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MATHAMATICS

Beyond Ramanujan

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The aura of Srinivasa Ramanujan, perhaps one of the greatest mathematicians of all time, has virtually eclipsed other Indian mathematicians.

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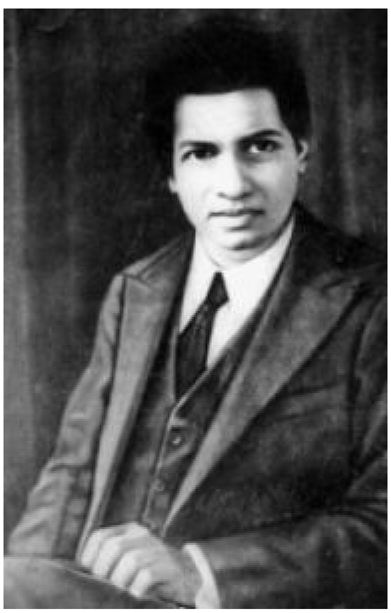

Srinivasa Ramanujan (1887-1920).

THE mathematical genius of Srinivasa Ramanujan (1887-1920), the continuing impact of his highly original methods and the many beautiful results he obtained during his short life make him one of the greatest mathematicians of the 20th century, if not of all time. The aura of Ramanujan has thus virtually eclipsed other Indian mathematicians, his contemporaries and those who followed him, many of whom were world class and made important contributions, particularly in his area of analytic number theory. It is little wonder, then, that the general public, including a majority of the non-mathematical scientific community, finds it hard to name an Indian mathematician who made a mark in the world of mathematics after Ramanujan.

Mathematicians who followed in Ramanujan's tradition include names such as K. Ananda Rau, S.S. Pillai, T. Vijayaraghavan, S. Chowla, L.G. Sathe, S. Minakshisundaram and K. Chandrasekharan. The broad influence of Ramanujan is very much evident in their work, especially through G.H. Hardy's continuing significant role in number theory and analysis in the decades after Ramanujan's untimely death. Their important contributions, in fact, led to the recognition by the world mathematics community of an Indian school of number theory. Though by the 1970s and 1980s the activities of Indian mathematicians did diversify to include other areas of modern mathematics, number theory continues to be an important component of Indian mathematics research even today. It is important, therefore, to make these names more widely known in the country and highlight and appreciate their contributions.

A significant step in this direction has been taken by the Ramanujan Mathematical Society (RMS), Chennai. This initiative has resulted in the recent publication by the Society of the Collected Works of S.S. Pillai, which includes some of his unpublished work. Compiled and edited by R. Balasubramanian, himself an eminent number theorist, and R. Thangadurai of the Harish-Chandra Research Institute (HRI), Allahabad, this volume should mark the beginning of a process of systematic archiving of Indian mathematical achievements, especially in the four decades or so immediately after Ramanujan for which the records seem to be sparse.

It is a commentary on the state of affairs that even good photographs of these mathematicians are not available. As Balasubramanian, Director of the Institute of Mathematical Sciences (IMSc), Chennai, remarked at the recently concluded International Congress of Mathematicians (ICM 2010) in Hyderabad – where he gave a plenary talk presenting an interesting overview of the contributions of the post-Ramanujan Indian number-theoretic school up to the 1980s – we do not even seem to have a record of L.G. Sathe's full name. According to Balasubramanian, much of the details of the work of such mathematicians that he presented were drawn from obituaries that K. Chandrasekharan (popularly known as KC) had written in various journals. Ninety-year-old KC lives in Zurich.

K. Ananda Rau

The story of Indian mathematics beyond Ramanujan actually begins with K. Ananda Rau, his contemporary, and it is essentially Ananda Rau's students and descendants who constituted this Indian school in number theory. Ananda Rau was born six years after Ramanujan, in 1893, in Madras (now Chennai) into “relative opulence”, in the words of the eminent mathematician M.S. Raghunathan, the present chairman of the Publications Committee of the RMS. His mathematical trajectory, unlike Ramanujan's, was very much a conventional one though there is the common mathematical link with Hardy. Ananda Rau attended the Hindu School and then Presidency College of the University of Madras. Following a brilliant academic record, he sailed to England in 1914 only a few months after Ramanujan. After finishing his Mathematical Tripos from King's College, Cambridge, in 1916, he became a student of Hardy's and was guided and initiated into active research by him.

According to Balasubramanian, though Ananda Rau may have met Ramanujan for the first time in England, he must have at least known of Ramanujan earlier through R. Ramachandra Rao, a relative of Ananda Rau's, to whom Ramanujan had presented some of his work in 1910. Ramachandra Rao, according to Hardy, was Ramanujan's “most devoted friend”. In any case, Ananda Rau and Ramanujan became good friends at Cambridge. It must, however, be emphasised that Ananda Rau decided to pursue a career in mathematics well before Ramanujan's mathematical prowess became known.

