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The role of shear in expanding cylindrical perfect fluid models

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Abstract

For orthogonal cylindrically symmetric expanding perfect fluid spacetime we prove that vanishing of shear implies vanishing of acceleration which further renders spacetime homogeneous. That means inhomogeneous spacetimes must always be shearing and anisotropic. Non-singular spacetimes will thus be both inhomogeneous and anisotropic.

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The remarkable feature of inhomogeneous cylindrically symmetric spacetimes has been that they admit non-singular perfect fluid models satisfying the strong energy condition $\rho+3p>0$ [1-3]. The spacetimes are of course required to be expanding. In these models, acceleration which is crucial for avoidance of singularity is generated through the gradient of shear. Hence they have to be both accelerating and shearing (inhomogeneous and anisotropic). For a metric separable in space and time in comoving coordinates, this result is true in general for it can be established by using only the kinematic parameters without reference to the field equations and matter distribution [4].

In this note we wish to establish this result by trading off separability of metric for perfect fluid matter distribution. It turns out that orthogonal cylindrical spacetimes if they are to represent shear free expanding perfect fluid, then they must be acceleration free (geodetic) as well. Geodeticity will in turn imply homogeneity leading to the Bianchi class. That means orthogonal cylindrically symmetric expanding perfect fluid models if they are inhomogeneous, then they must be anisotropic (non-zero shear) as well. There cannot exist shear free inhomogeneous perfect fluid models.

According to the Raychandhuri equation [6]

$$\frac{d\theta}{ds} = \dot{u}^a_{;a} + w^2 - \sigma^2 - \frac{1}{3}\theta^2 - R_{ab}u^a u^b \tag{1}$$

for a fluid satisfying the strong energy condition $R_{ik}u^i u^k \leq 0$, the collapse can be halted in the absence of vorticity only if acceleration is non-zero. All the symbols in (1) have the usual meaning. The non-zero acceleration is therefore crucial for non occurrence of singularity in a vorticity-free spacetime.

We begin with the general orthogonal cylindrically symmetric metric

$$ds^2 = D^2dt^2 - A^2dr^2 - B^2dz^2 - C^2d\phi^2 \tag{2}$$

where A, B, C, D are functions of r and t. The kinematic parameters θ, σ and $\dot{u}_\alpha$ are given by

$$\theta = \frac{1}{D}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) \tag{3}$$

$$\sigma^2 = \frac{1}{9D^2}\left[\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C} - \frac{2\dot{A}}{A}\right)^2 + \left(\frac{\dot{C}}{C} + \frac{\dot{A}}{A} - \frac{2\dot{B}}{B}\right)^2 + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} - \frac{2\dot{C}}{C}\right)^2\right] \tag{4}$$

$$\dot{u}_r = -\frac{D'}{D} \tag{5}$$

where $\dot{A} = \frac{\partial A}{\partial t}$ and $A' = \frac{\partial A}{\partial r}$.

The Ricci tensor for the metric (2) is as follows:

$$R_{01} = \frac{1}{AD}\left[\frac{\dot{B}'}{B} + \frac{\dot{C}'}{C} - \frac{\dot{A}}{A}\left(\frac{B'}{B} + \frac{C'}{C}\right) - \frac{D'}{D}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\right] \tag{6}$$

$$R^0_0 = \frac{1}{D^2}\left[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} - \frac{\dot{D}}{D}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\right] - \frac{1}{A^2}\left[\frac{D''}{D} + \frac{D'}{D}\left(\frac{B'}{B} + \frac{C'}{C} - \frac{A'}{A}\right)\right] \tag{7}$$

$$R^1_1 = -\frac{1}{A^2}\left[\frac{B''}{B} + \frac{C''}{C} + \frac{D''}{D} - \frac{A'}{A}\left(\frac{B'}{B} + \frac{C'}{C} + \frac{D'}{D}\right)\right] + \frac{1}{D^2}\left[\frac{\ddot{A}}{A} + \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C} - \frac{\dot{D}}{D}\right)\right] \tag{8}$$

