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STATIONARY STATES IN QUANTUM COSMOLOGY

(Based on the lectures given at the Bhavnagar Mini School)

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1. Approach to Quantum Gravity

Gravity is one of the four basic interactions of physics known today. The other three interactions are the strong interaction, the weak interaction and the electromagnetic interaction. Of these the first two are of short range, i.e. they are effective over sub-atomic distances of the order 10^{-12} cm. For their discussion one is necessarily driven to the use of quantum framework. The electromagnetic interaction is of long range and its effects are felt at macroscopic level. The successes of the electromagnetic theory at the classical level were not sufficient, however for the full understanding of its implications. Quantum electrodynamics was necessary for understanding several phenomena at the atomic level.

Take for example, the hydrogen atom. The electron circulating round the proton in the H-atom should not stay in the same orbit if the classical Maxwell-Lorentz electrodynamics is any guide. The classical electron is expected to circulate round in steadily smaller orbits, ultimately spiraling into the proton. The characteristic time scale for this to happen is very small :

$$\tau_e \sim \frac{e^2}{m_e c^3} \sim 10^{-24}\ \mathrm{s}. \qquad \ldots(1)$$

The quantum theory of the H-atom solves the problem satisfactorily. Instead of the electron moving in a continuously changing orbit (as the classical theory requires) it goes in one of the many discrete orbits. These orbits are *stationary* and the electron can continue moving along a stationary orbit unless it is disturbed by outside agency or unless it makes a spontaneous transition to an orbit of lower energy. Quantum theory tells us that even the lowest energy orbit is of finite size. The characteristic linear size to emerge from the quantum calculations was

$$a \sim \frac{\hbar^2}{m_e e^2} \sim 10^{-8}\ \mathrm{cm}. \qquad \ldots(2)$$

More sophisticated calculations show further small scale effects such as the Lamb shift which have been confirmed by experiments. Thus by now quantum electrodynamics is considered a well established, indeed the best understood, part of physics, What about gravity? Like electromangetic theory gravity is a long range interaction. Can it be quantized? Does it have to be quantized? If so what new effects may we expect from quantum gravity? Can these effects be measured by laboratory experiments as the effects of electromagnetic theory were measured and established?

Leaving the first question aside for the time being we will turn our attention to the later questions. Our guidelines from the rest of physics suggest that gravity should also be discussed within the quantum framework. A rule of the thumb for deciding when the quantum framework is relevant is the one suggested by Dirac (1) This is as follows.

Consider the classical action **A** describing any physical interaction. The classical behaviour of the interaction is given by the principle of stationary action

$$\delta \mathbf{A} = 0. \qquad \ldots(3)$$

For such a theory the quantum considerations become important if in the relevant space-time region V the quantity of action becomes *comparable* or *less* than $\hbar$;

$$\mathbf{A} \lesssim \hbar. \qquad \ldots(4)$$

Let us apply these considerations to the Einstien-Hilbert action of general relativity

$$\mathbf{A} = \frac{c^4}{16\pi G} \cdot \int_v R\sqrt{-g}\,d^4x. \qquad \ldots 5)$$

Suppose in a spherical region of radial size L the average density is ρ. Taking the time interval to be L/c we get for **A**,

$$\mathbf{A} \sim \frac{c^4}{16\pi G} \cdot \frac{8\pi G}{c^4} \rho c^2 \cdot \frac{4\pi}{3} L^3 \frac{L}{c}$$

$$= \frac{2\pi}{3} \rho L^4 c, \qquad \ldots(6)$$

where we have used the Einstein equations to estimate R in terms of ρ. In a case of strong gravity we would have the black hole limit:

$$L \lesssim \frac{2G}{c^2} \cdot \frac{4\pi}{3} \rho L^3,$$

i.e.,

$$\rho L^2 \lesssim \frac{3c^2}{8\pi G}. \qquad \ldots(7)$$

From (6) and (7) we get

$$\mathbf{A} \lesssim \frac{L^2 c^3}{4G}. \qquad \ldots(8)$$

Therefore the quantum condition (4) becomes an upper limit on the linear size

$$L \sim 2\sqrt{\frac{G\hbar}{c^3}} \sim 2L_P \qquad \ldots(9)$$

where

$$L_P = \sqrt{\frac{G\hbar}{c^3}} \qquad \ldots(10)$$

is called the Planck length. Since G. $\hbar$ and c are the only fundamental constants available to a quantum theory of gravity the Planck length inevitably turns up in any approach to quantum gravity.

