A novel approach to particle production in an uniform electric field

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We outline a different method of describing scalar field particle production in a uniform electric field. In the standard approach, the (analytically continued) harmonic oscillator paradigm is important in describing particle production. In the gauges normally considered, in which the four vector potential depends only on the time or space coordinate, the system reduces to a non-relativistic effective Schrödinger equation with an inverted oscillator potential. The Bogolubov coefficients are determined by tunnelling in this potential. In the Schwinger proper time method of determining the effective Lagrangian, the analytically continued propagator for the usual oscillator system is regarded as the correct propagator for the inverted oscillator system and is used to obtain the gauge invariant result.

However, there is another gauge in which the particle production process has striking similarities with the one used to describe Hawking radiation in black holes. This gauge we use to describe the electric field in is the *lightcone* gauge, so named because the mode functions for a scalar field are found to be singular on the lightcone. We use these modes in evaluating the effective Lagrangian using the proper time technique. The key feature of this analysis is that these modes can be explicitly "normalized" by using the criterion that they reduce to the usual flat space modes in the limit of the electric field tending to zero. This normalization procedure allows one to determine the Schwinger proper time kernel without using the analytical continuation of the harmonic oscillator kernel that is resorted to in the standard analysis. We find that the proper time kernel is not the same as the analytically continued oscillator kernel though the effective Lagrangian is the standard result as it should be.

We also consider an example of a confined electric field system using the lightcone gauge modes that has several features of interest. In particular, our analysis indicates that the Bogolubov coefficients, in taking the limit to the uniform electric field case, are multiplied by energy dependent boundary factors that have not been taken into account before.

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I. INTRODUCTION

We present a different derivation of particle production in an uniform background electric field in Minkowski spacetime. The problem addressed is that of a scalar field propagating in flat spacetime in such a background. The backreaction on the electric field due to particle production is not discussed but only the mechanism by which particles are produced is considered. The difference between the method described here and the standard analysis discussed in Refs. [2–6] arises in the gauge used. The electric field here is described using the *lightcone* gauge which has already been introduced in Ref. [1]. In this gauge, the mode functions are combinations of elementary functions and they are singular on the lightcone. The latter property is similar to the mode functions in spacetimes with a horizon like the Schwarzchild or Rindler spacetimes with the modes being singular on the horizon. This property of the lightcone gauge modes explicitly shows that particle production occurs in a similar fashion in both systems (though not exactly in the same way as will be subsequently shown). Note that these modes describe the same system as the parabolic cylinder functions do in the time and space dependent gauges used normally. The singularity present in the modes of the lightcone gauge manifests itself as the singular inverted oscillator potential in the other two gauges. These modes have the property that they can be "normalized" by a suitable physical criterion which allows one to calculate the Schwinger proper time kernel in a straightforward manner using an appropriate extension of the Feynman-Kac formula. Such a normalizability property circumvents the need to regard the analytically continued harmonic oscillator kernel as the correct propagator for the inverted oscillator kernel. This is required in the standard analysis because the parabolic cylinder functions cannot be normalized in a simple manner. The proper time kernel determined from

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Note that the solution for ϕ is an elementary function of the variable u. This is unlike the modes in the time or space dependent gauge which contain transcendental parabolic cylinder functions. This is the simplest mode function possible for any gauge of the electric field. This solution is singular on the null surface $t - x = \gamma/qE_0$ and it is for this reason that the gauge in Eq. (2.1) is referred to as the lightcone gauge. The Bogolubov coefficients can be easily calculated by constructing a tunnelling scenario for $\phi(u)$, calculating the transmission and reflection coefficients using the method of complex paths and then using the tunnelling interpretation to suitably interpret these coefficients. This has been done in detail in Ref. [1] and will not be repeated here. It was shown there that the standard result is obtained.

From the form of the lightcone gauge given in Eq. (2.1), it is clear that there is another equivalent gauge that also gives the simplest mode functions possible. This gauge is of the form

$$A^i = -\frac{E_0}{2}\left(t + x,\, t + x,\, 0,\, 0\right). \tag{2.8}$$

This gauge, however, will not be considered here separately since its properties are very similar to that of the light cone gauge.

In the next section, we calculate the effective Lagrangian using Schwinger's proper time approach.

III. EFFECTIVE LAGRANGIAN FOR ELECTRIC FIELD

The uniform electric field problem, in the time dependent gauge, can be essentially reduced to an effective Schrödinger problem with an inverted harmonic oscillator potential with mode functions that are transcendental parabolic cylinder functions. This effective quantum mechanical system has no ground state. But the basic formalism of the effective Lagrangian method requires that the system be in the vacuum state in the asymptotic past and future. It makes the implicit assumption that the electric field tends to zero in the asymptotic limits. This issue is resolved, in the path integral technique, by analytically continuining the simple harmonic oscillator kernel to imaginary frequencies [6]. This analytically continued kernel is assumed to be the correct kernel for the electric field system. Such a continuation also implies the boundary condition that the electric field tends to zero asymptotically.

In this section, we propose an alternative derivation of the effective Lagrangian result without using the harmonic oscillator kernel. The gauge we work in is the light cone gauge discussed in the previous section. The mode functions in this gauge are combinations of elementary functions and are singular on the light cone. This singular behaviour implies that the proper time mode functions cannot be normalized by the usual Schrödinger normalization condition. (This non-normalizability of the proper time modes also occurs in the standard approach because of the presence of the parabolic cylinder functions which do not have the required asymptotic behaviour.) However, this can be circumvented by demanding that these modes reduce to the usual flat space modes in the limit of the electric field $E_0 \to 0$. Imposing this normalization condition is equivalent to the analytic continuation that is resorted to in the standard approach and gives the correct result. The lightcone structure of these modes plays an important role in determining particle production as will be shown. The entire analysis is conveniently done in the (u, v, y, z) coordinate system where $u = t - x$ and $v = t + x$ are the usual null coordinates.

This section is divided into two parts. In the first part, the proper time kernel is calculated for the case $E_0 = 0$ in the (u, v, y, z) coordinate system in order to motivate the discussion for the case $E_0 \neq 0$ which will be considered in the next part. The effective Lagrangian will be calculated subsequently.

