

# Critical properties of spherically symmetric accretion in a fractal medium

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## ABSTRACT

Spherically symmetric transonic accretion of a fractal medium has been studied in both the stationary and the dynamic regimes. The stationary transonic solution is greatly sensitive to infinitesimal deviations in the outer boundary condition, but the flow becomes transonic and stable, when its evolution is followed through time. The evolution towards transonicity is more pronounced for a fractal medium than what is it for a continuum. The dynamic approach also shows that there is a remarkable closeness between an equation of motion for a perturbation in the flow, and the metric of an analogue acoustic black hole. The stationary inflow solutions of a fractal medium are as much stable under the influence of linearised perturbations, as they are for the fluid continuum.

**Key words:** accretion, accretion discs – hydrodynamics – ISM: structure

## 1 INTRODUCTION

Accretion processes involve the non-self-gravitating flow dynamics of astrophysical matter under the external gravitational influence of a massive astrophysical object, like an ordinary star or a compact object (Frank et al. 2002). A paradigmatic model of an astrophysical accreting system is that of spherically symmetric infall on to a central accretor. Ever since the seminal work published by Bondi (1952), which effectively launched the subject in the form in which it is recognised today, the problem of spherically symmetric flows has been revisited time and again from various angles (Parker 1958, 1966; Axford & Newman 1967; Balazs 1972; Michel 1972; Mészáros 1975; Blumenthal & Mathews 1976; Mészáros & Silk 1977; Begelman 1978; Cowie et al. 1978; Stellingwerf & Buff 1978; Garlick 1979; Brinkmann 1980; Moncrief 1980; Petterson et al. 1980; Vitello 1984; Bonazzola et al. 1987, 1992; Theuns & David 1992; Kazhdan & Murzina 1994; Markovic 1995; Tsuribe et al. 1995; Titarchuk et al. 1996; Zampieri et al. 1996; Titarchuk et al. 1997; Kovalenko & Eremin 1998; Das 1999; Malec 1999; Toropin et al. 1999; Das 2000; Das & Sarkar 2001; Ray & Bhattacharjee 2002; Ray 2003; Das 2004; Ray & Bhattacharjee 2005; Gaithe 2006; Mandal et al. 2007; Roy 2007). This abiding appeal of the spherically symmetric model is explained by the fact that almost always it lends itself to an exact mathematical analysis, and in the process it allows a very clear insight to be had into the underlying physical principles.

Ease of mathematical manipulations, however, is not the only reason why spherically symmetric flows are regularly invoked in

accretion-related literature. The details of the physics of many astrophysical flows are, in fact, very faithfully described and understood with the help of this relatively simple model. Accretion of the interstellar medium (ISM) is a case in point.

While formal fluid dynamical equations in the Newtonian construct of space and time — which would involve a momentum balance equation (with gravity as an external driving force), the continuity equation and an equation of state — suffice to a great extent in shedding light on the accretion of the ISM, it must at the same time be recognised that the ISM is not entirely to be seen as a fluid continuum. In fact, for many purposes essential to grasping the underlying details, the ISM is believed to possess a self-similar hierarchical structure over several orders of magnitude in scale (Larson 1981; Falgarone et al. 1992; Heithausen et al. 1998). Direct H I absorption observations and interstellar scintillation measurements suggest that the structure extends down to a scale of 10 au (Crovisier et al. 1985; Langer et al. 1995; Faison et al. 1998) and possibly even to sub-au scales (Hill et al. 2005). Numerous theories have attempted to explain the origin, evolution and mass distribution of these clouds and it has been established, from both observations (Elmegreen & Falgarone 1996) and numerical simulations (Burkert et al. 1997; Klessen et al. 1998; Semelin & Combes 2000), that the interstellar medium has a clumpy hierarchical self-similar structure with a fractal dimension in three-dimensional space. The main reason for this is still not properly understood, but it can be the consequence of an underlying fractal geometry that may arise due to turbulent processes in the medium.

A theoretical study of these astrophysical systems — either a fluid continuum or a fractal structure — will necessitate the application of the mathematical principles of nonlinear dynamics. This is the principal objective of this work. The physical processes in

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a fractal medium have been analysed by fractional integration and differentiation (Zaslavsky 2002, and references therein). To do so, the fractal medium has had to be replaced by a continuous medium and the integrals on the network of the fractal medium has had to be approximated by fractional integrals (Ren et al. 2003). The interpretation of fractional integration is connected with fractional mass dimension (Mandelbrot 1983). Fractional integrals can be considered as integrals over fractional dimension space within a numerical factor (Tarasov 2004). This numerical factor has been chosen suitably to get the right dimension of any physical parameter and to derive the standard expression in the continuum limit. The isotropic and homogeneous nature of dimensionality has also been incorporated properly. All of these will give a self-consistent description of the hydrodynamics in a fractal medium (Roy 2007).

Once the hydrodynamic equivalence has been established, it has then been a fairly easy exercise to model the steady fractal flow like a simple first-order autonomous dynamical system (Strogatz 1994; Jordan & Smith 1999). This has made it possible to gain an understanding of the critical aspects of the stationary phase portrait of the fractal flow, especially the behaviour of the transonic solution. The critical point in the phase portrait has been shown to be a saddle point, and the transonic solution that has to pass through this point has been shown to be infinitely sensitive to the choice of a boundary condition. While this bodes ill for the feasibility of transonicity itself within the stationary framework, all steady global solutions (transonic or otherwise) have been found to be stable under a time-dependent linearised perturbation. An interesting fact that has come to light is that the necessary mathematical conditions, which include an equation of motion for the dynamic perturbation and its relevant boundary conditions, to argue for the stability of the steady fractal flows, have been found to be entirely identical to what has been reported earlier regarding the stability of continuous spherically symmetric inflows (Petterson et al. 1980; Theuns & David 1992). This similarity is fortunate and armed with this knowledge, it can be safely claimed that fractal flows are just as stable as continuous flows under the effects of small linearised perturbations.