As a student, Ananda Rau wrote an essay under Hardy's guidance that fetched him the coveted Smith Prize in 1917, following which he was also elected a fellow of King's College. Hardy was a leading expert in the “summability” and convergence properties of infinite series, and his book Divergent Series (1949) became a classic of that period. Summability refers to a method of assigning values to the sequence of partial sums of an otherwise divergent series. Coming naturally under Hardy's influence, Ananda Rau too did a considerable amount of work in that area. In fact, a theorem named after Ananda Rau figures in Hardy's book. In his essay, Ananda Rau obtained a number of results for “generalised Dirichlet series” that generalises what was shown by Hardy and the other famous British mathematician of that period, J.E. Littlewood, on the “summability” of “ordinary Dirichlet series”. However he published this only in 1932, in a slightly modified form, in the Proceedings of the London Mathematical Society (PLMS).

Ananda Rau's first paper was actually a note published in 1918 in the PLMS, the topic of which stemmed from a 1913 paper of Hardy's on the summability and convergence of a certain class of infinite series. Ananda Rau begins the paper by pointing out the incompleteness of Hardy's argument and sets out to fill the gap and provide the complete proof. “What stands out,” points out Balasubramanian, “is the clarity of [Ananda Rau's] exposition and attention to detail, qualities that Rau came to be admired for later as a teacher.”

Ananda Rau returned to India in 1919 and was appointed as a professor of mathematics at Presidency College at the age of 26. Among the first papers he wrote after his return was on the subject of the Riemann zeta function, and he provided an elegant proof of a certain relation satisfied by the function. The technique devised by Ananda Rau was found to have wide applicability in other problems of number theory.


Trinity College
Cambridge
26 March

Dear Mr Ramanujan
Since I wrote to you last I have [illegible] Mr Littlewood to whom I sent your last letter to me, and I have considered further some of your results.
Your formula (3) is wrong ([illegible] formula

[illegible])

very remarkable [illegible] in any way [illegible] the results you [illegible] continued fractions and elliptic functions are enough [illegible] to show it (assuming they are correct) [illegible] I am quite willing to do, [illegible]

Mr Littlewood suggested to me also that your [illegible] was probably due to
1 + 1 + 1 + ...

and I am quite sure that the wisest thing you can do, in your own interests, [illegible] let me have one as soon as possible.
I am
Yours very sincerely
G.H. Hardy

A LETTER G.H. Hardy wrote Ramanujan.

Hardy's influence on Ananda Rau was so strong that he remained attracted to the topic of summability for a number of years after his return. His work and that of his students in summability and the related topic known as Tauberian theory acquired international reputation. In the later part of his career, Ananda Rau turned his attention to elliptic functions. He also did work in the representation of integers as sums of squares, and that naturally led to the related Waring's problem, which involves the representation of integers as sums of higher powers. Waring's problem attracted other Indian mathematicians as well, notably S.S. Pillai, and the case of representation in terms of fourth powers had remained unsolved for very long. This was finally solved by Balasubramanian in 1986 in collaboration with two French mathematicians.

Vijayaraghavan, S.S. Pillai, Chandrasekharan and Minakshisundaram, the torch-bearers of Indian mathematics in the post-Ramanujan era, were all Ananda Rau's students. Ananda Rau retired from Presidency College in 1948 at the mandated age of 55 but continued to work in the areas of his interest until his death in 1966 at the age of 73. Already in 1937, he had been awarded the title of Rao Bahadur by the Government of India.

S.S. Pillai

The real impact of Indian contributions to number theory in the post-Ramanujan period actually came in the mid-1930s with the arrival on the scene of S.S. Pillai. Born in 1901 in Tirunelveli (in present-day Tamil Nadu), he lost both his parents before he completed high school. But his talents had been noticed by one of his teachers, known as Sastriar, who took it upon himself to give S.S. Pillai financial assistance to complete his schooling. He also helped him earn a scholarship for higher studies.

S.S. Pillai first went to Scott Christian College in Nagercoil and then to Maharaja's College (now University College) in Thiruvananthapuram, where he obtained his B.A. He moved to the University of Madras in 1927 as a research student following the award of a stipend. It was here that S.S. Pillai came into contact with Ananda Rau. Under the inspiring guidance of Ananda Rau and R. Vaidyanathaswamy, S.S. Pillai began to carry out research in number theory and obtained his PhD under Ananda Rau. (It may be noted in passing that Vaidyanathaswamy was another well-known mathematician of the period who also went to England but whose interests were far removed from those of the Cambridge school. He was the earliest to venture into areas such as symbolic logic, lattice theory and topology.) During this time he wrote a number of papers and met, among others, Vijayaraghavan.