A. Proper time kernel for $E_0 = 0$

For the case $E_0 = 0$, the proper time effective Schrödinger equation in the (u, v, y, z) coordinate system for a scalar field of mass m is

$$\left(4\partial_u\partial_v - \nabla_\perp^2 + m^2\right)\Phi = E\Phi, \tag{3.1}$$

where E is the "energy" corresponding to the proper time s. The solution to the above equation is of the form

$$\Phi = N e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} e^{-i\alpha u} e^{-i\gamma v}, \tag{3.2}$$

where N is a normalisation constant to be determined and α, γ and $\mathbf{k}_\perp = (0, k_y, k_z)$ are arbitrary constants taking values in the range $(-\infty, \infty)$ such that

$$E = m^2 + \mathbf{k}_\perp^2 - 4\alpha\gamma. \tag{3.3}$$

We normalize Φ by the usual Schrödinger prescription to obtain the normalized wavefunctions

$$\Phi = \frac{\sqrt{2}}{(2\pi)^2} e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} e^{-i\alpha u} e^{-i\gamma v}. \tag{3.4}$$

The proper time kernel, using the Feynman-Kac expression, is

$$\begin{aligned} K(a,b;s) &= \frac{2}{(2\pi)^4}\int d^2\mathbf{k}_\perp d\alpha d\gamma e^{i\mathbf{k}_\perp\cdot(\mathbf{a}_\perp-\mathbf{b}_\perp)} e^{-i\alpha(u_a-u_b)} e^{-i\gamma(v_a-v_b)} e^{-iEs} \\ &= \frac{2\pi}{(2\pi)^4 is} e^{i(\mathbf{a}_\perp^2-\mathbf{b}_\perp^2)/4s} e^{-im^2 s}(2\pi)\int d\gamma\, \delta\left[4\gamma s - (u_a-u_b)\right] e^{-i\gamma(v_a-v_b)} \\ &= \frac{1}{16\pi^2 is^2} e^{-im^2 s} e^{i(\mathbf{a}_\perp^2-\mathbf{b}_\perp^2)/4s} e^{-i(u_a-u_b)(v_a-v_b)/4s} \\ &= \frac{1}{16\pi^2 is^2} e^{-im^2 s} e^{-i(a-b)^2/4s}, \end{aligned} \tag{3.5}$$

where we have substituted for E from Eq. (3.3), $a = (u_a, v_a, a_y, a_z)$, $b = (u_b, v_b, b_y, b_z)$, $\mathbf{a}_\perp = (0, a_y, a_z)$, $\mathbf{b}_\perp = (0, b_y, b_z)$ and $\delta(x)$ is the one dimensional Dirac delta function. The above result is seen to be the standard result for a complex scalar field.

B. Proper time kernel for $E_0 \neq 0$

Now, we are ready to consider the case of a uniform electric field pointing along the $\hat{\mathbf{x}}$ direction with a magnitude E_0. The gauge we consider is the light cone gauge. The proper time effective Schödinger equation in this case is

$$\left[(\partial_i + iqA_i)\left(\partial^i + iqA^i\right) + m^2\right]\Psi = E\Psi. \tag{3.6}$$

The lightcone gauge, in the (u, v, y, z) coordinate system, is

$$A^i = E_0\,(u,\, 0,\, 0,\, 0)\,. \tag{3.7}$$

The solutions to Eq. (3.6) are given by

$$\Psi = N e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} e^{-i\gamma v}\left[\sqrt{\frac{qE_0}{2}}u - \sqrt{\frac{2}{qE_0}}\gamma\right]^{i\varrho-\frac{1}{2}}, \tag{3.8}$$

where N is a normalisation constant to be determined, γ, ϱ and $\mathbf{k}_\perp$ are arbitrary constants and the energy E is given by the relation

$$E = m^2 + \mathbf{k}_\perp^2 - 2qE_0\varrho\,. \tag{3.9}$$

Since the above mode functions are singular on the surface $u = 2\gamma/qE_0$, it can easily be shown that they cannot be normalized by the usual Schrödinger prescription. To make progress, we impose the condition that, in the limit of $qE_0 \to 0$, Ψ reduces to the usual Minkowski mode functions given in (3.4). This implies that the "normalized" mode functions must be of the form

$$\Psi = \frac{\sqrt{2}}{(2\pi)^2}\left(\sqrt{\frac{2}{qE_0}}\gamma\right)^{-\frac{2i\alpha\gamma}{qE_0}+\frac{1}{2}} e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} e^{-i\gamma v}\left[\sqrt{\frac{2}{qE_0}}\gamma - \sqrt{\frac{qE_0}{2}}u\right]^{\frac{2i\alpha\gamma}{qE_0}-\frac{1}{2}}, \tag{3.10}$$

where a new constant α has been defined such that

$$E = m^2 + \mathbf{k}_\perp^2 - 4\alpha\gamma, \tag{3.11}$$

just as in (3.3) with $\varrho = 2\alpha\gamma/qE_0$. Extending the definition of the Feynman-Kac formula for such a system, we have

$$K(a,b;s) = \frac{2}{(2\pi)^4}\sqrt{\frac{2}{qE_0}}\int d^2\mathbf{k}_\perp d\alpha d\gamma\, \gamma e^{i\mathbf{k}_\perp\cdot(\mathbf{a}_\perp-\mathbf{b}_\perp)} e^{-i\gamma(v_a-v_b)} e^{-i(m^2+\mathbf{k}_\perp^2-4\alpha\gamma)s}$$
$$\times \left[\sqrt{\frac{2}{qE_0}}\gamma - \sqrt{\frac{qE_0}{2}}u_a\right]^{\frac{2i\alpha\gamma}{qE_0}-\frac{1}{2}} \left[\sqrt{\frac{2}{qE_0}}\gamma - \sqrt{\frac{qE_0}{2}}u_b\right]^{-\frac{2i\alpha\gamma}{qE_0}-\frac{1}{2}} . \tag{3.12}$$

