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Topology Selection through Quantum Cosmology

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ABSTRACT

We argue that, in order to obtain the causal semiclassical Einstein equation at the stages much later than the Planck time, we have to regard the in-in path-integral formalism as fundamental in quantum cosmology. We then deduce that any classical universe should allow at least one maximal surface. If the natural energy condition holds, this means that (1) the classical universe should be a Wheeler universe, i.e. it starts from and ends in a singularity, and (2) the possible topologies of the classical universe are strongly restricted.

The implications of the recent observation of the cosmic microwave background radiation is also discussed.

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When we want to investigate the very early stage of the universe, we are forced to take the quantum gravitational effects into account. One reason is that the universe is very small at that stage. As the action is not a conserved quantity, its typical value becomes comparable to $\hbar$ at that time, hence quantum theoretical treatment becomes necessary. The second reason is that we want to select the possible types of universe to explain the global properties of our universe. General relativity in itself is a set of second order partial differential equations with constraints, which allows infinite number of spacetimes as a solution. There are no means to compare these solutions with each other within a classical theory. Quantum theory, on the other hand, handles all trajectories together, including all possible classical solutions, compares them with each other, and selects suitable trajectories among them.

At present, since we do not have a satisfactory quantum theory of gravity at hand, it might be considered at first sight that quantum cosmological considerations have little meaning. However, we know that general relativity and quantum theory become valid at least at the stages of the universe much later than the Planck time. Only from this fact, the spacetimes can be selected to a great extent, as we shall see below. This kind of approach can be regarded as an extrapolation of our knowledge, and an extrapolation is sometimes quite effective as in the case of the pre-quantum theory.

We confine ourselves to the case where the universe is spatially closed, i.e. any spatial section is compact without boundary. We use the following notation: $S_g := \int R\sqrt{-g}$, matter action $S_M = 16\pi G/c^3 \times$(usual action), $\alpha := 16\pi G\hbar/c^3 = l_{pl}^2$. Then, the non-dimensional quantity, usually written as $S/\hbar$, is expressed as $(S_g + S_M)/\alpha$.

At the later stages of the universe, the semiclassical Einstein equation

$$G_{ab}(x) = 8\pi G/c^3 < \phi|T_{ab}(x)|\phi > \tag{1}$$

becomes valid, when, roughly speaking, the condition $|S_g| \gg \alpha$, $|S_M| \sim \alpha$ is satisfied. Let us pay attention to the fact that this equation possesses a causal character: the right-hand side of Eq. (1) is expressed as an expectation value and it depends only on the information, at most, in the region surrounded by the past light-cone of x and the spatial hypersurface on which the state $|\phi >$ was prepared. Thus, it has a power to predict the future.

The fundamental quantity in quantum cosmology which yields Eq. (1) at later stages of the universe is clearly,

$$(h\ \phi) = \int_{c(h,\phi)} [dg]\,[d\phi] \exp \frac{i}{\alpha}(S_g + S_M(\phi; g)), \tag{2}$$

where $c(h, \phi)$ implies the closed-time path, the boundary value being fixed at (h, ϕ). We parametrize the closed-time path as $\tau : 0 - T$ and $T - 2T$ for its forward and backward sections. We use the suffix " + " or " − " to indicate the quantity on + branch ($\tau : 0 - T$) or − branch ($\tau : T - 2T$). (For the in-in formalism (closed-time path formalism), see [1],[2] and [8].)

For our present purpose, there is no need to attach any probabilistic meaning to $(h\ \phi)$. It is enough to regard $(h\ \phi)$ as something which yields the expectation value for the matter part when the universe behaves classically.

In the usual in-out path integral formulation, we need to fix the in-configuration as well as the out-configuration. If we apply this to quantum cosmology, we have to fix the in-state of the universe in some manner, but there is no preferable choice for such a state. Any theory of quantum cosmology based on the in-out formalism faces this difficulty. On the other hand, the in-in formalism requires only the "present" configuration to be specified. Hence, it can be said that the in-in formalism is essential in quantum cosmology in a two-fold way: One is for causality and another is for the boundary value specification.

A convenient gauge fixing for Eq.(2) is provided by

$$\int_0^T N_+ d\tau_+ = \int_0^T N_- d\tau_-, \quad N_{i_+} = N_{i_-} = 0,$$

with arbitrary gauge-fixing for the remaining freedom. The matter-part integration in Eq.(2) yields the following quantity:

$$\begin{aligned}\exp\frac{i}{\alpha}W[h,\phi;g_+,g_-] &= \int_{c(h,\phi)}[d\phi]\exp\frac{i}{\alpha}S_M(\phi;g) \\ &= \int d\phi'\int[d\phi_+]|_{\phi,\phi'}[d\phi_-]|_{\phi,\phi'}\exp\frac{i}{\alpha}\{S_M(\phi_+;g_+)-S_M(\phi_-;g_-)\} \quad ,\end{aligned} \tag{3}$$

where "$|_{\phi,\phi'}$" indicates boundary values fixed for path-integration.

The metric $g_\pm$ behaves just like the source $J_\pm$ in the usual in-in formalism. Thus, $(h\ \phi)$ becomes

$$(h\ \phi) = \int_{c(h)}[dg]\exp\frac{i}{\alpha}(S_g + W[h,\phi;g]) \tag{4}$$

In the case of $|S_g| \gg \alpha$, we can perform the stationary phase approximation for gravity part. Thus, we obtain

$$\frac{\delta S_g}{\delta g_\pm} + \frac{\delta W}{\delta g_\pm} = 0.$$

This is equivalent to the following two equations,

$$G_{g+} - \frac{\alpha}{2\hbar}\,{}_{g-}\langle\phi|T_{g+}|\phi\rangle_{g+}/{}_{g-}<\phi|\phi>_{g+}= 0, \tag{5-a}$$

$$G_{g-} - \frac{\alpha}{2\hbar}\,{}_{g-}\langle\phi|T_{g-}|\phi\rangle_{g+}/{}_{g-}<\phi|\phi>_{g+}= 0. \tag{5-b}$$

Here, $|\phi>$ is some normalized matter state and ${}_{g-}\langle\cdot\rangle_{g+}$ implies the influence of the source $g_\pm$. As G_{g+} and G_{g-} are real, Eqs. (5-a,b) show that ${}_{g-}\langle\phi|T_{g+}|\phi\rangle_{g+}/{}_{g-}<\phi|\phi>_{g+}$ and ${}_{g-}\langle\phi|T_{g-}|\phi\rangle_{g+}/{}_{g-}<\phi|\phi>_{g+}$ should also be real at the stationary phase point. This indicates that $g_+ = g_-$ ($= g_0$, say), i.e., $+$ and $-$ branches follow the same classical trajectory. Then, Eqs.(5-a,b) reduce to Eq.(1).

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8. For more details on the subject depicted here as well as the detailed investigations on semiclassical gravity, see M. Seriu and T.P. Singh, preprint TIFR-TAP-2/93 "On the relation between causality and topology in the semiclassical universe".