Having said this, one will still have to confront the fact that the time-dependent perturbative analysis has done nothing to indicate the primacy of the transonic solution, and that the steady transonic inflow solution would not be possible without an infinite precision in prescribing a proper boundary condition. This obstacle has, however, been overcome by taking into consideration explicit dynamics in the flow system, and then evolving a physical flow through time, after having started with appropriate initial conditions. Transonicity becomes evident very soon, and it has been argued with a simplified analytical model in the “pressure-free” limit, that the guiding physical criterion to select the transonic solution is the one forwarded by Bondi (1952), i.e. the transonic solution will be chosen because it corresponds to a minimum energy configuration. While the “pressure-free” limit does not involve the fractal properties directly, it has been demonstrated through a numerical integration of the dynamic flow equations of the fractal medium that transonicity is very much the favoured mode of infall in this case too. And the most salient result to have emerged from this numerical study has been that transonic features becomes more pronounced, as the medium is more like a fractal.

It has already been mentioned that the perturbative treatment has been shown to give no clear-cut evidence to favour transonicity. Support for transonicity, however, has come indirectly from the perturbative angle too. The equation of motion for the propagation of the linearised perturbation has been shown to have subtle simi-

larities with an equation implying the metric of an acoustic black hole (Visser 1998). This hints at the fact that matter might cross the sonic horizon at the greatest possible rate, i.e. transonically, just as matter has to cross the event horizon of a black hole maximally.

## 2 THE EQUATIONS OF THE FLOW AND ITS FIXED POINTS

Considering the existence of a medium that has a fractal structure of mass dimensionality  $D = 3d$  (with  $d < 1$ ) embedded in a 3-dimensional space, the mass enclosed in a sphere of radius  $r$  can be written as (Roy 2007)

$$M_D = kr^D \sim \rho l_c^3 \left( \frac{r}{l_c} \right)^{3d}, \quad (1)$$

with  $D$  referring to the dimension,  $\rho$  to the constant density of the medium, and  $l_c$  to a characteristic inner length of the medium that can take an arbitrary value in the limit  $d \rightarrow 1$ . This is the scale below which the medium will be continuous. The fractional integrals are computed as integrals over fractional dimension space within a numerical factor. The fractional infinitesimal length for a medium with isotropic mass dimension will, therefore, be given by (Roy 2007)

$$d\bar{r} = \left( \frac{r}{l_c} \right)^{d-1} dr, \quad (2)$$

with the constant having been chosen to derive the standard expression in the limit  $d \rightarrow 1$ . It is to be noted that the infinitesimal area and volume elements in this “fractional continuous” medium of mass dimension  $D = 3d$  will be different, and hence the mass enclosed in a sphere of radius  $r$  for constant density  $\rho$  will be (Roy 2007)

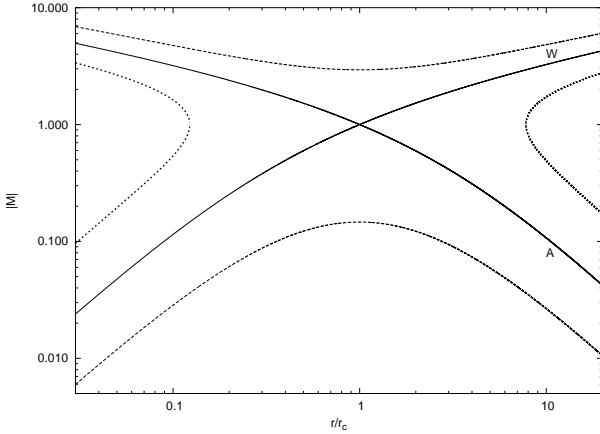
$$M_D = \int_V \rho d\bar{V} = \frac{4}{3} \pi \rho \left( \frac{l_c}{d} \right)^3 \left( \frac{r}{l_c} \right)^{3d} \sim r^D. \quad (3)$$

In this medium the inviscid Euler equation, describing the dynamics of the velocity field,  $v$ , can be expressed as (Roy 2007)

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} + \phi'(r) = 0, \quad (4)$$

where  $\phi(r)$  is the gravitational potential of the central accretor that drives the flow (with the prime denoting the spatial derivative of the potential). This is a local conservation law and, as it is to be expected, this has exactly the same form as that of the equation for the continuous medium. In the case of stellar accretion, the flow is driven by the Newtonian potential,  $\phi = -GM/r$ . On the other hand, frequently in studies of black hole accretion, it becomes convenient to dispense completely with the rigour of general relativity, and instead make use of an “effective” pseudo-Newtonian potential that will imitate general relativistic effects in the Newtonian construct of space and time (Paczynski & Wiita 1980; Nowak & Wagoner 1991; Artemova et al. 1996). The choice of a particular form of the potential will not affect the general arguments overmuch.

The pressure,  $p$ , is related to the local density,  $\rho$ , through a polytropic equation of state  $p = K\rho^\gamma$ , in which  $K$  is a constant, and  $\gamma$  is the polytropic exponent, whose admissible range is given by  $1 < \gamma < 5/3$ . This range is restricted by the isothermal and the adiabatic limits, respectively (Chandrasekhar 1939). The evolution of  $\rho$  is described by the equation of continuity (Roy 2007),



**Figure 1.** Integral solutions of the stationary fractal flow, driven by the Newtonian potential, for  $D = 2.55$  and  $\gamma = 1.4$ . The continuous curves, A and W represent “accretion” and “wind”, respectively. The fixed point is at  $r = r_c$  and  $|M| = 1$ . This point is a saddle point.

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^\alpha} \frac{\partial}{\partial r} (\rho v r^\alpha) = 0, \quad (5)$$

in which  $\alpha = 3d - 1$ .