Doing the integrals over k_y and k_z and defining new dimensionless variables

$$\alpha' = \sqrt{\frac{2}{qE_0}}\alpha, \qquad \gamma' = \sqrt{\frac{2}{qE_0}}\gamma, \tag{3.13}$$

the expression for the kernel becomes

$$K(a,b;s) = \frac{2}{(2\pi)^3 is}\left[\frac{qE_0}{2}\right]^{3/2} e^{-im^2 s} e^{i(\mathbf{a}_\perp-\mathbf{b}_\perp)^2/4s}\int d\alpha' d\gamma'\, \gamma' e^{-i\gamma'\sqrt{qE_0/2}(v_a-v_b)}$$
$$\times\, e^{2iqE_0 s\alpha'\gamma'}\left[\left(\gamma' - u_a\sqrt{qE_0/2}\right)\left(\gamma' - u_b\sqrt{qE_0/2}\right)\right]^{-\frac{1}{2}}$$
$$\times\, \exp\left(i\alpha'\gamma' \ln\left[\frac{\gamma' - u_a\sqrt{qE_0/2}}{\gamma' - u_b\sqrt{qE_0/2}}\right]\right). \tag{3.14}$$

When $u_a \neq u_b$ and assuming further that $u_a > u_b$ for definiteness, the integral over α' gives a Dirac delta function which can be easily evaluated to give the result

$$K(a,b;s) = \frac{1}{16\pi^2 is}\frac{qE_0}{\sinh(qE_0 s)} e^{-im^2 s} e^{i(\mathbf{a}_\perp-\mathbf{b}_\perp)^2/4s}$$
$$\times \exp\left(-\frac{iqE_0\left(u_a - u_b e^{-2qE_0 s}\right)(v_a - v_b)}{2\left(1-e^{-2qE_0 s}\right)}\right). \tag{3.15}$$

In the limit of $qE_0 \to 0$, it is easily checked that the above result reduces to the free field result in Eq. (3.5). This result is clearly not the same as the proper time kernel of the analytically continued simple harmonic oscillator. This difference merely reflects the choice of gauge in each case.

The effective Lagrangian can be calculated using the above kernel by setting $v_a = v_b$, $\mathbf{a}_\perp = \mathbf{b}_\perp$ and taking the limit $u_a \to u_b$. This gives

$$L_{\rm eff} = -i\lim_{a\to b}\int_0^\infty \frac{ds}{s}K(a,b;s)$$
$$= -\frac{1}{16\pi^2}\int_0^\infty \frac{ds}{s^2}e^{-im^2 s}\frac{qE_0}{\sinh(qE_0 s)}, \tag{3.16}$$

which indeed is the standard result [6]. The imaginary part of the effective Lagrangian can be calculated in the usual way by using standard contour integration techniques. Note that the final answer is seen to be valid even if $u_a \neq u_b$ (but with $v_a = v_b$ and $\mathbf{a}_\perp = \mathbf{b}_\perp$) and so is perfectly well defined in the limit $u_a \to u_b$. Therefore, we see that the standard result is recovered in a rather simple and straightforward manner. The role of the light cone structure appears to make no difference to the production of particles as to be expected of a gauge invariant result. The following subtlety however, is worth noting. Let us impose the condition $u_a = u_b$ *before* the evaluation of the integral over α' in Eq. (3.14). In this case, the last exponential containing the logarithmic term does not contribute. Evaluating the integral over α', one has

$$K(a,b;s) = \frac{2}{(2\pi)^2 is}\left[\frac{qE_0}{2}\right]^{3/2} e^{-im^2 s} e^{i(\mathbf{a}_\perp-\mathbf{b}_\perp)^2/4s}\int d\gamma'\, \gamma' e^{-i\gamma'\sqrt{qE_0/2}(v_a-v_b)}$$
$$\times \left(\gamma' - \sqrt{qE_0/2}u_a\right)^{-1}\delta(2qE_0 s\gamma')$$
$$\equiv 0. \tag{3.17}$$

The kernel vanishes with the consequent result that the effective Lagrangian is *identically zero.* A possible way of understanding this sensititivity to the order of operations is as follows. The presence of the electric field produces a singularity on the light cone at each spacetime point (with the singularity at a point x occuring at a time $t = x + (2\gamma/E_0))$. In order to have pair production, the electric field modes have to propagate past this lightcone singularity. Imposing the condition $u_a = u_b$ before the evaluation of the integral over α' in Eq. (3.14) implies that this propagation across the singularity does not take place with the result that the kernel and the effective Lagrangian do not acquire an imaginary part. Hence no particles are produced. The zero result is primarily the consequence of the normalization criteria used to normalize the modes and just means that there is no vacuum polarization term present. When $u_a \neq u_b$, the evaluation of the Dirac Delta function to give the result in Eq. (3.15) ensures propagation of these modes across the singularity. Since the contribution to the kernel from the singularity, which results in the appearance of an imaginary term, is independent of u_a or u_b (this is analogous to a tunnelling situation where the tunnelling coefficients are independent of the initial and final coordinates and arise only from the singularities and turning points present in the potential in the complex plane; see, for example, Ref. [1]), the final answer is well defined even in the limit of $u_a \to u_b$. We can therefore conclude that particle production in an uniform electric field is dependent on the light cone structure of the electric field modes. This clearly shows that electric field particle production is essentially a tunnelling process. In the time and space dependent gauges, the singular potential was responsible for particle production while in the light cone gauge, it is the singularity present on the lightcone.

IV. A CONFINED ELECTRIC FIELD SYSTEM

It was mentioned earlier that the uniform electric field problem, in the purely time dependent gauge, can be essentially reduced to an effective Schrödinger problem with an inverted harmonic oscillator potential. Since this system does not possess the required asymptotic properties, the effective Lagrangian has to be calculated in a suitable fashion (in the path integral method, the analytic continuation of the proper time kernel of the simple harmonic oscillator to imaginary frequencies provides the solution). In order to explicitly justify the method used to compute the effective Lagrangian, one has to consider a system where the electric field is temporally bounded. That is, one should assume a continuous four vector potential that corresponds to zero electric field everywhere in the distant past and future. This four vector potential should also contain a parameter that enables this system to tend to the uniform field case in a suitable limit. By appropriate mode matching at the boundaries (or by determining the exact solution for a smoothly varying electric field) and calculating the Bogolubov coefficients and subsequently the effective Lagrangian, it ought to be possible to verify if the methods used in the standard calculation are justified by taking an appropriate limit to the uniform field case.