The smallness of Lp suggests that the quantum effects of gravity are not going to be important at a microscopic level. This explains why unlike quantum electrodynamics, quantum gravity has not come up with any laboratory experiment. Indeed there is no gravitational phenomenon known today which would cease to exist if $\hbar$ were zero. Contrast this situation with that in electrodynamics. If $\hbar$ were zero, there would be no spectroscopy, no Compton effect, no photoelectric effect....

Why then concern ourselves with a subject which has no demonstrable effect? The justification for studying quantum gravity can be given on two grounds: (i) The *aesthetic* reason that for completeness we should develop the quantized version of gravity. After all the approach to unifying different physical interactions is also guided by an aesthetic motivation, for the so called grand unified theories are really tested at energies far beyond those of any man made accelerators. (ii) High energy astrophysics and cosmology may provide scenarios where quantum gravity becomes relevant. We will discuss two such scenarios in this paper.

Having established the 'why', the next question of 'how' is more difficult to answer. Several different approaches to quantum gravity exist, none of them being entirely satisfactory. Here we will adopt the path integral approach which has proved useful in other areas of quantum theory. We will begin with a brief description of this approach.

2. The Path Integral Formalism

Let us consider a simple problem in classical mechanics, the problem of the free particle moving in one dimension. The action describing this motion in the Newtonian framework is

$$\mathbf{A} = -\int \tfrac{1}{2} m \dot{x}^2 \, dt. \qquad \text{...(11)}$$

Here m is the mass of the particle, x its displacement from origin at time t. $\delta\mathbf{A}=0$ gives us the equation of motion

$$m\ddot{x} = 0. \qquad \text{...(12)}$$

If we are given that the particle was at $P_1(x_1, t_1)$ to start with and at $P_2(x_2, t_2)$ to end with, the path of the particle is uniquely fixed as one given by the equation

$$\overline{x}(t) = x_1 + \frac{x_2 - x_1}{t_2 - t_1}(t - t_1), \qquad \text{...(13)}$$

for $t_1 \leqslant t \leqslant t_2$ Denote this path in the space-time diagram by $\overline{\Gamma}$, and refer to it as the *classical path*.

A general path from P_1 to P_2 may be denoted by Γ, and it is given by a C^2 function $x(t)$ with $x(t_1) = x_1$ and $x(t_2) = x_2$. The action evaluated along Γ is therefore a functional of $x(t)$. We will denote it by $\mathbf{A}(\Gamma)$. Thus, we have

$$\mathbf{A} = \mathbf{A}(\overline{\Gamma}) = -\frac{m(x_2 - x_1)^2}{2(t_2 - t_1)}. \qquad \text{...(14)}$$

It was Feynman [2] who gave a prescription which relates classical mechanics to quantum mechanics and which gives a more precise expression to the Dirac's concept described earlier. The Feynman rule is to attach a probability amplitude $\Upsilon(\Gamma)$ to each path Γ from P_1 to P_2 and to define a propagator K $[P_2, P_1]$ as a sum of $\Upsilon(\Gamma)$ over all paths Γ. Thus we have

$$\Upsilon(\Gamma) = \exp\left\{\frac{iA(\Gamma)}{\hbar}\right\}, \qquad \ldots(15)$$

$$K[P_2;P_1] = \sum_{\Gamma} \Upsilon(\Gamma). \qquad \ldots(16)$$

The sum over paths is in practice an integral over the continuum of all paths. In the particular case of the free particle the path integral can be explicitly evaluated to give for $t_2 > t_1$

$$K[x_2,t_2;\ x_1,t_1] = \sqrt{\frac{im}{2\hbar(t_2-t_1)\pi}}\exp\left\{-\frac{im(x_2-x_1)^2}{2h(t_2-t_1)}\right\} \qquad \ldots(17)$$

$K=0$ for $t_2 < t_1$. Thus all paths are supposed to be future directed.