The flow system will, therefore, be specified by equations (4) and (5), along with the polytropic equation of state. Of particular interest are the steady state solutions for the case where the fractal medium is at rest at a large distance from the accretor. Since transonic flows are of primary concern here, it is required that the static flow should evolve from  $v \rightarrow 0$  as  $r \rightarrow \infty$  (the outer boundary condition) to  $v > a(r)$  for small  $r$ , where  $a$  is the speed of sound, given by  $a^2 = \partial p / \partial \rho = \gamma K \rho^{\gamma-1}$ . The stationary state implies  $\partial v / \partial t = \partial \rho / \partial t = 0$ . Consequently, equations (4) and (5) will be reduced to their steady state forms as

$$v \frac{dv}{dr} + \frac{1}{\rho} \frac{dp}{dr} + \phi'(r) = 0 \quad (6)$$

and

$$\frac{d}{dr} (\rho v r^\alpha) = 0, \quad (7)$$

respectively. It is easily seen that equations (6) and (7) remain invariant under the transformation  $v \rightarrow -v$ , i.e. the mathematical problem for inflows ( $v < 0$ ) and outflows ( $v > 0$ ) is identical in the stationary state (Choudhuri 1999).

It is in principle possible to eliminate either  $v$  or  $\rho$  and solve for the other variable as a function of  $r$ . However, adopting a slightly different approach, it is possible to recast equations (6) and (7) in a combined form as

$$\frac{d}{dr} (v^2) = \frac{2v^2}{r} \left[ \frac{\alpha a^2 - r \phi'(r)}{v^2 - a^2} \right], \quad (8)$$

whose integral solutions have been shown in Fig. 1, with  $\phi(r) = -GM/r$ , and with the vertical axis scaled by the absolute value of the Mach number,  $M = v/a$ . The two continuous curves labelled “A” and “W” refer to accretion and wind, respectively. The critical points in the flow will be derived from the standard requirement that the flow solutions will have a finite gradient when they will cross the sonic horizon (where the bulk flow velocity exactly matches the speed of acoustic propagation), which will mean that both the numerator and the denominator of equation (8) will have to vanish simultaneously and non-trivially (Chakrabarti 1990, 1996). This can only happen when

$$v_c^2 = a_c^2 = \frac{r_c \phi'(r_c)}{\alpha}, \quad (9)$$

which gives the critical point (or the sonic point in this particular case) conditions, with the subscript “c” labelling the critical point values.

It is not a difficult exercise to integrate equation (6) and then transform the variable  $\rho$  in it to  $a$  with the help of the equation of state. This, with the critical point conditions as given by equation (9), will give a relation for fixing the critical point coordinates in terms of the flow parameters,  $\mathcal{E}$  (which is actually Bernoulli’s constant),  $\alpha$  and  $\gamma$  as

$$\left( \frac{\gamma + 1}{\gamma - 1} \right) \frac{r_c \phi'(r_c)}{2\alpha} + \phi(r_c) = \mathcal{E}. \quad (10)$$

The form of  $\phi(r)$  will obviously determine the number of the critical points, and for the Newtonian potential only one root of  $r_c$  will be obtained from equation (10). This root, for  $\mathcal{E}$  fixed by the outer boundary condition of the transonic inflow solution, will be given as

$$r_c = \left[ \frac{(\gamma + 1) - 2\alpha(\gamma - 1)}{2\alpha} \right] \frac{GM}{a_\infty^2}, \quad (11)$$

with  $a_\infty$  being the speed of sound at the outer boundary of the flow (Chakrabarti 1990, 1996), where the influence of gravity is negligibly weak.

It should be worth mentioning here that although with the choice of a pseudo-Newtonian potential multiple roots for  $r_c$  would result, practically speaking only one of these roots would be a physically meaningful critical point, through which an integral solution may pass. For spherically symmetric flows in the general relativistic framework, this issue has been discussed by Mandal et al. (2007).

### 3 THE FLOW AS AN AUTONOMOUS DYNAMICAL SYSTEM

So far the flow variables have been ascertained only at the critical points. Since the flow equations are in general nonlinear differential equations, short of carrying out a numerical integration, there is no completely rigorous analytical prescription for solving these differential equations to determine the global nature of the flow variables. Nevertheless, some analytical headway could be made after all by taking advantage of the fact that equation (8), which gives a complete description of the  $r - v^2$  phase portrait of the flow, is an autonomous first-order differential equation, and as such, could easily be recast into the mathematical form  $\dot{x} = X(x, y)$  and  $\dot{y} = Y(x, y)$ , which is that of the very familiar coupled first-order dynamical system (Strogatz 1994; Jordan & Smith 1999). Quite frequently for any nonlinear physical system, a linearised analytical study of the properties of the fixed points of a first-order dynamical system, affords a robust platform for carrying out an investigation to understand the global behaviour of integral solutions in the phase portrait.

And so it is that to investigate the nature of the critical point, equation (8) will have to be decomposed in terms of a mathematical parameter,  $\tau$ , to read as

$$\begin{aligned} \frac{d}{d\tau} (v^2) &= 2v^2 [\alpha a^2 - r \phi'(r)] \\ \frac{dr}{d\tau} &= r (v^2 - a^2), \end{aligned} \quad (12)$$

in both of which the parameter  $\tau$  does not make an explicit appearance in the right hand side, something of an especial advantage that derives from working with autonomous systems. This kind of parametrization is quite common in fluid dynamics (Bohr et al. 1993), and in accretion studies especially, this approach has been made before (Ray & Bhattacharjee 2002; Afshordi & Paczyński 2003; Chaudhury et al. 2006; Mandal et al. 2007; Goswami et al. 2007). Some earlier works in accretion had also made use of the general mathematical aspects of this approach (Matsumoto et al. 1984; Muchotrzeb-Czerny 1986; Abramowicz & Kato 1989). A further point that has to be noted is that the function  $a^2$  in the right hand side of equation (12) can be expressed entirely in terms of  $v^2$  and  $r$ , with the help of the equation of state and equation (7). This will exactly satisfy the criterion of a first-order autonomous dynamical system.

