac{r_{min}^2}{\eta} \mu^2\right)^{3/2} d\mu = (1/4)\pi \cos\theta_o \frac{am^2}{r_{min}^3}$$

This expression is of the third order $\frac{am^2}{r_{min}^3}$ which is of the same order as the result given in [4].

6 Faraday rotation of gravitational waves

It has been shown that for gravitational waves of small amplitude propagating in a curved background, one can develop the geometric optics in such a way that the wave and polarization 4-vectors satisfy the same relations as given by equations (2) [8]. So it can easily be seen that all the main relations that we have found for the gravitational Faraday rotation of light rays are also applicable to gravitational waves of small amplitude.

Discussion

We have shown that using the 1+3 formulation of stationary spacetimes one can cast the gravitational Faraday rotation in exactly the same mathematical form as the usual Faraday effect i.e the gravitational Faraday rotation

is proportional to the gravomagnetic field of the spacetime along the propagation path of the light ray. One should note that the origins of these two effects are completely different. The usual Faraday effect originates from the interactions between the electrons in plasma in one hand with the electromagnetic field of the light ray and the external magnetic field on the other hand and therefore depends on the frequency of the light ray. But the gravitational Faraday effect is a purely geometrical one originating from the structure of the spacetime under consideration and it is normally attributed to the behaviour of the reference frames outside the spacetime of a rotating body. Using the quasi-Maxwell form of the vacuum Einstein equations we showed that there is an easy way to calculate the effect by transforming the line integral of \mathbf{B}_g to a surface integral of the gravitational analogue of the Poynting vector. More importantly the order of the effect can be seen without going through the detailed calculation (as in equation (23) for the Kerr case) and in some cases like NUT space just a simple observation reveals that there is no effect at all. It was also shown that there is gravitational Faraday rotation acting on the gravitational waves of small amplitude in exactly the same way as for the light rays.

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Appendix A

The equation governing the projection of the orbit in the (r, θ) plane for Kerr metric is given by [1]

$$\int^r \frac{dr}{\sqrt{r^4 + (a^2 - \xi^2 - \eta)r^2 + 2m[\eta + (\xi - a)^2]r - a^2\eta}} = \int^\theta \frac{d\theta}{\sqrt{\eta + a^2 \cos^2\theta - \xi^2 \cot^2\theta}} \quad (A1)$$

where ξ and η are the constants of the motion and we choose the case in which $\eta > 0$, which corresponds to the null geodesics which intersect the equatorial plane and are symmetric about it [1]. We perform the above

integrations for the case when $a/r \ll 1$ and $m/r \ll 1$ i.e. for weak deflections and indeed as we will see for a case in which there is no deflection in the (r, θ) plane. First we evaluate the LHS of the above equation which can be written in the following form (after discarding the small terms)

$$\int dr/r^2 \sqrt{1 - r_{min}^2/r^2} = (1/r_{min}) \arccos(r_{min}/r) \quad (A2)$$

where $r_{min} = \sqrt{\xi^2 + \eta}$ is the leading term (in the expansion) of the largest root of $r^4 + (a^2 - \xi^2 - \eta)r^2 + 2m[\eta + (\xi - a)^2]r - a^2\eta = 0$ for small deflection [4].

Now we evaluate the RHS of the equation (A1) in the same limit. This integral can be written in the following form

$$RHS = - \int \frac{d\mu}{\sqrt{\eta + \mu^2(a^2 - \xi^2 - \eta) - a^2\mu^4}}$$

Now using the fact that $a/r_{min} \ll 1$ and $r_{min} = \sqrt{\xi^2 + \eta}$ we can approximate and evaluate the above integral as follows

$$RHS = - \int \frac{d\mu}{\sqrt{\eta - \mu^2 r_{min}^2}} = -(1/r_{min}) \arcsin(\mu \sqrt{r_{min}^2/\eta}) \quad (A3)$$

Equating the equations (A2) and (A3) we will have

$$r_{orb} = \frac{r_{min}}{\sqrt{1 - (r_{min}^2/\eta) \cos^2 \theta}} \quad (A4)$$

which is the projection of the orbit in the (r, θ) plane for small deflections and in this case in fact no deflection because there is no term depending on m or a . As one can see $r \rightarrow \infty$ when $\cos \theta = \pm \frac{\sqrt{\eta}}{r_{min}}$ where plus and minus signs correspond to the position angles θ_o and θ_s of the observer and the source respectively.

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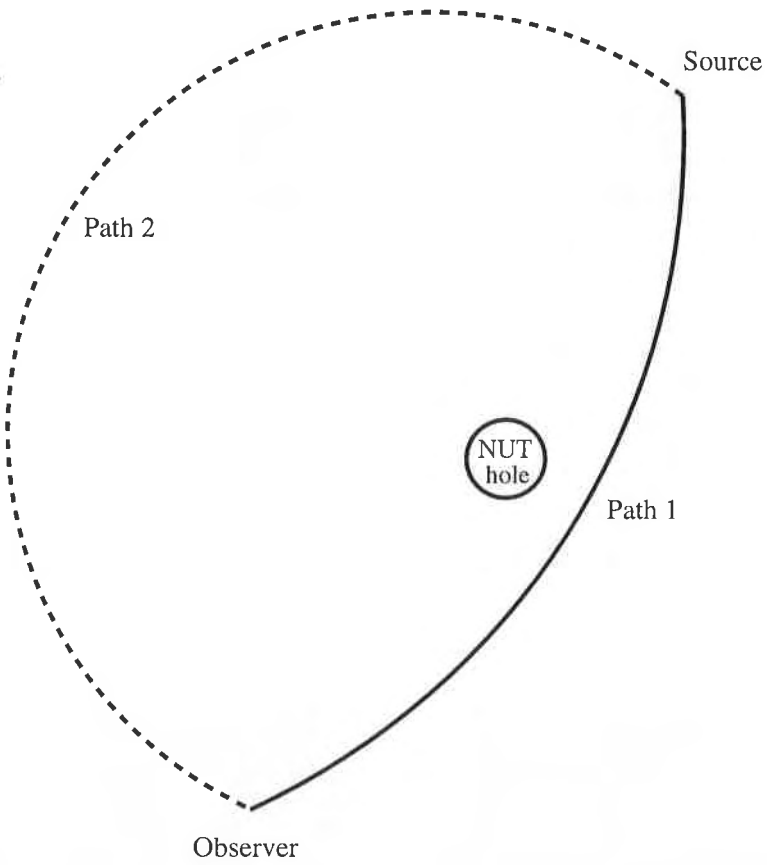


Figure 1: The NUT hole and a closed path \mathcal{C} around it.