The two-point function gives the probability amplitude for the particle to be at P_2 given that it was at P_1. Quantum mechanically speaking we no longer have a unique path to go from P_1 to P_2. All paths Γ contribute towards K which describes the over-all effect of interference of probability amplitudes along the different paths. The fact that we cannot definitely say which path the particle took from P_1 to P_2 is indicative of the lack of determinism in the quantum framework.

The quantum uncertainty principle tells us that we cannot even assert definitely that the particle was at P_1 or that it will be at P_2. All we can say is that there was a probability amplitude $\Psi(x_1, t_1)d\,x_1$ that it was in the range (x_1, x_1+dx_1) at t_1 and that similarly $\Psi(x_2, t_2)dx_2$ describes the probability amplitude for the particle to be in the range (x_2, x_2+dx_2) at t_2. The functions Ψ are called wave functions and they are related by the propagator K as follows :

$$\Psi(x_2,t_2) = \int_{-\infty}^{\infty} K[x_2,t_2;\ x_1\ t_1]\,\Psi(x_1,t_1)dx_1. \qquad \ldots(18)$$

This interpretation follows directly from Feynman's rule of sum over paths.

Let us consider now the 'classical limit' implied by $\hbar \to 0$. In this limit the exponential in (15) oscillates on the unit circle $|z| = 1$ in the complex plane so that with the exception of a few paths the contributions of all paths tend to cancel. The exceptional paths lie in the neighbourhood of the classical path $\bar{\Gamma}$ along which $\delta A=0$ For, by virtue of the stationarity of action the path $\bar{\Gamma}$ contribute the same value of the exponential, and these contributions add coherently to give

$$K \to \bar{K} \sim \exp\{ iA(\bar{\Gamma})/\hbar\} \qquad \ldots(19)$$

as $\hbar \to 0$. This explains why the classical principle of least action holds in the classical limit.

Consider next the application of (18) to the stationary wavepacket. We may idealize the wavepacket by the wavefunction

$$\psi(x_1, t_1 \mid \triangle_1) = (2\pi\triangle_1{}^2)^{-1/4} \exp\left\{-\frac{x_1{}^2}{4\triangle_1{}^2}\right\}, \qquad \ldots(20)$$

where $\triangle_1$ is a constant showing the characteristic spread of the wavepacket at $t=t_1$. The probability of finding the particle at $x \in [x_1, x_1+dx_1]$ given by the Gaussian

$$|\Psi(x_1, t_1 \mid \triangle_1)|^2 \, dx_1 = (2\pi\triangle_1{}^2)^{-1/2} \exp\left\{-\frac{x_1{}^2}{2\triangle_1{}^2}\right\} dx_1, \qquad \ldots(21)$$

with the mean position at the origin $x_1=0$ We will have occasion to use such a Gaussian wavepacket in later work.

Applying (18) to (20) gives us another wavepacket at t_2 The probability in (21) is then given by

$$|\psi(x_2, t_2 \mid \triangle_2)|^2 \, dx_2 = (2\pi\triangle_2{}^2)^{-1/2} \exp\left\{-\frac{x_2{}^2}{2\triangle_2{}^2}\right\} dx_2 \qquad \ldots(22)$$

where

$$\triangle_2{}^2 = \triangle_1{}^2 + \frac{\hbar^2 (t_2 - t_1)^2}{4m^2 \triangle_1{}^2}. \qquad \ldots(23)$$

Thus (23) shows that the wavepacket steadily spreads around the mean value $x_2=0$. In other words, the quantum uncertainty grows with time, albeit at a steady rate.