The next task would be to make a linearised approximation about the fixed point coordinates and extract a linear dynamical system out of equations (12). This will give a direct way to establish the nature of the critical points (or fixed points). Expanding about the fixed point values, a perturbation of the kind  $v^2 = v_c^2 + \delta v^2 = v_c^2(1 + \epsilon_1)$  and  $r = r_c + \delta r = r_c(1 + \epsilon_2)$  can be applied. Using the continuity equation and the equation of state, this perturbation scheme, when linearised, will also give  $\delta a/a_c^2 = -(\gamma - 1)(\epsilon_1 + 2\alpha\epsilon_2)/2$ . Applying this perturbative expansion on equation (12), and linearising in  $\epsilon_1$  and  $\epsilon_2$  will give,

$$\begin{aligned} \frac{d\epsilon_1}{d\tau} &= \alpha v_c^2 \left\{ -(\gamma - 1)\epsilon_1 \right. \\ &\quad \left. - 2 \left[ \alpha\gamma - \alpha + 1 + \frac{\phi''(r_c)r_c}{\phi'(r_c)} \right] \epsilon_2 \right\} \\ \frac{d\epsilon_2}{d\tau} &= \alpha v_c^2 \left[ \left( \frac{\gamma + 1}{2\alpha} \right) \epsilon_1 + (\gamma - 1)\epsilon_2 \right]. \end{aligned} \quad (13)$$

Using solutions of the type  $\epsilon_1 \sim \exp(\Omega\tau)$  and  $\epsilon_2 \sim \exp(\Omega\tau)$  in equations (13), the eigenvalues of the stability matrix associated with the critical points will be derived as

$$\Omega^2 = \alpha a_c^4 \left[ (2\alpha - 1) - \gamma(2\alpha + 1) - (\gamma + 1)r_c \frac{\phi''(r_c)}{\phi'(r_c)} \right] \quad (14)$$

with  $a_c$  itself being a function of  $r_c$ , as given by equation (9).

Once the position of a critical point,  $r_c$ , has become known from equation (10), it is then quite easy to determine the nature of that critical point by using  $r_c$  in equation (14). Since  $r_c$  is a function of  $\mathcal{E}$  and  $\gamma$ , it effectively implies that  $\Omega^2$  can, in principle, be regarded as a function of the flow parameters. From the form of  $\Omega^2$  in equation (14), a generic conclusion that can be immediately drawn is that any critical point, as it may be expected for a conservative system, will be either a saddle point (for  $\Omega^2 > 0$ ) or a centre-type point (for  $\Omega^2 < 0$ ). For the particular case of the Newtonian potential,  $\phi = -GM/r$ , the eigenvalues of the stability matrix will be given by

$$\Omega_N^2 = \alpha a_c^4 [(2\alpha + 1) - \gamma(2\alpha - 1)]. \quad (15)$$

For the values of  $\gamma$  and  $\alpha$  lying in the range of physical interest, it can always be shown that  $\Omega_N^2 > 0$ . Hence in this case the critical point is a saddle point, and the curves which have been labelled ‘‘accretion’’ and ‘‘wind’’ in Fig. 1 are in fact separatrices of a dynamical system, rather than physical solutions.

To understand the full import of this line of reasoning, what has to be borne in mind is that saddle points are inherently unsta-

ble, and to make a solution pass through such a point, after starting from an outer boundary condition, will entail an infinitely precise fine-tuning of that boundary condition (Ray & Bhattacharjee 2002). This can be demonstrated through simple arguments. Going back to equations (13), the coupled set of linear differential equations in  $\epsilon_1$  and  $\epsilon_2$  can be set down as

$$\frac{d\epsilon_1}{d\epsilon_2} = \frac{d\epsilon_1/d\tau}{d\epsilon_2/d\tau} = \frac{Q_1\epsilon_1 + Q_2\epsilon_2}{Q_3\epsilon_1 + Q_4\epsilon_2}, \quad (16)$$

in which the constant coefficients  $Q_1$ ,  $Q_2$ ,  $Q_3$  and  $Q_4$  are to be determined simply by an inspection of equations (13). It is also to be easily seen that  $Q_1 = -Q_4$ . This makes the integration of equation (16) a straightforward exercise, and it yields

$$Q_2\epsilon_2^2 + 2Q_1\epsilon_1\epsilon_2 - Q_3\epsilon_1^2 + \mathcal{C} = 0 \quad (17)$$

with  $\mathcal{C}$  being an integration constant. Generally speaking equation (17) is the equation of a conic section in the  $\epsilon_2 - \epsilon_1$  plane. If the origin of this plane were to be considered to have been shifted to the saddle point, then the condition for solutions passing through the origin, i.e.  $\epsilon_1 = \epsilon_2 = 0$ , would be  $\mathcal{C} = 0$ , which would reduce equation (17) to a pair of straight lines intersecting each other through the origin itself. All other solutions in the vicinity of the origin will, therefore, be hyperbolic in nature, a fact that is given by the condition  $(Q_1^2 + Q_2Q_3) > 0$ . For the case of  $\phi = -GM/r$ , this contention can be verified completely analytically, and this shows that even a very minute deviation from a precise boundary condition for transonicity (i.e. a boundary condition that will generate solutions to pass *only* through the origin,  $\epsilon_1 = \epsilon_2 = 0$ ) will take the stationary solution far away from a transonic state. This extreme sensitivity of transonic solutions to boundary conditions is entirely in keeping with the nature of saddle points. It may be imagined that in a proper astrophysical system such precise fulfillment of a boundary condition will make the transonic solution well-nigh physically non-realizable. Indeed, this difficulty, for any kind of accreting system, is readily appreciated by anyone trying to carry out a numerical integration of equation (8) to generate the transonic solutions, which can only be obtained when the numerics is first biased in favour of transonicity by using the saddle point condition itself as the boundary condition for numerical integration.