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S.S. Pillai's meeting with Vijayaraghavan resulted in an interesting development, which has come to light from a recently discovered unpublished manuscript of S.S. Pillai's. Hardy and Ramanujan had worked on a problem known as “rarity of round numbers”. (A round number is the product of a considerable number of comparatively small factors, and these, as Hardy noted, are very rare contrary to what one would naively expect.) A certain equation related to the problem had been stated by E. Landau on heuristic grounds. Hardy, who was at the University of Oxford from 1919 to 1931, had posed a generalisation of the equation to his class. Vijayaraghavan was in Oxford during 1925-28 working with Hardy and had learnt of the problem, which he narrated to S.S. Pillai in Chennai where he had moved to on his return from England after short stints in Aligarh and Dacca.

In 1929, S.S. Pillai was appointed as a lecturer at Annamalai University, Chidambaram. In the same year, he made a breakthrough in the solution of Hardy's problem but, for reasons unknown, did not publish his results. He remained at Annamalai University for 12 years, during which he was awarded a DSc by the University of Madras, the first to be awarded by the university in mathematics. It is during this period that his best work seems to have been done, including his remarkable result on Waring's problem, which he solved for all powers equal to or greater than six.

In 1941, S.S. Pillai moved to Travancore and left the following year for the University of Calcutta. It was here he met the bright young Sathe, who became his student and to whom he suggested Hardy's problem in 1943 and put at his disposal all his manuscripts on the problem. In less then two years, Sathe solved the problem, on the basis of a complex induction argument, which ran into 134 pages. This, in fact, did much more than solve Hardy's problem, according to Balasubramanian.

However, the paper, which was sent to Transactions of the American Mathematical Society, was not published on the grounds that the proof was a “rather complicated and involved one” though, in the opinion of the referee, A. Selberg, Sathe's results were “very beautiful and interesting”. It was finally published in four parts between 1953 and 1954 in the Journal of the Indian Mathematical Society of which KC was the editor at that time. Selberg's observations and a modern analytic proof by Selberg were also included as a note along with the paper. The Pillai-Sathe association, however, did not last long as, soon after his work on Hardy's problem, Sathe was struck by an illness that incapacitated him. The final blow, of course, came on August 31, 1950, when the plane carrying S.S. Pillai on his way to the ICM of 1950 in Harvard crashed near Cairo killing all on board. Pillai was just 49.

S. Chowla

Sarvadaman Chowla was, as his biography notes, “one of the best known number theorists from India to have followed in the tradition of Ramanujan”. Since he carried on his research abroad for a much longer period than the others, his collected works, numbering as many as 350 papers, and a number of commentaries on his work are available. Chowla was born in 1907 in London and was brought up in Lahore. He went to the University of Cambridge in 1929 for his doctoral work, which he completed in 1931 under Littlewood. On his return he was appointed as a lecturer at St Stephen's College in Delhi and subsequently taught at Banaras Hindu University and Andhra University, Waltair, before returning to Lahore as a professor in 1936. In 1948, he emigrated to the United States. Following a stay at the Institute for Advanced Study (IAS) in Princeton, he served at various U.S. universities. He died in 1995 at the age of 88.

Starting in the late 1920s, and up to one month before his death, S.S. Pillai maintained regular contact with Chowla, according to Balasubramanian. They published joint papers, the first of which appeared in 1930. This was on the topic of the Euler function. Chowla had also made an important conjecture with regard to a series relation satisfied by the function, which went against the prevailing wisdom. Chowla was proved right in 1951 by Paul Erdos and H.S. Shapiro. Chowla and S.S. Pillai continued in their collaborative work on Euler's function and related areas. The second theme that they collaborated on was a problem that had roots in the history of Indian mathematics. It concerned solutions to the Brahmagupta-Pell equation. In 1931, they improved on a result of Vijayaraghavan's, who had also made important contributions to the subject. Thereafter, their collaborative work moved to Waring's problem, a theme in which both Chowla and S.S. Pillai were greatly interested. While S.S. Pillai solved the case of sixth and higher power representations of integers, Chowla (along with F.C. Auluck) focussed on fourth powers, which was solved much later by Balasubramanian and others.

T. Vijayaraghavan

Vijayaraghavan was born in 1902. He did his schooling, in which he did very well, in a number of places. He then went to Pachaiyappa's College in Chennai, but as in Ramanujan's case, serious mathematics distracted him from the unexciting curricular mathematics. As a result, he did not do well in his degree course. But Ananda Rau recognised his talent and arranged to have him admitted in Presidency College to complete his honours course. He published his first work even before he completed his college studies. Parallels with Ramanujan continue. Around 1921, Vijayaraghavan began sending his results to Hardy. Hardy did not respond initially, but when he did, he urged the University of Madras to provide Vijayaraghavan a fellowship so that he could visit him in England. Following this, Vijayaraghavan worked with Hardy in 1925-28.

