

Higher Dimensional Analogue of McVittie Solution

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Abstract

A generalization of the McVittie solution, representing spacetime of a mass particle placed in $(n + 2)$ dimensional Robertson-Walker universe is reported.

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McVittie [1] obtained a solution of the Einstein's field equations which is described by the metric

$$ds^2 = \left[\frac{1 - \mu(r, t)}{1 + \mu(r, t)} \right]^2 dt^2 - \frac{[1 + \mu(r, t)]^4}{(1 + k\frac{r^2}{4})^2} R^2(t) (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta^2 + r^2 \sin^2 \theta d\phi^2) \quad (1)$$

where

$$\mu(r, t) = \frac{m}{2rR(t)} \left(1 + k\frac{r^2}{4}\right)^{1/2}, \quad (2)$$

m is a constant and $k = 0, \pm 1$. This solution which is important for historical reasons is a superposition of the Schwarzschild exterior solution and the Robertson-Walker solution. It represents the gravitational field of mass particle placed in a Robertson Walker Universe. Several aspects of the McVittie solution, of geometrical and physical relevance have been investigated by many authors [2]. Raychaudhuri [3] established the uniqueness of McVittie solution under following assumptions:

- (i) The metric is spherically symmetric with a singularity at the centre.
- (ii) The matter distribution is a perfect fluid.
- (iii) The metric must asymptotically go over to the isotropic cosmological form.
- (iv) The fluid flow is shear-free.

Developments in superstring theories [4-5] have stimulated the study of physics in higher dimensional spacetimes [6]. Subsequently the implications of relativistic field equations in which dimensions of the spacetime are $1 + D, 3 \leq D \leq 9$, in spite of the lack of direct observational evidence in favour of extra dimensions so far, have been studied by several authors [7-9]. Solutions of Einstein field equations in higher dimensional spacetimes are believed to be of physical relevance possibly at the extremely early times before the universe underwent the compactification transitions. Liddle et al [10] have discussed the consequences of extra dimensions on the structure and maximum mass of a star.. However to our knowledge non-static models describing spacetime of a star placed in a higher dimensional, homogeneous and isotropic universe have not been studied so far. Therefore we thought it worthwhile to obtain a higher dimensional counterpart of the McVittie solution.

We begin with the non-static spherically symmetric metric

$$ds^2 = e^{\nu(r,t)} dt^2 - e^{w(r,t)} \{ dt^2 + r^2 d\Omega_n^2 \} \quad (3)$$

where

$$d\Omega_n^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \dots + \sin^2\theta_1 \sin^2\theta_2 \dots \sin^2\theta_{n-1} d\theta_n^2 \quad (4)$$

of a $(n + 2)$ - dimensional spacetime in isotropic coordinates and

$$T_k^i = (\rho + p) u^i u_k - p \delta_k^i. \quad (5)$$

the energy-momentum tensor for perfect fluid describing its physical content. Adopting co-moving coordinates so that

$$u_i = e^{\nu/2} \delta_i^t, \quad (6)$$

the system of Einstein's field equations reduces to the following set of equations :

$$8\pi\rho = \frac{n(n+1)}{8} e^{-\nu} \dot{w}^2 - \frac{n}{2} e^{-w} \left[w'' - \frac{(n-1)}{4} w'^2 + \frac{n}{2} \frac{w'}{r} \right]. \quad (7)$$

$$8\pi p = \frac{n}{2} e^{-w} \left[\nu' \left(\frac{w'}{2} + \frac{1}{r} \right) + \frac{(n-1)w'}{r} + \frac{(n-1)}{4} w'^2 \right] - \frac{n}{2} e^{-\nu} \left[\ddot{w} - \frac{\dot{w}\dot{\nu}}{2} + \frac{(n+1)}{4} \dot{w}^2 \right] \quad (8)$$

$$\dot{w}' - \frac{\dot{w}\nu'}{2} = 0 \quad (9)$$

and

$$\nu'' + (n-1)w'' + \frac{\nu'^2}{2} - \frac{(n-1)}{2} w'^2 - w'\nu' - \frac{\nu'}{r} - (n-1)\frac{w'}{r} = 0. \quad (10)$$

Here and in what follows an overhead prime and an overhead dot respectively denote differentiations with respect to r and t .