Apart from this, there is also a mathematical aspect of the physical non-realizability of transonic solutions. Using the condition  $\mathcal{C} = 0$  will make it easy to express  $\epsilon_1$  in terms of  $\epsilon_2$  and vice versa. Going back to the set of linear equations given by equations (13) and choosing the second one of the two equations (the choice of the first would also have led to the same result), one gets

$$\frac{d\epsilon_2}{d\tau} = \pm \left( \sqrt{Q_1^2 + Q_2Q_3} \right) \epsilon_2, \quad (18)$$

which can be integrated for both the roots from an arbitrary initial value of  $\epsilon_2 = \epsilon_2^*$  lying anywhere on the transonic solution, to a point  $\epsilon_2 = \Delta$ , with  $\Delta$  being very close to the critical point given by  $\epsilon_2 = 0$ . Using the equivalence that  $\Omega^2 = Q_1^2 + Q_2Q_3$ , it can be shown that

$$\tau = \pm \frac{1}{\Omega} \int_{\epsilon_2^*}^{\Delta} \frac{d\epsilon_2}{\epsilon_2} = \pm \frac{1}{\Omega} \ln \left| \frac{\Delta}{\epsilon_2^*} \right|, \quad (19)$$

from which it is easy to see that  $|\tau| \rightarrow \infty$  for  $\Delta \rightarrow 0$ . This implies that the critical point may be reached along either of the separatrices, only after  $|\tau|$  has become infinitely large. This divergence of the parameter  $\tau$  indicates that in the stationary regime, solutions passing through the saddle point are not actual solutions,

but separatrices of various classes of solutions (Strogatz 1994; Jordan & Smith 1999). This fact, coupled with the sensitivity of the stationary transonic solutions to the choice of an outer boundary condition, makes their feasibility a seriously questionable matter.

#### 4 A TIME-DEPENDENT PERTURBATIVE APPROACH

It has been demonstrated in the foregoing section that the steady transonic accretion solution is unstable under infinitesimal deviations from the precise outer boundary condition needed to generate the solution. In the astrophysical context, such precision is quite impossible, and, therefore, the very feasibility of transonicity becomes a matter of grave doubt. This difficulty can, however, be dispelled if one is mindful of the fact that the real astrophysical flow is not static but dynamic in character, i.e. it will have an explicit dependence on time. This, of course, will mean that the time-dependent terms involving both the velocity and the density fields in equations (4) and (5) will have to be retained.

While full-fledged time-dependence of the flow variables will undoubtedly reveal many new interesting mathematical facets (all of them involving the mathematics of partial differential equations) of the physical problem, it would still be worthwhile to go back to studying the properties of the background stationary flow under the influence of a linearised perturbative effect. As a preliminary exercise in accounting for explicit time-dependence, this will at least shed some light on the global stability of the flow solutions.

To that end, it will first be necessary to define, closely following a perturbative procedure prescribed by Petterson et al. (1980) and Theuns & David (1992), a new physical variable  $f = \rho v r^\alpha$ . It is quite obvious from the form of equation (5) that the stationary value of  $f$  will be a global constant,  $f_0$ , which can be closely identified with the matter flux rate. A perturbation prescription of the form  $v(r, t) = v_0(r) + v'(r, t)$  and  $\rho(r, t) = \rho_0(r) + \rho'(r, t)$ , will give, on linearising in the primed quantities,

$$f' = f_0 \left( \frac{\rho'}{\rho_0} + \frac{v'}{v_0} \right), \quad (20)$$

with the subscript “0” denoting stationary values in all cases. From equation (5), it then becomes possible to set down the density fluctuations,  $\rho'$ , in terms of  $f'$  as

$$\frac{\partial \rho'}{\partial t} + \frac{v_0 \rho_0}{f_0} \left( \frac{\partial f'}{\partial r} \right) = 0. \quad (21)$$

Combining equations (20) and (21) will then render the velocity fluctuations as

$$\frac{\partial v'}{\partial t} = \frac{v_0}{f_0} \left( \frac{\partial f'}{\partial t} + v_0 \frac{\partial f'}{\partial r} \right) \quad (22)$$

which, upon a further partial differentiation with respect to time, will give

$$\frac{\partial^2 v'}{\partial t^2} = \frac{\partial}{\partial t} \left[ \frac{v_0}{f_0} \left( \frac{\partial f'}{\partial t} \right) \right] + \frac{\partial}{\partial t} \left[ \frac{v_0^2}{f_0} \left( \frac{\partial f'}{\partial r} \right) \right]. \quad (23)$$

From equation (4) the linearised fluctuating part could be extracted as

$$\frac{\partial v'}{\partial t} + \frac{\partial}{\partial r} \left( v_0 v' + a_0^2 \frac{\rho'}{\rho_0} \right) = 0 \quad (24)$$

with  $a_0$  being the speed of sound in the steady state. Differentiating equation (24) partially with respect to  $t$ , and making use of equations (21), (22) and (23) to substitute for all the first and second-order derivatives of  $v'$  and  $\rho'$ , will deliver the result

$$\begin{aligned} \frac{\partial}{\partial t} \left[ \frac{v_0}{f_0} \left( \frac{\partial f'}{\partial t} \right) \right] + \frac{\partial}{\partial t} \left[ \frac{v_0^2}{f_0} \left( \frac{\partial f'}{\partial r} \right) \right] + \frac{\partial}{\partial r} \left[ \frac{v_0^2}{f_0} \left( \frac{\partial f'}{\partial t} \right) \right] \\ + \frac{\partial}{\partial r} \left[ \frac{v_0}{f_0} (v_0^2 - a_0^2) \frac{\partial f'}{\partial r} \right] = 0. \end{aligned} \quad (25)$$

A little readjustment of terms in equation (25) will finally give an equation of motion for the perturbation as

$$\frac{\partial^2 f'}{\partial t^2} + 2 \frac{\partial}{\partial r} \left( v_0 \frac{\partial f'}{\partial t} \right) + \frac{1}{v_0} \frac{\partial}{\partial r} \left[ v_0 (v_0^2 - a_0^2) \frac{\partial f'}{\partial r} \right] = 0, \quad (26)$$

which is an expression that is exactly the same as what can be derived upon perturbing the stationary solutions of spherically symmetric inflows in a continuous medium (Petterson et al. 1980; Theuns & David 1992). Another aspect of equation (26) is that its form has no explicit dependence on the potential — Newtonian or pseudo-Newtonian — that is driving the flow. This is entirely to be expected, because the potential, being independent of time, will only lend its direct presence to the stationary background flow. Arguments regarding stability will, therefore, be more dependent on the boundary conditions of the steady flow. As the form of the equation of motion for the linearised perturbation remains unchanged even for a flow in a fractal medium, and as the physical boundary conditions are also not altered in this case, the general conclusions reached by both Petterson et al. (1980) and Theuns & David (1992) regarding flows in a continuous medium, will carry over here, and it can be safely claimed that under all reasonable boundary conditions, both the transonic and subsonic solutions will be stable.