“Vijayaraghavan was a problem solver in the best sense of the phrase,” says Balasubramanian. He had no great fascination for constructing theories or being confined to a given branch of mathematics. He worked on any problem that interested him. His most important work was related to analytic number theory, in particular summability. He worked in the related area of Tauberian theorems and produced high-quality work. He also worked on Diophantine approximations – approximating real numbers with rational numbers. The set of numbers known as Pisot-Vijayaraghavan numbers, of which the golden ratio is one, bears his name. Given his close association with Chowla and S.S. Pillai, he would off and on return to analytic number theory. Chowla and Vijayaraghavan collaborated on three important papers. As mentioned earlier, after his stints at Aligarh and Dacca, Vijayaraghavan moved to Chennai to head the newly formed Ramanujan Institute. He died in 1955 at a relatively young age of 53.

Chandrasekharan has this to say of Vijayaraghavan: “No one who knew him intimately – as a working mathematician, as a genial host or as an affectionate father – could fail to say here was an intellectual of whom the country could be proud.” Vijayaraghavan loved lecturing and was known to be a “lucid, effective and sometimes brilliant lecturer”, especially on topics of immediate interest to him. He used to say that one could not claim to know a theorem unless one could give not less than three different proofs for it, at least one of which was one's own.

With Chowla fleeing to the U.S. from Lahore in the wake of Partition in 1948 and the death of S.S. Pillai in 1950, post-Ramanujan Indian mathematics can be said to have entered its second epoch. While Ramanujan's influence continued, the emphasis began to shift to more analytic and modular parts of his work, says Balasubramanian. “If a single event can be said to mark the beginning of this epoch,” he says, “it is the emergence of the school of mathematics at the Tata Institute of Fundamental Research [TIFR], Mumbai, under the stewardship of KC.”

K. Chandrasekharan

Chandrasekharan was born in 1920 at Machilipatnam in present-day Andhra Pradesh. After attending school in Bapatla, he studied at Presidency College from where he obtained his M.A. and came under the influence of Ananda Rau. He wrote his thesis under Ananda Rau's guidance and obtained his PhD in 1942.

After teaching at Presidency College for a few years, he moved to the IAS, Princeton, in 1947. Chandrasekharan's early work, true to Ananda Rau's school at Presidency, was in summability theory, a topic in which continued to work throughout the 1940s. In 1950, he authored an important monograph with Minakshisundaram on “Riesz's typical means”. While at Princeton, his interests moved to topics such as Fourier transform theory. His interest in analytic number theory remained and surfaced later at the TIFR, where he moved on an invitation from Homi J. Bhabha in 1949. During his stay there, Chandrasekharan wrote many expository books on analytic number theory. He retired from the TIFR in 1965 and moved to Zurich.

S. Minakshisundaram

The case of Minakshisundaram, popularly known as Minakshi, is slightly different from that of the others described above. He was born in Thrissur in 1913. After his early education in Chennai, he took his B.A. degree from Loyola College and then became a research student at Madras University under Ananda Rau. This naturally steered his interest towards summability and Tauberian theorems, wherein he produced quality work. During 1937-38 at Loyola College, he came under the influence of Father Charles Racine, S.J., who brought with him the French mathematical heritage of the likes of Elie Joseph Cartan and Jacques Hadamard, and was also a friend of famous mathematicians such as Henri Paul Cartan (Elie's son) and André Weil (who became Vijayaraghavan's close friend at Aligarh and Dacca). Minakshi was Father Racine's first major success in his attempts to wean Indian students away from the traditional areas handed down by the Cambridge school.

Following the British school, in particular Hardy, summability was an area pursued by many Indian mathematicians, including Ananda Rau and his students, who made substantial contributions. But it continued to be researched upon by many Indians long after it had ceased to offer exciting challenges, says Raghunathan. This, in fact, had prompted Weil to remark: “Hardy spoilt many Indian mathematicians, but of course Ramanujan was too great to be spoilt.” Though Minakshi retained his interest in summability until the end of his life, he was perhaps the first to move away into more modern areas, and his most important work relates to the problem of “eigenvalues of the Laplacian operator on compact Riemannian manifolds”. He died in 1968 at the Indian Institute of Advanced Study, Shimla, without completing the book on spectral theory that he had been working on, but his period marked the beginning of what may be termed as the third epoch in Indian mathematics – the emergence of top-class Indian mathematicians in other areas of modern mathematics that have come out of research in places such as Paris and Gottingen, Germany.

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