where Q is defined in Eq. (5). Equations (27b), (24), and (28) yield

$$(G^{\mu}{}_{\nu} n_{\mu} v^{\nu})_b^{+} = 2y_b^{-2}(dM_b/d\tau)(d\tau/dT)^2 \quad (31)$$

Junction condition (26b) and Eq. (28) imply that[2]

$$8\pi p_b = -2Q_b/B_b \quad (32)$$

In the interior, with use of Eqs. (5) and (7), the boundary is obtained as a solution of (32). When heat flow ceases ($Q \to 0$), the boundary reverts to a zero-pressure surface with a Schwarzschild exterior.[3]

6. CONCLUSION

Exact collapse solutions with shear and heat flow have been generated. The boundary between the collapsing fluid and the Vaidya region is determined from Eq. (32). Since the generating technique is naive, it is unlikely that the equation of state, determined *a posteriori*, will be physically valid. However, there are arbitrary radial and time functions in the generating equations and generated solutions which can be constrained by imposing physical validity. The physical properties of the solutions generated here, and others, will be discussed elsewhere.

REFERENCES

1. Misner, C. W., and Sharp, D. H. (1965). *Phys. Lett.*, **15**, 279.
2. Glass, E. N. (1981). *Phys. Lett.*, **86A**, 351.
3. Vaidya, P. C. (1953). *Nature*, **171**, 260.
4. Lindquist, R. W., Schwartz, R. A., and Misner, C. W. (1965). *Phys. Rev.*, **137**, B1364.

[2] An error in Ref. [2], wherein the boundary was said to be at zero pressure, is corrected. There is a misprint in Eq. (6) of Ref. 2. In the last term, $2\dot{B}/B$ should read $2\ddot{B}/B$.
[3] The first match of an interior to the Vaidya metric is given in J. L. Synge (1957). *Proc. Roy. Irish Acad.* **59A**, 1. In this work an incoherent shell of radiation is joined to Vaidya.

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Conserved Quantities from Piecewise Killing Vectors

Tevian Dray[1,2] **and T. Padmanabhan**[1]

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In the presence of symmetries, conserved quantities can be obtained by contracting the stress-energy tensor with a Killing vector. We generalize this result to piecewise Killing vectors by giving sufficient conditions for the construction of an associated conserved quantity. A typical example, namely, two stationary space-times joined together in such a way that the resulting space-time is not stationary, is treated in detail.

1. INTRODUCTION

Given an n-dimensional space-time (M, g_{ab}) with Killing vector ξ^a, so that

$$\mathscr{L}_\xi g_{ab} \equiv \xi_{(a;b)} = 0 \quad (1)$$

then it is well known that contracting ξ^a with the stress-energy tensor T^{ab} yields a conserved quantity, i.e., the vector

$$P_a := T^{ab}\xi_b \quad (2)$$

is divergence-free. The associated integral conservation law is

$$\oint P^a d\Sigma_a = 0 \quad (3)$$

where the integral is over any closed $(n-1)$-dimensional hypersurface without boundary and where

$$d\Sigma_a = \varepsilon_{ab\cdots n} dx^b \cdots dx^n \quad (4a)$$

[1] Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India.
[2] Permanent address: Department of Mathematics, Oregon State University, Corvallis, Oregon 97331.

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where $\varepsilon_{ab\cdots n}$ denotes the Levi-Civita tensor (volume element) of (M, g_{ab}). It is important to realize that $d\Sigma_a$ is *not*, in general, the intrinsic volume element $d\Sigma$ of the hypersurface. Rather, at least if the hypersurface is not null, one has

$$d\Sigma_a = n_a d\Sigma \tag{4b}$$

where n_a is the unit normal to the hypersurface.

In Section 2 we generalize this result to piecewise Killing vectors, i.e., vectors which satisfy Killing's equation (1) almost everywhere. We give a sufficient condition for the vector analogous to (2) to be conserved despite the absence of a global symmetry. In Section 3 we present in detail the example which led us to this result, namely, glueing two Schwarzschild space-times together along a null cylinder representing a shell of massless dust [1]. (This result has recently been generalized [2] to the case of Reissner–Nordstrøm space-times joined by charged shells of matter.) The corresponding conserved quantity turns out to be the total energy of the shell and is just the difference of the Schwarzschild masses.

2. PIECEWISE KILLING VECTORS

We consider the situation where space-time is separated into two regions by an $(n-1)$-dimensional hypersurface N. This naturally arises when two space-times are glued together [3], in which case the differentiability of the metric could be as low as C^0 (piecewise C^2). Let u be a function such that $N = \{u = \alpha\}$, and denote by θ the usual step function, i.e.,

$$\begin{aligned} \theta &= 1 \qquad \text{for} \quad u > \alpha \\ \theta &= 0 \qquad \text{for} \quad u < \alpha \end{aligned} \tag{5}$$

A *piecewise Killing vector* is a vector which satisfies Killing's equation almost everywhere. If $\xi^a_\pm$ are Killing vectors on the regions $\{u > 0\}$ and $\{u < 0\}$, respectively, then

$$\xi^a = (1-\theta)\,\xi^a_- + \theta \xi^a_+ \tag{6}$$

is piecewise Killing. ξ^a is, in general, not itself a Killing vector, since

$$\xi_{(a;b)} = \Delta\xi_{(a} u_{,b)}\delta \tag{7a}$$

where δ denotes the derivative of θ, i.e., the Dirac delta function, and

$$\Delta\xi^a \equiv (\xi^a_+ - \xi^a_-)|_{u=\alpha} \tag{7b}$$

As an example of a piecewise Killing vector, consider the vector

$$\xi = [1-\theta(t)]\,\partial_t + \theta(t)\,\partial_\tau \tag{8a}$$

in Minkowski space, where ∂_t is the usual time translation and

$$\partial_\tau = x\partial_t + t\partial_x \tag{8b}$$

is the boost Killing vector associated with Rindler time. Similar considerations apply to the coordinate patches in de Sitter space-time.