While this does nothing to cause any immediate worry, it also does not reveal anything in particular either about the physical feasibility of any solution from a perturbative point of view. This is in keeping with the conventional wisdom about spherically symmetric flows (Garlick 1979) that the natural preference of the system for any particular solution — especially the transonic solution — cannot be justified by a linear stability analysis, but by the more fundamental arguments forwarded by Bondi (1952).

For all that, some positive hint about the primary status of the transonic solution can actually be derived if the whole issue of a linear stability analysis is viewed from a different perspective. It is known that there is a close one-to-one correspondence between certain features of black hole physics and the physics of supersonic acoustic flows (Visser 1998). In this very context, a compact rendering of equation (25) can be obtained as

$$\partial_\mu (f^{\mu\nu} \partial_\nu f') = 0, \quad (27)$$

in which the Greek indices are made to run from 0 to 1, with the identification that 0 stands for  $t$ , and 1 stands for  $r$ . An inspection of the terms in the left hand side of equation (25) will then allow for constructing the symmetric matrix

$$f^{\mu\nu} = \frac{v_0}{f_0} \begin{pmatrix} 1 & v_0 \\ v_0 & v_0^2 - a_0^2 \end{pmatrix}. \quad (28)$$

Now in Lorentzian geometry the d'Alembertian for a scalar field in curved space is given in terms of the metric  $g_{\mu\nu}$  by (Visser 1998)

$$\Delta \psi \equiv \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \psi) \quad (29)$$

with  $g^{\mu\nu}$  being the inverse of the matrix implied by  $g_{\mu\nu}$ . Comparing equation (27) with equation (29), it would be tempting to look for an exact equivalence between  $f^{\mu\nu}$  and  $\sqrt{-g} g^{\mu\nu}$ . This, however, cannot be done in a general sense. What can be appreciated, nevertheless, is that equation (27) gives a relation for  $f'$  which is of the type given by equation (29). The metrical part of equation (27),

as given by equation (28), may then be extracted, and its inverse will incorporate the notion of a sonic horizon of an acoustic black hole when  $v_0^2 = a_0^2$ . This point of view does not make for a perfect acoustic analogue model, but it has some similar features to the metric of a wave equation for a scalar field in curved space-time, obtained through a somewhat different approach, in which the velocity of an irrotational, inviscid and barotropic fluid flow is first represented as the gradient of a scalar function  $\psi$ , i.e.  $\mathbf{v} = -\nabla\psi$ , and then a perturbation is imposed on this scalar function (Visser 1998).

The foregoing discussion indicates that the physics of supersonic acoustic flows closely corresponds to many features of black hole physics. This closeness of form is very intriguing. For a black hole, infalling matter crosses the event horizon maximally, i.e. at the greatest possible speed. By analogy the same thing may be said of matter crossing the sonic horizon of a spherically symmetric fluid flow, falling on to a point sink. That this fact can be appreciated for the spherically symmetric accretion problem, through a perturbative result as given by equation (25), is quite remarkable. This is because it is universally recognised that that no insight into the special status of any inflow solution may possibly be derived solely through a perturbative technique (Garlick 1979). It is the transonic solution that crosses the sonic horizon at the greatest possible rate (Bondi 1952), and the similarity of form between equations (25) and (29) may very well be indicative of the primacy of the transonic solution. If such an insight were truly to be had with the help of the perturbation equation, then the perturbative linear stability analysis might not have been carried out in vain after all.

## 5 DYNAMIC EVOLUTION TOWARDS TRANSONICITY

Much more direct and robust evidence in favour of transonicity could be obtained if the accreting system were to be made to evolve through time, as opposed to making it suffer small linearised perturbations in time. Having said this, it must also be stressed that equations (4) and (5), which govern the temporal evolution of the flow, are not amenable to a ready mathematical analysis; indeed, in the matter of incorporating both the dynamic and the pressure effects in the equations, short of a direct numerical treatment, the mathematical problem, is very aptly described as “insuperable” (Bondi 1952). Therefore, to have a preliminary appreciation of the governing mechanism that underlies any possible selection of a transonic flow, it should be necessary to adopt some simplifications. This will pave the way for a more complete physical understanding of the evolutionary properties of the flow.

The evolutionary dynamics is, therefore, to be studied first in the regime of what is understood to be the “pressureless” motion of a fluid in a gravitational field (Shu 1991), as opposed to dropping the dynamic effects to study a much simplified stationary picture (Bondi 1952). Simplification of the mathematical equations, however, is not the only justification for such a prescription. A greater justification lies in the fact that the result delivered is in conformity with, what Garlick (1979) calls “the more fundamental arguments” of Bondi (1952); that it is the criterion of minimum total energy associated with a solution, that will accord it a principal status over all the others.