We will find the above concepts useful when considering the role of quantum uncertainty in the description of space-time geometry, The above example from mechanics can evidently be generalized and in principle applied to any system describable by the action principle. Such applications may be found in the book by Feynmann and Hibbs [3].

Two results from the path intergral theory will be used in our applications to quantum gravity. The first result gives a generalization of the simple example described above, and is stated as follows. Suppose the action is of the following form :

$$\mathbf{A} = \int_{t_1}^{t_2} [\alpha(t)\dot{x}^2 + 2\{\beta(t) + \gamma(t)x\}\dot{x} + \lambda(t)x^2 + 2\mu(t)x + \nu(t)]dt \qquad \ldots(24)$$

and suppose that the solution of $\delta\mathbf{A}=0$ gives us the classical path

$$\overline{\Gamma} : x = \overline{x}(t). \qquad \ldots(25)$$

It then follows that the propagator is given by

$$\mathrm{K}[x_2, t_2 ; x_1, t_1] = f(t_1, t_2) \exp\{ i\mathbf{A}[\overline{x}(t)]/\hbar \}. \qquad \ldots(26)$$

Thus so long as the action is quadratic in $x, \dot{x}$ the resulting path integral is explicitly solvable. The function $f(t_1, t_2)$ is usually determinable from the transitive property of the propagator :

$$\mathrm{K}[x_3, t_3 ; x_1, t_1] = \int_{-\infty}^{\infty} \mathrm{K}[x_3, t_3 ; x_2, t_2]\, \mathrm{K}[x_2, t_2 ; x_1, t_1] dx_2. \qquad \ldots(27)$$

The second result is the relationship between the path integration and the Schrodinger equation. If we introduce potential functions in the action and construct a hamiltonian $H(x, p, t)$ for the position (x) and momentum (p) variables then the function H satisties the Schrodinger equatien

$$\left[i\hbar\frac{\partial}{\partial t_2} - H\left(x_2, -i\hbar\frac{\partial}{\partial x_2}, t_2 \right) \right] K[x_2, t_2; x_1, t_1] = \delta(x_2 - x_1). \qquad ...(28)$$

The wavefunction Ψ satisfies the source-free form of the above equation. The stationary states are given by the eigen-states of the energy operator $i\hbar\,\partial / \partial t$:

$$i\hbar\frac{\partial}{\partial t_2}\Psi(x_2, t_2) = E\Psi(x_2, t_2). \qquad ...(29)$$

The stationary state with the wavefunction $\phi(x_2)\exp(-iE\,t_2 / \hbar)$ has the energy E. The function H satisfies the differential equation

$$H\phi = E\phi. \qquad ...(30)$$

3. The Thin Sandwhich and Superspace

The classical equations of general relativity are deriv d from the action

$$A = \frac{c^4}{16\pi G}\int_V R\sqrt{-g}\, d^4x + A_m \qquad ...(31)$$

where R = scalar curvature and A_m is the action describing the matter present in the space-time region V. The Einstein equations

$$R_{ik} - \tfrac{1}{2} g_{ik} R = -\frac{8\pi G}{c^4} T_{ik}, \qquad ...(32)$$

follow from the variation $g_{ik} \rightarrow g_{ik} + \delta g_{ik}$ of the space-time metric tensor.

We will denote the classical solution of these equations with prescribed boundary conditions by the metric tensor g_{ik}. How can we bring in path integration into this picture? What are the 'points' which these paths are supposed to connect?