One such electric field which is mathematically tractable is a time varying homogeneous electric field system of the form

$$\mathbf{E} = \frac{E_0}{\cosh^2(\omega t)}\hat{\mathbf{x}}, \tag{4.1}$$

which tends to zero in the infinite past and future [7] (see also Refs. [8,9] for related work). The above example admits an exact solution for the mode functions in terms of hypergeometric functions (see [11], Part 2, pp.1651-1660). In the limit of $\omega \to 0$, it is clear that the system tends to an uniform electric field system. This system can be reduced to an effective Schrödinger equation in the t coordinate. By analysing this effective Schrödinger system, it can be shown that, in the limit $\omega \to 0$, the transmission and reflection coefficients tend to the standard values thus showing that these values are obtainable using a well defined limiting process. Thus, the effective Lagrangian can be obtained in a consistent manner which is free of the issue raised in the previous paragraph about the non-asymptotic behaviour of the uniform electric field system.

In the example discussed above, the boundary conditions at termporal infinity were not imposed properly. Though the complete solution is given in terms of hypergeometric functions which have the required asymptotic behaviour, it should be noted that an extra normalization factor arises if these modes are matched to the standard Minkowski mode functions as mentioned in the introduction. This extra term modifies the expressions for the Bogolubov coefficients and thus the number of particle pairs created is different. However, the relative probability of pair creation, which is quantified by the reflection coefficient R in the temporally varying electric field [7], remains unchanged. This occurs because both the Bogolubov coefficients are modified in exactly the same way by a multiplicative term.

We shall now study an example of a confined electric field system that is conveniently described using the lightcone gauge. Recall that the modes in this gauge are singular on the lightcone surface. This is very similar to the black hole system where the modes are singular on the horizon which is also a null surface. Particle production described in such a gauge appears to be remarkably similar to that occuring in a black hole system with the presence of a null surface playing an important role (also see the concluding paragraphs of section (III)). This system is constructed such that, in the limit of a suitable parameter tending to infinity, it tends to a uniform electric field system. It also clarifies the issue raised in the previous paragraph by showing that the Bogolubov coefficients, in this limit, are modified by an extra factor that arises due to mode matching at the boundaries.

Consider a vector potential that is continuous in the null coordinate $u = t - x$ of the form

$$A^i = \begin{cases} (0,\,0,\,0,\,0) & u \le u_1 \quad \text{(in region)} \\ E_0(u-u_1,\,0,\,0,\,0) & u_1 < u < u_2 \quad \text{(region II)} \\ E_0(u_2-u_1,\,0,\,0,\,0) & u \ge u_2 \quad \text{(out region)} \end{cases}, \tag{4.2}$$

where u_1 and u_2 are constants. The electric field, charge density ρ and current density $\mathbf{j}$ for this system are

$$\mathbf{E} = E_0\theta(u-u_1)\theta(u_2-u)\hat{\mathbf{x}},$$
$$\rho = \frac{E_0}{4\pi}\left[\delta(u_2-u) - \delta(u-u_1)\right], \qquad \mathbf{j} = (\rho,\,0,\,0)\,, \tag{4.3}$$

where $\theta(x)$ the step function. The above electric field propagates along the null geodesic $u = constant$. At any particular point in space, the electric field switches on and off for a finite time interval starting from some particular time that is dependent on the location of this point. That is, the electric field, at any fixed point x in space, switches on at $t_i = x + u_1$ and switches off at $t_f = x + u_2$ which therefore implies that the interval during which the electric field is on at any point in space is $t_f - t_i = u_2 - u_1 = T$. During this time interval, the electric field at that point is constant. The charge and current density configuration required to set up such a system is clearly unfeasible since it involves positive and negative charges moving at the speed of light along the null lines $u = u_2$ and $u = u_1$ respectively. Though such a system is physically unrealizable, it is neverthless worth studying since it is mathematically simple and tends to a uniform electric field system by a concrete limiting procedure.

From the form of A^i in Eq. (4.2), we see that our calculations can be conveniently done in the $(u.v.y,z)$ coordinate system where $u = t - x$ and $v = t + x$. The advantage is that the scalar wave equation reduces to solving first order equations in the u and v variables. Therefore, imposing boundary conditions in the u variable involves just matching the modes at the boundary and *not* the first derivatives. The scalar wave equation in the (u,v,y,z) coordinate system with a vector potential of the form $A^i = (f(u),\,0,\,0,\,0)$ is

$$\left[4\partial_u\partial_v + 2iqf(u)\partial_u + iqf'(u) - \nabla_\perp^2 + m^2\right]\Psi = 0, \tag{4.4}$$

where $f'(u) = df/du$, $\nabla_\perp^2 = \partial_y^2 + \partial_z^2$ and q and m are the charge and mass of the scalar field respectively.