An immediate consequence of adopting dynamic equations is that the invariance of the stationary solutions under the transformation  $v \rightarrow -v$ , is lost. As a result, one will have to separately consider either the inflows ( $v < 0$ ) or the outflows ( $v > 0$ ), a choice that has to be imposed upon the system at  $t = 0$ . Euler’s equation,

tailored according to the simplified requirements of a “pressureless” field, is rendered as

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{GM}{r^2} = 0 \quad (30)$$

which can be solved by the method of characteristics (Debnath 1997). The characteristic solutions are obtained from

$$\frac{dt}{1} = \frac{dr}{v} = \frac{dv}{-GM/r^2}. \quad (31)$$

First solving the  $dv/dr$  equation will give

$$\frac{v^2}{2} - \frac{GM}{r} = \frac{c^2}{2} \quad (32)$$

in which  $c$  is an integration constant that derives from the spatial part of the characteristic equation. This result is to be used to solve the  $dr/dt$  equation from equation (31), which will finally lead to

$$\frac{2vr}{cr_s} - \ln r - \ln \left( \frac{v}{c} + 1 \right)^2 - \frac{2ct}{r_s} = \tilde{c} \quad (33)$$

with  $\tilde{c}$  being another integration constant, and  $r_s$  being a length scale in the system, defined as  $r_s = 2GM/c^2$ .

A general solution of equation (31) is given by the condition,  $f(\tilde{c}) = c^2/2$ , with  $f$  being an arbitrary function, whose form is to be determined from the initial condition. The general solution can, therefore, be set down as

$$\frac{v^2}{2} - \frac{GM}{r} = f \left[ \frac{2vr}{cr_s} - \ln r - \ln \left( \frac{v}{c} + 1 \right)^2 - \frac{2ct}{r_s} \right], \quad (34)$$

to determine whose particular form, the initial condition that will have to be used is  $v = 0$  at  $t = 0$  for all  $r$ . This will lead to

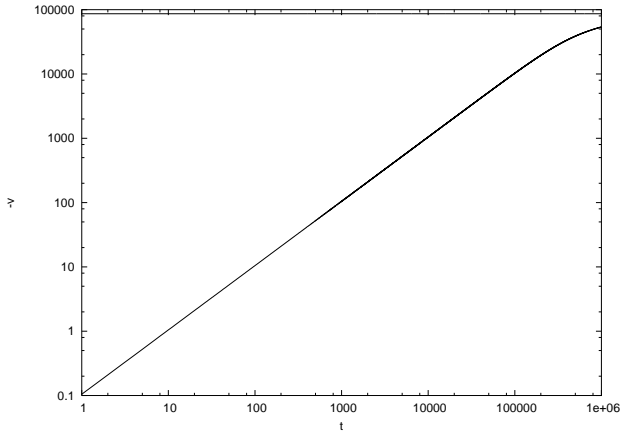
$$\frac{v^2}{2} - \frac{GM}{r} = -\frac{GM}{r} \left( \frac{v}{c} + 1 \right)^{-2} \exp \left( \frac{2vr}{cr_s} - \frac{2ct}{r_s} \right) \quad (35)$$

from which it is easy to see that for  $t \rightarrow \infty$ , what is approached is the stationary solution,

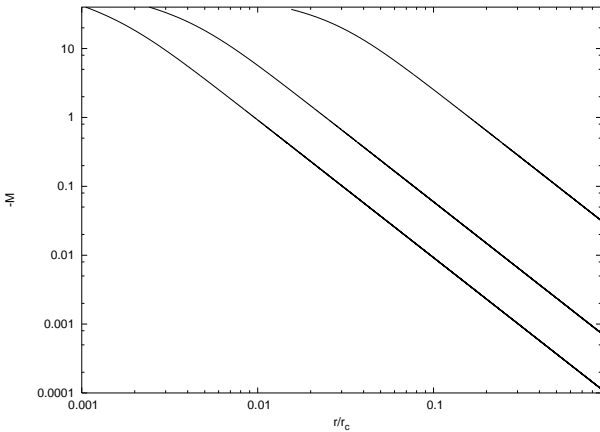
$$\frac{v^2}{2} - \frac{GM}{r} = 0. \quad (36)$$

Corresponding to the given initial condition, this is evidently the stationary solution associated with the lowest possible total energy, and the temporal evolution selects this solution from among all the others. The whole picture could be conceived of as one in which a system with a uniform velocity distribution  $v = 0$  everywhere, suddenly has a gravity mechanism switched on in its midst at  $t = 0$ . This induces a potential  $-GM/r$  at all points in space. The system then starts evolving to restore itself to another stationary state, with the velocity increasing according to equation (35), so that for  $t \rightarrow \infty$ , the total energy at all points,  $E = (v^2/2) - (GM/r) = 0$ , remains the same as at  $t = 0$ .

This contention has been borne out by a numerical integration of equation (30) by the finite differencing technique. The mass of the accretor has been chosen to be  $M_\odot$ , while its radius is  $r_\odot$ . The evolution through time has been followed at a fixed length scale of  $51r_\odot$ . The result of the numerical evolution of the velocity field,  $-v$  (for inflows  $v$  is actually negative), through time,  $t$ , has been plotted in Fig. 2. The limiting value of the velocity, as the evolution progresses towards the long-time limit, is evidently  $\sqrt{2GM/r}$  (with  $M = M_\odot$  and  $r = 51r_\odot$ ), as equation (36) would suggest. This is what the plot in Fig. 2 shows, as  $-v$  approaches its terminal value for  $t \rightarrow \infty$ . The slope of this logarithmic plot also indicates that in the early stages of the evolution there is a linear growth of the velocity field through time, but on later times, conspicuous deviation from linearity sets in.