The points are 3-geometries $^{(3)}\mathcal{G}$ in an abstract 'superspace'. To understand the concept imagine a region v of space-time sandwiched between two space-like hypersurfaces Σ_1 and Σ_2. If we consider an arbitrary space-like hypersurface Σ between Σ_1 and Σ_2, we can describe by $^{(3)}G$the geometry on it, the geometry beiug that of a 3-space. To get the total picture we need the intrinsic and the extrinsic curvatures at any point on Σ. Alternatively, given two 3-geometries. $^{(3)}G_1$ on Σ_1 and $^{(3)}G_2$ on Σ_2 the Einstein equations determine (supposedly uniquely) the sequence of 3-geometries from Σ_1 to Σ_2 along such intermediate space-like hypersurfaces as Σ. For a full description of the boundary value problem see Ref [4]

Quantization consists of imagining other sequences than the classical one, which start from a given $^{(3)}G_1$ on Σ_1 and end with a given $^{(3)}G_2$ on Σ_2. Each sequence may be called a 'path' Γ and the classical sequence called the path $\bar{\Gamma}$ The amplitude along each path is then given by Feynman's prescription and we end up with the propagator

$$K[{}^{(3)}G_2, \Sigma_2; {}^{(3)}G_1, \Sigma_1] = \sum_F \exp[iA(F)/\hbar], \qquad \text{.. (33)}$$

describing the probability amplitude of arriving at the 3-geometry ${}^{(3)}G_2$ on Σ_2, starting from a 3 geometry ${}^{(3)}G_1$ on Σ_1.

Although the above prescription is concisely stated, it is not easy to translate into practice. How do we sum over paths in superspace? What is the relevant measure ? Since these questions are largely unresolved, we will adopt a more modest aim in our approach to quantization.

4. Conformal Fluctuations

From a given classical solution generate another geometry by a conformal transformation

$$g_{ik} = \Omega^2 \overline{g}_{ik} \qquad \text{...(34)}$$

where Ω is a general function of space and time. Writing $\Omega_i = \partial\Omega/\partial x^i$ where x^i are the co-ordinates and using $\overline{g}_{ik}$ to raise or lower suffices we get

$$\int_v R\sqrt{-g}\, d^4x = \int_v (\Omega^2 \overline{R} - 6\Omega_i\ \Omega^i)\left\{-\overline{g}\right\}^{\frac{1}{2}} d^4x + \int_{\partial v} 6\Omega^i\ \Omega_i \left\{\overline{g}\right\}^{\frac{1}{2}} d\Sigma_i \ \text{...(35)}$$

Here the surface integral has appeared because R contains second derivatives of the metric tensor. Normally the action is supposed to contain only the first derivatives of the dynamical variables. To 'remove' the unwanted derivatives Hawking and Gibbons [5] suggested that an extra surface term be added to the Hilbert action, a term whose variation would cancel out the 'unwanted' surface term of (35). Thus henceforth we will ignore the surface integral over ∂v.

Notice that the action (35) is quadratic over Ω, a circumstance which makes it relatively easy [in view of its similarity to (24)] to write down the propagator explicitly. I give the following example.

In Friedmann cosmology, near the classical big bang the dust solution is of the form

$$d\overline{s}^2 = c^2\, dt^2 - \left(\frac{t}{t_0}\right)^{\frac{4}{3}} [\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,] \qquad \text{...(36)}$$

where t_0 is a constant. The density of matter is given by

$$\rho = \rho_0 \frac{t_0^2}{t^2}, \qquad \rho_0 = \text{constant}. \qquad \text{...(37)}$$

In this universe consider a region of co-ordinate volume v, and write the conformal *departure* from the classical solution by

$$\Omega - 1 = \phi \qquad \text{...(38)}$$

The propagator now describes the probability amplitude to go from a state of $\Phi = \Phi_1$ on $\Sigma = \Sigma_1$ to a state of $\Phi = \Phi_2$ on $\Sigma = \Sigma_2$ How are the surfaces Σ_1, Σ_2 denoted ?