Higher dimensional analogue of the Robertson-Walker metric follows on setting

$$\nu = 0, \quad e^w = \frac{R^2(t)}{(1 + k\frac{r^2}{4})^2}. \quad (11)$$

The higher dimensional analogue of the exterior schwarzschild metric in isotropic coordinates is obtained on setting

$$e^\nu = \left(\frac{1 - \frac{m}{2(n-1)r^{n-1}}}{1 + \frac{m}{2(n-1)r^{n-1}}} \right)^2, \quad e^w = \left(1 + \frac{m}{2(n-1)r^{n-1}} \right)^{4/(n-1)} \quad (12)$$

The expressions (1), (11) and (12) suggest that the metric (3) with

$$e^\nu = \left\{ \frac{1 - B}{1 + B} \right\}^2, \quad e^w = (1 + B)^{4/(n-1)} \frac{R^2}{(1 + \frac{k}{4}r^2)^2} \quad (13)$$

may lead to the higher dimensional version of McVittie's solution. Here B is a function of r and t . In view of (13) equations (9) and (10) become

$$(1 - B) \frac{\dot{B}'}{B'} + \dot{B} + (n - 1) \frac{\dot{R}}{R} = 0 \quad (14)$$

and

$$-\frac{B''}{B} + \frac{n+1}{n-1} \frac{B'^2}{B^2} + \left(\frac{1 - \frac{3}{4}kr^2}{1 + \frac{k}{4}r^2} \right) \frac{B'}{rB} = 0. \quad (15)$$

The equations (14) and (15) admit the solution

$$B = \frac{m}{2(n-1)} r^{1-n} R^{1-n} \left(1 + \frac{k}{4}r^2 \right)^{(n-1)/2} \quad (16)$$

where m is an arbitrary constant. Subsequently the metric

$$ds^2 = \left(\frac{1 - B}{1 + B} \right)^2 dt^2 - (1 + B)^{\frac{4}{n-1}} \frac{R^2(t)}{(1 + \frac{k}{4}r^2)^2} (dr^2 + r^2 d\Omega_n^2) \quad (17)$$

with B as given by (16) represents higher dimensional analogue of the McVittie solution. The pressure and density of the fluid filling this universe are

$$\frac{16\pi p}{n} = \frac{2(1+B)}{(1-B)} \frac{\ddot{R}}{R} - \frac{(n-1) - (n+3)B}{1-B} \frac{\dot{R}^2}{R^2} - \frac{k(n-1)}{R^2(1-B)(1+B)^{(n+3)/(n-1)}} \quad (18)$$

and

$$\frac{16\pi\rho}{n(n+1)} = \frac{\dot{R}^2}{R^2} + \frac{k}{R^2(1+B)^{\frac{n+3}{n-1}}}. \quad (19)$$

The metric (17) asymptotically approaches the isotropic cosmological form in higher dimensions. The fluid flow is shear-free and irrotational. The metric has a singularity at the origin. It represents an inhomogeneous centrosymmetric cosmological model with

$$\Theta = (n+1)\frac{\dot{R}}{R}, \quad \dot{u}_r = -\frac{2(n-1)}{r(1+k\frac{r^2}{4})} \frac{B}{1-B^2} \quad (20)$$

as respectively kinematical parameters of expansion and acceleration. The expressions (18) - (20) show the dimensional dependence of the physical and kinematical parameters.

In the metric (17) the parameter B refers to the mass particle and its evolution in time is linked to the cosmological evolution through its dependence on the scale factor in eqn. (16). For $n > 1$, which is always the case, it will go as inverse power of the scale factor. For the expanding phase it would decrease and the opposite would be the case for contracting phase. It also includes the following particular cases: (i) when $k = 0$, $R(t) = const.$, it degenerates into the $(n+2)$ dimensional version of the Schwarzschild exterior metric, (ii) when $n = 2$, it reduces to the McVittie solution, (iii) when $m = 0$, it represents $(n+2)$ dimensional Robertson-Walker universe and (iv) when $k = 1$, $R(t) = const.$, it describes mass particle placed in the $n+2$ dimensional static Einstein universe.

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