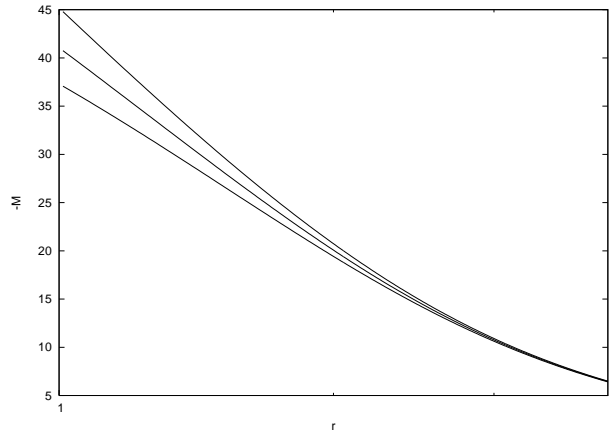
**Figure 2.** For the temporal evolution of velocity, the slope of this logarithmic plot shows that in the early stages of the evolution,  $-v$  varies linearly with  $t$ . Deviation from linear growth sets in later. The horizontal line on top shows the terminal value of the velocity field, whose evolution is being followed at a fixed length scale.



**Figure 3.** The log-log plot of the Mach number versus the radial distance, with the latter being scaled by the sonic radius,  $r_c$ , after 4000 seconds of evolution. Going from left to right each curve corresponds to  $d = 0.7$ , 0.85 and 1.0, respectively. The slope of the curves indicates a power law behaviour for the evolution on intermediate length scales. All the curves approach a saturation velocity scale closer to the accretor.

The argument presented above, with the effects of pressure taken into account, can now be extended to understand the dynamic selection of the transonic solution. The inclusion of the pressure term in the dynamic equation, fixes the total energy of the system accordingly at  $t = 0$ . A physically realistic initial condition should be that  $v = 0$  at  $t = 0$ , for all  $r$ , while  $\rho$  has some uniform value. The temporal evolution of the accreting system would then non-perturbatively select the transonic trajectory, as it is this solution with which is associated the least possible energy configuration. This argument is in conformity with the assertion made by Bondi (1952) that it is the criterion of minimum total energy that should make a particular solution (the transonic solution in this case) preferred to all the others. However, this selection mechanism is effective only through the temporal evolution of the flow.

To test this contention a numerical study has been carried out once again using finite differencing, but this time using both the dynamic equations for the velocity and the density fields, as



**Figure 4.** Velocity field, scaled as the Mach number,  $M$ , after 4000 seconds, for various values of  $d$ . Moving from the bottom to the top, successive solutions have been shown for  $d = 1.0$ , 0.85 and 0.7, respectively. The radial distance has been normalised with respect to  $r_\odot$ , and plotted logarithmically over a length scale of  $r_\odot$  to  $4r_\odot$ .

given by equations (4) and (5). The accretor has been chosen to have a mass,  $M_\odot$ , and radius,  $r_\odot$ . The “ambient” conditions are  $a_\infty = 10 \text{ km s}^{-1}$  and  $\rho_\infty = 10^{-21} \text{ kg m}^{-3}$ . The polytropic exponent,  $\gamma$ , has been set as  $n \equiv (\gamma - 1)^{-1} = 1.61$ . For these values of the physical constants, transonicity becomes apparent even at the very early stages of the evolution. This is shown in Fig. 3 in which the velocity field (scaled as the Mach number) has been plotted after it has evolved for 4000 seconds. The horizontal distance has been scaled by the sonic radius (which, going by equation (11), effectively makes this scaling dependent on the fractal dimension), and it shows a saturation behaviour for the inflow velocity at smaller length scales. This saturation scale for the velocity field is roughly the same for any value of  $d$ , although at larger length scales, a curve placed higher in the plot, corresponds to a higher value of  $d$ . Besides this, all curves, as it is apparent from the slope of each of them, display the same kind of power-law behaviour on larger length scales. All this is somewhat reminiscent of the growth processes exhibited by the ballistic deposition model, which is applied to generate a nonequilibrium interface (Barabási & Stanley 1995).

The comparative properties of the velocity field, at smaller length scales, for various values of  $d$ , have been exhibited in Fig. 4. Here the radial distance has been scaled in terms of the radius of the accretor, which in this case is  $r_\odot$ . What is interesting to note in this plot is that for all other conditions remaining the same, on length scales close to the accretor, solutions corresponding to lower values of the fractal dimension,  $d$ , grow faster in time than solutions with higher values of  $d$ . It is possible to argue that this is exactly how it should be. The pressure of the infalling gas, in so far as it is connected to the density through a polytropic equation of state, builds up resistance against gravity, because of the growth of the density field on small length scales. Transonicity can only be achieved when gravity wins over pressure on length scales smaller than the sonic radius. This will be all the more true near the accretor, where the velocity field will evolve under free-fall conditions, and, therefore, the more dilute the gas, the more efficient will be the drive towards the transonic state. Now a fractal medium may be viewed equivalently as a continuum with an effective lesser density. In this situation a system with a lesser value of  $d$  will be more prone to losing against gravity than a system with higher value of

$d$ , and so the race towards transonicity will be more successful as  $d$  decreases. It is exactly this state of affairs that Fig. 4 graphically represents.

## 6 CONCLUDING REMARKS

An earlier work reported by Roy (2007) was carried out under the implicit assumption that the accretion process would take place transonically. The present treatment bears out this assumption self-consistently. It was also discussed by Roy (2007) that the rate of accretion in a fractal medium can vary significantly from the Bondi (1952) rate for more massive accretors. This is very much in conformity with the conclusions derived in this work, through the numerical evolution of transonicity, and it would be judicious to account for this fact, while studying the accretion of a fractal medium on to a black hole. Black hole accretion is necessarily transonic (Chakrabarti 1990, 1996), but even for accretion from a molecular cloud on to a star, in the absence of any inner boundary condition being imposed on the flow, the flow is expected to be transonic (Petterson et al. 1980). Therefore, whatever be the nature of the accretor, both transonicity and the quantitative modifications arising due to the fractal nature of the accreting medium, will be very much relevant for studies in spherically symmetric accretion.

It has also been discussed here that transonic properties manifest themselves more noticeably with an increase in the fractal properties of the flow. The dynamic evolution has shown that the growth rate of the velocity field (as scaled against the speed of acoustic propagation) becomes significantly higher in this case. This, of course, will have a direct bearing on the mass accretion rate, and it is worth conjecturing that there might be some observational evidence for this kind of behaviour.

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