Of course the Weyl hypersurfaces t-constant are the natural choices for the family Σ. It is convenient, however, to define a new time co-ordinate τ by

$$\tau = t^{1/3}. \qquad \ldots(39)$$

The propagator is then given by

$$K[\Phi_2, \tau_2; \Phi_1, \tau_1] = \left[\frac{3iV\rho_0 \tau_1^2 \tau_2^2}{4\pi\hbar(\tau_1 - \tau_2)}\right]^{\frac{1}{2}} \exp Q \qquad \ldots(40)$$

where

$$Q = \frac{3iV\rho_0 c^2}{4\hbar(\tau_1 - \tau_2)}\left\{\tau_1^3(\tau_1 - 2\tau_2)\Phi_1^2 + 2\tau_1^2\tau_2^2\Phi_1\Phi_2 + (\tau_2 - 2\tau_1)\tau_2^3\Phi_2^2\right\} \qquad \ldots(41)$$

For details see Ref. [6].

We will assume $\tau_2 < \tau_1$ and study the implications of this as $\tau_2 \rightarrow 0$, the so called classical big bang. Using the wavepacket (20) and the relation (18) for the above propagator leads to the following analogue of (23)

$$\triangle_2 = \frac{\hbar}{3V\rho_0 c^2 \tau_1 \triangle_1}\left\{1 + \left(\frac{3V\rho_0 \triangle_1^2 c^2}{\hbar}\tau_1^3\right)^2\right\}^{\frac{1}{2}} \tau_2^{-2}. \qquad \ldots(42)$$

Notice that $\triangle_2$ diverges as $\tau_2 \rightarrow 0$. Our stationary wave packet therefore becomes infinitely dispersed as the classical singularity is approached. In otherwords, the quantum uncertainty becomes so large that the classical solution ceases to have any significance.

This result was advanced as a conjecture by Hoyle and Narlikar [7] in 1970. More specifically these authors had argued that the nonclassical cosmologies $(\Gamma \neq \overline{\Gamma})$ may permit horizonless models. The presence of a particle horizon in the classical Friedmann cosmology has always been something of an embarrassment. For, a particle horizon inhibits the transmission of physical signals over arbitrarily large distances and therefore, it is not possible to argue that the homogeneity presently observed in the Universe arose from physical processes, just as one argues that the early Universe reached thermodynamic equilibrium through frequent scatterings. Attempts to obtain horizonless cosmologies within the classical framework have failed. The above quantum treatment shows that such models can come out of quantum cosmology with finite probability.

The above result can be generalized to include the conformal fluctuations of a general space-time manifold containing dust (see Ref. [8]). It can be shown that the conformal fluctuations with Ω a function of all four x^i, diverge at the classical space-time singularity provided the Green's functions of the scalar wave operator

$$\Box + \frac{1}{6}R \qquad \ldots(43)$$

diverges at the singularity. Plausible arguments and explicit demonstrations in specific (but fairly general) scenarios show that such a divergence does occur. Therefore, we have every reason to believe that quantum fluctuations diverge at the classical space-time singularity.

5. Stationary States of the Closed Friedmann Universe

The above result suggests that just as quantum theory avoided the singularity of the H-atom discussed in § 1, quantum gravity avoids the space-time singularity of classical relativistic cosmology. In that case can we expect quantum cosmology to produce stationary states like those of the H-atom ? The following analysis of Padmanabhan and Narlikar [9] shows that stationary states are indeed possible.

Consider the closed Robertson-Walker line element in the form

$$ds^2 = c^2 dt^2 - S^2(t) \left[d\chi^2 + \sin^2\chi (d\theta^2 + \sin^2\theta d\phi^2) \right] \qquad \ldots (44)$$

with $S(t)$ as a quantum variable. That is, we want to construct the propagator $K[S_2, t_2 ; S_1, t_1]$ by the path integral method. Using the Hilbert action we get

$$\mathbf{A} = \frac{3\pi c^2}{4G} \int_{t^1}^{t_2} S(c^2 - \dot{S}^2) dt. \qquad \ldots (45)$$

(The spatial integral is supposed to give $2\pi^2$.)