The flat space modes in the "in region", $u < u_1$, are

$$\Psi_{\rm in} = N_{\rm in} e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} e^{-i\gamma_{\rm in}v} e^{-i\alpha_{\rm in}u}, \qquad m^2 + \mathbf{k}_\perp^2 = 4\alpha_{\rm in}\gamma_{\rm in}, \tag{4.5}$$

where $\gamma_{\rm in}$ and $\alpha_{\rm in}$ are arbitrary constants satisfying the relation on the right and $N_{\rm in}$ is a normalization constant to be determined. The other independent mode is $\Psi_{\rm in}^*$ which has an equivalent expression. We would now like to determine the conditions on $\alpha_{\rm in}$ and $\gamma_{\rm in}$ so that $\Psi_{\rm in}$ can be identified as a positive frequency mode. This can be done by noting that the normalized Minkowski positive frequency mode Φ in the (t,x,y,z) coordinate system in the same region can be written in the alternative form

$$\Phi_{\rm in} = \frac{1}{\sqrt{(2\pi)^3 2\omega_{\rm in}}} e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} e^{ik_x x} e^{-i\omega_{\rm in}t}$$
$$= \frac{1}{\sqrt{(2\pi)^3 2\omega_{\rm in}}} e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} e^{-i(\omega_{\rm in}-k_x)v/2} e^{-i(\omega_{\rm in}+k_x)u/2}, \tag{4.6}$$

where $\omega_{\rm in} = \sqrt{k_x^2 + \mathbf{k}_\perp^2 + m^2} > 0$. Comparing the modes in Eq. (4.5) and Eq. (4.6), one sees that, for $\Psi_{\rm in}$ to be a positive frequency mode, one must have

$$\gamma_{\rm in} = \frac{1}{2}(\omega_{\rm in} - k_x) > 0, \quad \alpha_{\rm in} = \frac{1}{2}(\omega_{\rm in} + k_x) > 0, \quad N_{\rm in} = \frac{1}{\sqrt{(2\pi)^3 2\omega_{\rm in}}}, \tag{4.7}$$

which therefore implies that $4\alpha_{\rm in}\gamma_{\rm in} = \omega_{\rm in}^2 - k_x^2$. A similar argument shows that $\Psi_{\rm in}^*$ can be identified with negative frequency modes.

In the previous section, a suitable normalization criterion for the electric field modes was introduced. We apply the same to the electric field modes in region II i.e. we demand that these modes reduce to the standard Minkowski modes in the limit $E_0 \to 0$. Keeping in mind, the relations in Eq. (4.7), the normalized electric field modes in the region $u_1 < u < u_2$ (denoted by $\Psi_{\rm II}$) are given by

$$\Psi_{\rm II} = N_{\rm in}\left(\sqrt{\frac{2}{qE_0}}\gamma_{\rm in}\right)^{-\frac{2i\alpha_{\rm in}\gamma_{\rm in}}{qE_0}+\frac{1}{2}} e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} e^{i\alpha_{\rm in}u_1} e^{-i\gamma_{\rm in}v}$$
$$\times \left[\sqrt{\frac{2}{qE_0}}\gamma_{\rm in} - \sqrt{\frac{qE_0}{2}}(u-u_1)\right]^{\frac{2i\alpha_{\rm in}\gamma_{\rm in}}{qE_0}-\frac{1}{2}}. \tag{4.8}$$

The other independent mode is $\Psi^*_{\rm II}$ which has an equivalent expression. Since the normalization criterion for $\Psi_{\rm II}$ has been chosen such that, in the limit of $E_0 \to 0$, they reduce to the standard modes in Eq. (4.5), we identify $\Psi_{\rm II}$ as the *electric field positive frequency vacuum mode.* Similarly, $\Psi^*_{\rm II}$ can be identified as the electric field negative frequency vacuum mode.

Note that $\Psi_{\rm II}$ and $\Psi^*_{\rm II}$ are singular on the light cone surface $u = u_s = u_1 + 2\gamma_{\rm in}/qE_0$. For particle production to take place, it is *necessary* that the condition $u_1 < u_s < u_2$ hold. This arises as follows: If the boundary condition, that for $u < u_s$, only positive frequency electric field modes are present, is imposed, then the electric field modes for $u > u_s$ are not pure positive frequency modes but are a combination of both positive and negative frequency modes because of the singularity at $u = u_s$. This implies particle production and is possible only if u_s lies between u_1 and u_2. Substituting for $\gamma_{\rm in}$ from Eq. (4.7), this condition can be written as

$$\omega_{\rm in} - k_x < qE_0T. \tag{4.9}$$

For this electric field, only those vacuum modes with wave vectors (k_x, k_y, k_z) satisfying the above condition are excited and hence contribute to particle production. This condition will be used when the effective Lagrangian is calculated.

Finally, consider the "out region". The normalized Minkowski positive frequency modes in the (t, x, y, z) coordinate system in this region are

$$\Phi_{\rm out} = \frac{1}{\sqrt{(2\pi)^3\,(2\omega_{\rm out} - qE_0T)}} e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} e^{i\bar{k}_x x} e^{-i\omega_{\rm out}t}, \tag{4.10}$$

where $\omega_{\rm out}$, $\bar{k}_x$ and (k_y, k_z) satisfy the relation

$$(\omega_{\rm out} - qE_0T/2)^2 - \left(\bar{k}_x + qE_0T/2\right)^2 = \mathbf{k}_\perp^2 + m^2. \tag{4.11}$$

Thus, by analogy with that done for the "in region", the normalized positive frequency modes in the (u, v, y, z) coordinate system are

$$\Psi_{\rm out} = N_{\rm out} e^{i\mathbf{k}_\perp\cdot\mathbf{r}_\perp} e^{-i\gamma_{\rm out}v} e^{-i\alpha_{\rm out}u}, \tag{4.12}$$

with the identifications

$$\gamma_{\rm out} = \frac{1}{2}(\omega_{\rm out} - \bar{k}_x), \quad \alpha_{\rm out} = \frac{1}{2}(\omega_{\rm out} + \bar{k}_x), \quad N_{\rm out} = \frac{1}{\sqrt{(2\pi)^3\,(2\omega_{\rm out} - qE_0T)}}. \tag{4.13}$$

The constant $\gamma_{\rm out}$, or equivalently $\bar{k}_x$, is determined later by using the matching conditions at the boundary $u = u_2$. The other independent mode is $\Psi^*_{\rm out}$ which has an equivalent expression.