$$R^2_2 = -\frac{1}{A^2}\left[\frac{B''}{B} + \frac{B'}{B}\left(\frac{C'}{C} + \frac{D'}{D} - \frac{A'}{A}\right)\right] + \frac{1}{D^2}\left[\frac{\ddot{B}}{B} + \frac{\dot{B}}{B}\left(\frac{\dot{A}}{A} + \frac{\dot{C}}{C} - \frac{\dot{D}}{D}\right)\right] \tag{9}$$

$$R^3_3 = -\frac{1}{A^2}\left[\frac{C''}{C} + \frac{C'}{C}\left(\frac{B'}{B} + \frac{D'}{D} - \frac{A'}{A}\right)\right]$$
$$+\frac{1}{D^2}\left[\frac{\ddot{C}}{C} + \frac{\dot{C}}{C}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} - \frac{\dot{D}}{D}\right)\right]. \quad (10)$$

Theorem: An orthogonal cylindrically symmetric expanding perfect fluid spacetime can have non-zero acceleration only when shear is non-zero. The vanishing of shear implies vanishing of acceleration which further makes spacetime homogeneous. That means inhomogeneous spacetime must therefore be always shearing and anisotropic.

Proof: The vanishing of shear from (4) implies

$$\dot{A}/A = \dot{B}/B = \dot{C}/C. \quad (11)$$

For perfect fluid, Einstein's equations read as

$$R_{ik} = -8\pi\left[(\rho + p)u_i u_k - \frac{1}{2}(\rho - p)g_{ik}\right] \quad (12)$$

which in the comoving coordinates with $u_i = D\delta^0_i$ imply $R_{01} = 0, R^1_1 = R^2_2 = R^3_3$. In view of (11) $R_{01} = 0$ from (7) will give

$$A = a(r)e^{\psi},\ B = b(r)e^{\psi},\ C = rc(r)e^{\psi} \quad (13)$$

with

$$\psi = \int D(r,t)dt. \quad (14)$$

From (9) and (10) $R^2_2 = R^3_3$ yields

$$\psi' + \frac{D'}{D} = f(r)$$

which on differentiation w.r.t. t gives

$$D' + \left(\frac{D'}{D}\right)^{\bullet} = 0 \tag{15}$$

while $R^1_1 = R^2_2$ from (8) and (9) will give

$$D'' + \left(\frac{D''}{D}\right)^{\bullet} - 2\frac{D'^2}{D} = 0 \tag{16}$$

where (15) has been used. Differentiating (15) w.r.t. r and using (16) we get

$$D' - \left(\frac{D'}{D}\right)^{\bullet} = 0 \tag{17}$$

which in view of (15) leads to $D' = 0$. That means vanishing of acceleration from (5). We have thus shown that shear free implies acceleration free.

We can further see that vanishing of acceleration implies homogeneity. From (12), we can write

$$-16\pi\rho = R^0_0 - 3R^1_1 \tag{18}$$

$$-16\pi p = R^0_0 + R^1_1 \tag{19}$$

Since $\dot{u}_r = 0$ sets $D = 1$ which from (14) renders the metric separable. Then $(R^0_0)' = 0$ and we have $p' = (R^0_0 + R^1_1)' = 0$ from the conservation equation which implies $(R^1_1)' = 0$. Thus both ρ, p are homogeneous and the spacetime will belong to one of the Bianchi models.

This proves the theorem.

Thus all expanding cylindrically symmetric perfect fluid models are both shearing and accelerating, i.e. anisotropic and inhomogeneous. The absence of shear ultimately leads to homogeneity and spacetime will belong to the Bianchi class. That means there cannot exist shear free inhomogeneous spacetimes. It is important to note that this result is not true for spherical symmetry as there exist shear free inhomogeneous perfect fluid models [6]. This property is hence specific to cylindrical symmetry.

In the context of non-singular character of spacetime, the presence of acceleration is essential for which shear must be non-zero always. Non-singular perfect fluid cylindrically symmetric models will thus have ot have non-zero shear and hence inhomogeneous and anisotropic. Note that this result does not depend upon separability of the metric. So far non-singular perfect fluid solutions of Einstein's field equations have been obtained only when the metric is separable [4, 7]. There must exist such solutions in the non-separable case as well. It would be difficult but interesting to find them. As a matter of fact we wonder whether any solution at all has been found in the non-separable case. It would hence be worthwhile even to find a solution whether non-singular or not.

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