Unlike the case of conformal fluctuations, this action does not have a quadratic integrand, Thus explicit determination of K is not possible. Nevertheless consider the following transformation

$$S = \lambda q^{2/3} \qquad \ldots (46)$$

$$m = \frac{2\lambda^3 c^2}{3G}, \; \phi(q) = \frac{3\lambda c^2}{4G} q^{2/3}, \qquad \ldots (47)$$

to rewrite (45) as

$$\mathbf{A} = -\int_{t_1}^{t_2} \left[\tfrac{1}{2} m \dot{q}^2 - \Phi(q) \right] dt. \qquad \ldots (48)$$

Apart from the negative sign, (48) resembles the dynamical action for a particle of mass m moving under a potential Φ. If $\Psi(q)$ denotes the wavefunction of the particle we have the corresponding Schrodinger equation given by

$$-\frac{\hbar^2}{2m} \frac{d^2 \Psi}{dq^2} + \Phi(q)\Psi = E\Psi. \qquad \ldots (49)$$

[Compare with our earlier discussion leading to (30)].

A discussion of the Sturm-Liouville problem for (49) shows that stationary states exist for the system described here and that the 'lowest' energy state has

$$< S^2 > \geq \frac{Gh}{c^3} = L_P^2. \qquad \ldots (50)$$

In other words L_P, the Planck length appears as the lower bound to the root mean square value of the scale factor.

2

What does the above result mean? It means that stationary, i.e. time independent states exist for the Universe provided it has small enough scale factor. Since $<S^2>$ is non-zero we can argue that these states are non-singular.

It it possible to generalize this result?

6. Bianchi Universes

Recently Padmanabhan [10] has generalized the above result to Bianchi models, i.e., to models wherein the Universe is homogenous but not isotropic. The metrics of such models have isometry groups of $\geqslant 3$ parameters. It is convenient to write the general metric in the form (of Ref. [11] for details).

$$ds^2 = c^2\, dt^2 - S_0{}^2\,(t) e^{-2\Omega\,(t)} \exp\left[\, 2\beta_{\lambda\mu}(t) \right] \omega^\lambda \omega^\mu \qquad \ldots(51)$$

Here ω^λ are the basis one forms and their differentials satisfy the relations

$$d\omega^\lambda = \tfrac{1}{2} c^\lambda{}_{\mu\nu}\, \omega^\lambda \wedge \omega^\nu, \qquad \ldots(52)$$

$c^\lambda{}_{\mu\nu}$ being the structure constants of the isometry group. So is a constant while $\Omega(t)$ is a function describing how the spatial volume element changes with time. The matrix $\beta_{\lambda\mu}$ may be expressed as similar to the diagonal matrix

$$\beta_d = \mathrm{diag}\,[\, \beta_+ + \sqrt{3}\,\beta_-,\ \beta_+ - \sqrt{3}\,\beta_-,\ -2\beta_+ \,] \qquad \ldots(53)$$

through the relation

$$\beta = M^{-1}\, \beta_d\, M \qquad \ldots(54)$$

$$M = e^{\Phi K_3}\, e^{\theta K_1}\, e^{\Psi K_3} \qquad \ldots(55)$$

$$K_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \qquad K_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \ldots(56)$$

The models are therefore described by the six parameters $(\Omega, \beta_+, \beta_-, \Psi, \theta, \phi)$.

To discuss stationary states and other quantum behaviour of such space-times it is convenient to use the framework of superspace. In this framework a supermetric is defined, which measures the 'distance' between the neighbouring 3-geometries. The supermetric can of course have arbitrary forms. However, it is found [12] that the following supermetric is simple and yet significant:

$$d\sigma^2 = G_{\lambda\mu\nu\rho}\, dg^{\lambda\mu}\, dg^{\nu\rho} \qquad \ldots(57)$$

$$G_{\lambda\eta\nu\rho} = \tfrac{1}{2}\left(g_{\lambda\nu} g_{\mu\rho} + g_{\lambda\rho} g_{\mu\nu} - 2 g_{\lambda\mu} g_{\nu\rho} \right), \qquad \ldots(58)$$

is symmetric in (λ, μ), (ν, ρ) and for the interchange of the pair (λ, μ) and (ν, ρ)