Thus, in all three regions, we have

$$\Psi = \begin{cases} N_1^{\rm in}\Psi_{\rm in} + N_2^{\rm in}\Psi^*_{\rm in} & u \le u_1 \\ N_1^{\rm II}\Psi_{\rm II} + N_2^{\rm II}\Psi^*_{\rm in} & u_1 < u < u_s \\ N_3^{\rm II}\Psi_{\rm II} + N_4^{\rm II}\Psi^*_{\rm in} & u_s < u < u_2 \\ N_1^{\rm out}\Psi_{\rm out} + N_2^{\rm out}\Psi^*_{\rm out} & u \ge u_2 \end{cases}, \tag{4.14}$$

where $N_1^{\rm in}$, $N_2^{\rm in}$, ..., $N_2^{\rm out}$ are constants to be determined. We now impose the boundary condition that, for $u < u_1$, only positive frequency Minkowski modes are present. This immediately implies that $N_2^{\rm in} = N_2^{\rm II} = 0$. Matching the modes at the boundary $u = u_1$ gives $N_1^{\rm II} = N_1^{\rm in}$. These positive frequency electric field modes propagate past the singularity at $u = u_s$ to become a combination of positive and negative frequency modes with amplitudes $N_3^{\rm II}$ and $N_4^{\rm II}$. For convenience, we set

$$N_3^{\rm II} = C_1 N_1^{\rm II} \quad ; \quad N_4^{\rm II} = C_2 N_1^{\rm II}. \tag{4.15}$$

These amplitudes C_1 and C_2 can be exactly determined by matching the modes and their derivatives at the singularity. However, since only their modulus squares are finally required, these can be determined from the expressions for the modulus square of the Bogolubov coefficients for these modes (see Ref. [1]) without doing any complicated calculation.

Now, at the boundary $u = u_2$, it is clear that positive frequency electric field modes go over to the positive frequency Minkowski modes and similarly for the negative frequency modes. Since the mode matching must hold for arbitrary v, one must have $\gamma_{\rm in} = \gamma_{\rm out}$. Using this, we get

$$N_1^{\rm out} = C_1 N_1^{\rm in} \left(\frac{N_{\rm in}}{N_{\rm out}}\right) e^{i(\alpha_{\rm in}u_1 + \alpha_{\rm out}u_2)} \left(\sqrt{\frac{2}{qE_0}}\gamma_{\rm in}\right)^{-\frac{2i\alpha_{\rm in}\gamma_{\rm in}}{qE_0}+\frac{1}{2}} \times \left[\sqrt{\frac{2}{qE_0}}\gamma_{\rm in} - \sqrt{\frac{qE_0}{2}}T\right]^{\frac{2i\alpha_{\rm in}\gamma_{\rm in}}{qE_0}-\frac{1}{2}} \tag{4.16}$$

and

$$N_2^{\rm out} = C_2 N_1^{\rm in} \left(\frac{N_{\rm in}}{N_{\rm out}}\right) e^{-i(\alpha_{\rm in}u_1 + \alpha_{\rm out}u_2)} \left(\sqrt{\frac{2}{qE_0}}\gamma_{\rm in}\right)^{\frac{2i\alpha_{\rm in}\gamma_{\rm in}}{qE_0}+\frac{1}{2}} \times \left[\sqrt{\frac{2}{qE_0}}\gamma_{\rm in} - \sqrt{\frac{qE_0}{2}}T\right]^{-\frac{2i\alpha_{\rm in}\gamma_{\rm in}}{qE_0}-\frac{1}{2}} . \tag{4.17}$$

From the expression for the Bogolubov coefficients, we have

$$|C_1|^2 = 1 + e^{-\pi\lambda} \quad ; \quad |C_2|^2 = e^{-\pi\lambda}, \tag{4.18}$$

where $\lambda = (m^2 + \mathbf{k}_\perp^2)/qE_0$ as usual. The final expressions for the Bogolubov coefficients are

$$|\alpha_{\mathbf{k}}|^2 = \left|\frac{N_1^{\rm out}}{N_1^{\rm in}}\right|^2 = \left(\frac{\gamma_{\rm in}}{\omega_{\rm in}}\right)\left(\frac{(qE_0T - 2\gamma_{\rm in})^2 + \mathbf{k}_\perp^2 + m^2}{(qE_0T - 2\gamma_{\rm in})^2}\right)\left(1 + e^{-\pi\lambda}\right) \tag{4.19}$$

and

$$|\beta_{\mathbf{k}}|^2 = \left|\frac{N_2^{\rm out}}{N_1^{\rm in}}\right|^2 = \left(\frac{\gamma_{\rm in}}{\omega_{\rm in}}\right)\left(\frac{(qE_0T - 2\gamma_{\rm in})^2 + \mathbf{k}_\perp^2 + m^2}{(qE_0T - 2\gamma_{\rm in})^2}\right) e^{-\pi\lambda}. \tag{4.20}$$

The expression for $|\beta_{\mathbf{k}}|^2$ gives the number of pairs produced for this electric field system. Note that the relative probability of particle production which is

$$\frac{|\beta_{\mathbf{k}}|^2}{|\alpha_{\mathbf{k}}|^2} = \frac{e^{-\pi\lambda}}{1 + e^{-\pi\lambda}}, \tag{4.21}$$

is *independent* of T. Thus, in the limit of $T \to \infty$ (or equivalently $u_1 \to -\infty$ and $u_2 \to \infty$), which corresponds to an uniform electric field existing over all space and time, it is clear that the standard results are recovered. Moreover, the Bogolubov coefficients also tend to the standard values upto a multiplicative factor i.e.

$$\lim_{T\to\infty} |\alpha_{\mathbf{k}}|^2 = \left(\frac{\omega_{\rm in} - k_x}{\omega_{\rm in}}\right)\left(1 + e^{-\pi\lambda}\right), \qquad \lim_{T\to\infty} |\beta_{\mathbf{k}}|^2 = \left(\frac{\omega_{\rm in} - k_x}{\omega_{\rm in}}\right) e^{-\pi\lambda}. \tag{4.22}$$

This difference can be traced to the fact that, in the standard calculation, boundary effects at $t \to \pm\infty$ due to mode matching were not taken into account. Therefore, the above expressions can be considered to be the correct ones with the extra factors arising due to these boundary effects at $t \to \pm\infty$. Note that both the coefficients have been modified by the same factor. It therefore follows that the relative probability of pair production, which is the ratio $\left(|\beta_{\mathbf{k}}|^2/|\alpha_{\mathbf{k}}|^2\right)$, is independent of this factor.