If we now define P^a by (2), then

$$P^a{}_{;a} = T^{ab}\Delta\xi_b u_{,a}\delta \tag{9}$$

so that P^a is conserved if and only if

$$T^{ab}\Delta\xi_b u_{,a}|_{u=\alpha} = 0 \tag{10}$$

A natural condition on ξ^a at N is

$$\Delta\xi_a = k u_{,a} \qquad (\text{at } u = \alpha) \tag{11}$$

so that the tangential components of $\xi^a_\pm$ agree. Then condition (10) becomes

$$T^{uu}|_{u=\alpha} = 0 \tag{12}$$

which says that the energy density at N as seen by an observer transverse to N must vanish. (We use transverse to refer to the normal vector to N unless N is null, in which case we mean the dual null direction. When N is not spacelike, the phrases "energy density" and "observer" are, of course, inappropriate.) We have thus shown that (11) and (12) are sufficient conditions for the quantity P^a associated with the piecewise Killing vector ξ^a to be conserved.

3. EXAMPLE

We consider two Schwarzschild spacetimes with different masses joined along a null cylinder Σ representing a spherical shell of massless dust. The corresponding metric is [1]

$$ds^2 = \begin{cases} -\dfrac{32m^3}{r} e^{-r/2m} du\, dv + r^2\, d\Omega^2 & (u \leqslant \alpha) \\ -\dfrac{32M^3}{r} e^{-r/2M} dU\, dV + r^2\, d\Omega^2 & (u \geqslant \alpha) \end{cases} \tag{13a}$$

where $U = U(u)$, $U(\alpha) =: \beta$, $V = V(v)$, and

$$uv = -\left(\frac{r}{2m} - 1\right) e^{r/2m} \qquad (u \leqslant \alpha)$$

$$UV = -\left(\frac{r}{2M} - 1\right) e^{r/2M} \qquad (u \geqslant \alpha) \tag{13b}$$

Continuity of the metric requires that on N we have [1]

$$\frac{\alpha}{m} = \frac{\beta}{MU'(\alpha)} =: \gamma \tag{14}$$

where we have used (13b) to express V as a function of v at $u = \alpha$. The only nonzero component of the stress-energy tensor is

$$T_{uu} = \frac{\delta}{\gamma \pi r^2} (M - m) \tag{15}$$

Since the Schwarzschild timelike Killing vector is

$$\partial_t = \frac{v\partial_v - u\partial_u}{4m} \tag{16}$$

we obtain the piecewise Killing vector

$$\xi = (1 - \theta) \frac{v\partial_v - u\partial_u}{4m} + \theta \frac{V\partial_V - U\partial_U}{4M} \tag{17}$$

But (14) implies that on N

$$\frac{U\partial_U}{M} \equiv \frac{u\partial_u}{m} \qquad (u = \alpha) \tag{18}$$

so that

$$\Delta\xi \sim \partial_V \qquad (u = \alpha) \tag{19}$$

and (11) is satisfied, while (12) is trivially satisfied by virtue of (15) and the dual null form of the metric.

This leads to an integral conservation law of the form (3). Since the support of T_{ab} is on N, we obtain a conserved quantity Q by integrating (3) over *any* hypersurface intersecting N only once. We choose

$$\Sigma = \begin{cases} \{t = \text{constant}\} & (u \leqslant \alpha) \\ \{T = \text{constant}\} & (u \geqslant \alpha) \end{cases} \tag{20}$$

where the constants are chosen so that Σ is continuous and where t and T denote Schwarzschild time in the regions $u \leqslant \alpha$ and $u \geqslant \alpha$, respectively. We obtain the conserved quantity

$$-Q = \int_\Sigma [(1 - \theta)\, T^t_t + \theta T^T_T]\, r^2 \sin\theta \, dr \, d\theta \, d\varphi \tag{21}$$

Although this integral appears to involve the distributional product $\theta\delta$, it turns out that starting from (15) yields

$$T^T_T \equiv T^t_t = -\frac{\delta(r - r_0)}{4\pi r^2} (M - m) \tag{22}$$

where r_0 is the radius of the shell where Σ intersects N and where we have used

$$\frac{\partial u}{\partial r} = \frac{u}{4m} \frac{1}{1 - r/2m} \tag{23}$$

to replace $\delta(u - \alpha)$ by $\delta(r - r_0)$. (This argument needs to be slightly modified if $\alpha = 0$.) Inserting (22) into (21) finally leads to

$$Q = M - m \tag{23}$$

which shows that the energy of the shell is precisely the difference of the two Schwarzschild masses.

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REFERENCES

1. Dray, T., and 't Hooft, G. (1985). *Commun. Math. Phys.*, **99**, 613.
2. Dray, T., and Joshi, P. S. (1989). *Class. Quant. Grav.*, (to appear).
3. Clarke, C. J. S., and Dray, T. (1987). *Class. Quant. Grav.*, **4**, 265.