Consider the following simple action in the superspace

$$\mathbf{A}_s = -\int\left(\frac{d\sigma}{dt}\right)^2 d\tau + \int \mathbf{R}\, dt. \qquad \text{...(59)}$$

Here τ is a parameter along a superspace path and R the curvature parameter of the 3-geometry at a typical point. $\mathbf{R}$ is given by

$$\mathbf{R}=\left[{}^{-(3)}\,g\right]^{(3)}\mathrm{R} \qquad \text{...(60)}$$

where ${}^{(3)}g$ is the determinant of the 3-metric tensor and ${}^{(3)}R$ the scalar curvature for the 3-geometry. It turns out that with the supermetric given by (58), the action principle above gives Einstein's equations.

For the Bianchi models considered above the superspace has essentially 6 dimensions and its supermetric may be written as

$$\begin{aligned} d\sigma^2 = 24[d\Omega^2 - d\beta_+^2 - d\beta_-^2 &- \tfrac{1}{3}\sinh^2(2\sqrt{3}\,\beta_-)(d\Psi + \cos\theta\, d\Phi)^2 \\ &-\tfrac{1}{3}\sinh^2(3\beta_+ + \sqrt{3}\,\beta)(\sin\Psi\, d\theta - \cos\Psi\sin\theta\, d\Phi)^2 \\ &-\tfrac{1}{3}\sinh^2(3\beta_+ - \sqrt{3}\,\beta_-)(\cos\Psi\, d\theta - \sin\Psi\sin\theta\, d\Phi)^2]. \end{aligned} \qquad \text{...(61)}$$

For the diagonal Bianchi models Ψ, θ, Ψ are zero and (61) becomes

$$d\sigma^2 = 24\,[\, d\Omega^2 - d\beta_+^2 - d\beta_-^2 \,] \qquad \text{...(62)}$$

$$\mathbf{R} = S_0^4\, e^{-4\Omega}\,[\,V(\beta_+, \beta_-)-1]. \qquad \text{...(63)}$$

The 'potential' function V is known explicitly for all Bianchi type I–IX models.

If we further restrict to isotropic models, we are back to the Robertson–Walker case with

$$d\sigma^2 = 24\, d\Omega^2. \qquad \text{...(64)}$$

We will, however consider (62).

The superspace aetion (59) is accordingly given by

$$\mathbf{A}_s = \int\left[\, 24(\dot{\beta}_+^2 + \dot{\beta}_-^2 - \dot{\Omega}^2) - S_0^4\, e^{-4\Omega}\,(V-1)\right] d\tau, \qquad \text{...(65)}$$

where the overhead dot denotes differentiation with respect to τ. Corresponding to (65) we can at once write down the Schrodinger equation in the form

$$\left(\frac{\partial^2}{\partial\Omega^2} - \frac{\partial^2}{\partial\beta_+^2} - \frac{\partial^2}{\partial\beta_-^2}\right)\Psi + 96\, S_0^4\, e^{-4\Omega}(V-1)\,\Psi = E\Psi \qquad \text{...(66)}$$

The wavefunction Ψ depends on Ω and $\beta_\pm$.

This wave equation can be shown to have stationary states [10]. The above analysis using superspace metric agrees with our stationary state calculation of §5 in the simple case of (64), i.e..for the isotropic universe. This is satisfying since our superspace metric and its relationship to Einstein's equations seems to carry over in the quantum domain also.

7. Conclusions

This gives a brief glimpse of what might be possible by taking limited but clearcut sorties beyond the classical domain and into the quantum domain. We still need to know answers to a number of questions. Among them are (i) the interpretation of the energy levels E which appear in our stationary state calculations, (ii) the possibility of transitions between different stationary states and (iii) the details of the process which describes transition of the Universe from the quantum domain $A \lesssim h$ to the classical domain $A \gg$, h etc. etc.

The conerete results obtained from the preliminary work described here will, it is hoped, lead to the tackling of these more interesting problems.

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