The imaginary part of the effective Lagrangian for this system follows. Omitting several straightforward intermediate steps (see, for example Ref. [5,6]), one has

$$\int d^4x {\rm Im}\mathcal{L}_{\rm eff} = \frac{V}{2(2\pi)^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} e^{-\frac{\pi m^2}{qE_0}n}\int dk_x \int 2\pi|\mathbf{k}_\perp|\, d|\mathbf{k}_\perp| e^{-\frac{\pi n}{qE_0}\mathbf{k}_\perp^2}, \tag{4.23}$$

where the limits over k_x and $|\mathbf{k}_\perp|$ are to be determined from the condition in Eq. (4.9). From Eq. (4.9), we have,

$$\mathbf{k}_\perp^2 < (qE_0T)^2 + (2qE_0T)k_x - m^2 = L(k_x). \tag{4.24}$$

Thus, the limits for the $|\mathbf{k}_\perp|$ integral are $\left(0, \sqrt{L(k_x)}\right)$. From the condition that $L(k_x) > 0$ always, one must have

$$k_x > -\left(\frac{(qE_0T)^2 - m^2}{2qE_0T}\right) = -M. \tag{4.25}$$

Consider only the integral over k_x and $|\mathbf{k}_\perp|$. Integrating over $|\mathbf{k}_\perp|$, we get

$$\begin{aligned} I &= \int_{-M}^{\infty} dk_x \int_0^{\sqrt{L(k_x)}} 2\pi|\mathbf{k}_\perp|\, d|\mathbf{k}_\perp| e^{-\frac{\pi n}{qE_0}\mathbf{k}_\perp^2} \\ &= \frac{qE_0}{n}\int_{-M}^{\infty} dk_x \left[1 - e^{-\frac{\pi n}{qE_0}L(k_x)}\right]. \end{aligned} \tag{4.26}$$

The integral over the first term in the square brackets gives a formally divergent term. Denoting this by Z and doing the integration over k_x, one obtains

$$I = \frac{qE_0}{n}\left\{Z + \left(\frac{(qE_0T)^2 - m^2}{2qE_0T}\right) - \left(\frac{1}{2\pi nT}\right)\right\}. \tag{4.27}$$

Substituting the above into (4.23), we have

$$\begin{aligned} \int d^3\mathbf{r}\, dt\, \mathrm{Im}\mathcal{L}_{\mathrm{eff}} &= V\sum_{n=1}^{\infty} \frac{1}{2}\frac{(qE_0)^2}{(2\pi)^3}\frac{(-1)^{n+1}}{n^2}\exp\left[-\frac{\pi m^2}{qE_0}n\right] \\ &\quad \times \left\{Z + \frac{1}{2}T - \frac{m^2}{2(qE_0)^2T} - \frac{1}{2\pi qE_0 nT}\right\}. \end{aligned} \tag{4.28}$$

(Note that in the definition of the vacuum persistence amplitude, the limits $t_1 \to -\infty$ and $t_2 \to \infty$ were considered. Since two independent limits are being taken, it follows that, when differentiating with respect to T in Eq. (4.28), the right hand side has to be multiplied by a factor 1/2.) Differentiating both sides with respect to T and discarding the dimensionless divergent term Z/T , one gets the result

$$\mathrm{Im}\mathcal{L}_{\mathrm{eff}} = \sum_{n=1}^{\infty} \frac{1}{2}\frac{(qE_0)^2}{(2\pi)^3}\frac{(-1)^{n+1}}{n^2} e^{-\frac{\pi m^2}{qE_0}n}\left\{1 + \frac{m^2}{(qE_0)^2T^2} + \frac{1}{\pi qE_0 nT^2}\right\}. \tag{4.29}$$

This expression for the effective Lagrangian shows boundary effects occurring as a correction to the standard result. Taking the limit of $T \to \infty$ does reproduce the standard result. The positive sign for these correction terms implies that the vacuum persistence probability is *smaller* than that for the uniform electric field case.

We conclude this section by making a few general remarks regarding the importance of the light cone gauge in calculating the effective Lagrangian. The use of this gauge suggests a way of calculating the effective Lagrangian for a constant electric field system. (By a constant electric field system one means any spacetime region that has a constant electric field present in it.) The following points may be deduced from the effective Lagrangian calculation presented in this section: (1) The light cone gauge, as discussed earlier, has the property that the electric field modes are singular at finite spacetime points that are located on null lines. (2) Each singularity indicates the energy of the vacuum mode that is excited by the electric field. Particle production occurs in that mode when the vacuum modes propagate past this singularity. The set of all these singularities determines the range of energies for which particles are produced (see the discussion leading to condition Eq. (4.9)). (3) In the calculation of the Bogolubov coefficients, the boundary conditions present in the system modified both the coefficients in exactly the same way (see Eq. (4.20)). This implies that the relative probability of pair creation which is the ratio $|\beta_\mathbf{k}|^2/|\alpha_\mathbf{k}|^2$ is *independent* of the extra factors introduced by the boundary conditions. It is dependent solely on the presence of the singularities. This point is justified for any constant electric field system if one accepts that particle production in an electric field is essentially a tunnelling process which arises, in this case, because of the singularities present on the light cone. (In any other gauge, like the time dependent gauge for example, the inverted oscillator nature of the effective potential is responsible for particle production.)

We can use these three points to determine a procedure to calculate the effective Lagrangian for an arbitrary constant electric field system in the following manner. Consider a region of spacetime that has a constant electric field of magnitude E_0 along, say, the $\hat{\mathbf{x}}$ direction (without any loss of generality). The field modes in this region can always be described in terms of the modes of the light cone gauge (multiplied by a suitable gauge factor). The only singularities that occur in the mode functions are due to the light cone gauge modes (we assume that singular gauge transformations are not allowed). Now, identify the set of null rays $u = constant$ passing through this region. Since singularities occur on these rays, this set determines the set of possible modes described by the wave vector $\mathbf{k}$ that can be excited by the electric field. That is, the possible range of values that can be taken by $\mathbf{k} = (k_x, k_y, k_z)$ are determined. Using the fact that the relative probability of pair production in a particular mode, given by

$$\frac{|\beta_{\mathbf{k}}|^2}{|\alpha_{\mathbf{k}}|^2} = \frac{e^{-\pi\lambda}}{1 + e^{-\pi\lambda}}, \tag{4.30}$$

is independent of extra factors arising from boundary conditions, the effective Lagrangian can be calculated using the formula in Eq. (4.23) by evaluating the integrals over $\mathbf{k}$ over the range of values determined above. Any formally divergent terms that occur have to be discarded.

We apply the procedure outlined above to two simple electric field systems. The first is the finite time constant electric field configuration. Here, the electric field is switched on over all spatial points for a finite time interval T. It is easy to see that all null rays $u = constant$ intersect this region. Hence, the range of possible values for $\mathbf{k} = (k_x, k_y, k_z)$ is from $(-\infty, \infty)$ for all the three components. Carrying out the integration over k_y and k_z, we obtain

$$\int_0^T dt \, \mathrm{Im}\mathcal{L}_{\mathrm{eff}} = \sum_{n=1}^{\infty} \frac{1}{2} \frac{(qE_0)^2}{(2\pi)^3} \frac{(-1)^{n+1}}{n^2} e^{-\frac{\pi m^2}{qE_0} n} \int_{-\infty}^{\infty} \frac{dk_x}{qE_0} \tag{4.31}$$

Since this expression is formally divergent, we use the following regularization procedure to obtain a finite result. We set

$$\int_{-\infty}^{\infty} \frac{dk_x}{qE_0} = Z + \int_0^T dt \tag{4.32}$$

where Z is a formally divergent term. Differentiating both sides in (4.31) with respect to T and neglecting the dimensionless divergent term Z/T, it is easy to show that one obtains the standard result. This result is not too surprising because to set up a constant electric field over all space requires an infinite amount of energy. Hence, vacuum modes of arbitrary energy are excited. The second system is an uniform electric field that is bounded along the x-axis (but not along the y or z axes) to a region of width x_0 but existing for all time. This system would correspond to a pair of capacitor plates (of infinite area) separated by a distance x_0. Here too, it is clear that all null rays pass through this region implying that all vacuum energy modes are excited. The effective Lagrangian for this sytem too is the standard result. Notice that in both the above examples, the actual number of pairs produced per mode cannot be determined easily since the boundary conditions are non-trivial and difficult to impose. More complicated examples can be studied in this fashion.

V. CONCLUSIONS

Summarising the analysis in this paper, we see that particle production in an uniform electric field has been described differently from the standard method. The lightcone gauge in Eq. (2.1), clearly indicates the presence of a null surface which is responsible for particle production. This is very similar to the black hole case where again a null surface present in the manifold is responsible for particle creation. Both the black hole modes and the electric field modes (in the light cone gauge) possess a *logarithmic* singularity at the null surface which determines pair creation. However, the crucial difference between the two cases is that the null surface in the black hole case is a *one way* surface which is the horizon. To obtain particle production, we need the semi-classical prescription discussed in [1] or some other equivalent prescription that takes into account this one-way nature of the horizon. In the electric field case, the modes can be described either by the gauge in Eq. (2.1) which is written in terms of the "right moving" null coordinate $u = t - x$ (as done in this chapter) or by the gauge in Eq. (2.8) which uses the "left moving" null coordinate $v = t + x$ (both these gauges are appropriate light cone gauges). Whether the propagation of the electric field modes occurs from left to right or right to left across the singularity makes no difference to the final result. This implies that particle production in an uniform electric field is a genuine tunnelling phenomena (in the quantum mechanical sense) which is not the case for the black hole system.

The effective Lagrangian, when calculated in the lightcone gauge (see section (III)), is found to be the same as the standard result. The modes in this gauge were also explicitly "normalizable" by a suitable physically reasonable criterion (that they reduce to the standard Minkowski modes in the limit of the field tending to zero). Because of this "normalizable" property we can look upon the lightcone gauge as a more *natural* gauge to describe the uniform electric field in. (This property also prompts the question as to whether one can determine indirectly a suitable "normalization" constant for the transcendental scalar field modes in the time or space dependent gauges.) One of the assumptions made in calculating the proper time kernel is regarding the applicability of the Feynman-Kac formula to such a singular system. However, this can be justified since, in the limit of $E_0 \to 0$, the free space result is obtained.

The lightcone gauge was also used to calculate the Bogolubov coefficients and the effective Lagrangian for a finite time, but spatially non-uniform, electric field. The Bogolubov coefficients, in the limit of $T \to \infty$, which implies a uniform electric field over space and time, were found to reduce to the usual results upto a multiplicative factor which was due to extra boundary effects. The property of the light cone gauge that mode functions in this gauge are singular on null surfaces was used to develop a procedure to calculate the effective Lagrangian for a constant electric field present in an arbitrary region of spacetime. This procedure was based crucially on the assumption that the relative probability of pair creation depends only on the presence of the singularities and not on the specific boundary conditions present. This can be justified because pair creation in the uniform electric field is essentially a tunnelling process. In the light cone gauge, only tunnelling across the singularities produces particles and not otherwise. This procedure can be extended to arbitrary electric field systems which have the property that, in a sufficiently small neighbourhood of an arbitrary spacetime point, the electric field can be regarded as constant. This is very similar to the case of Reimannian manifolds which are locally flat. It would be therefore be of interest to ask if a suitable formalism can be developed on the lines of general relativity to describe arbitrary electric fields in the lightcone formalism and extend it to electromagnetic fields that satisfy the condition $\mathbf{E}^2 - \mathbf{B}^2 > 0$ (which must hold for particle production to take place). Such a formalism can be expected to shed light on the role of gauge transformations in particle production and would probably have relevance in describing backreaction on the electric field system. These issues will be considered in a future publication.

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