

Collapsing Shells of Radiation in Higher Dimensional Spacetime and Cosmic Censorship Conjecture

S.G. Ghosh*

Department of Mathematics, Science College, Congress Nagar,
Nagpur-440 012, INDIA

R.V. Saraykar†

Department of Mathematics, Nagpur University,
Nagpur-440 010, INDIA

and

A. Beesham‡

Department of Mathematical Sciences, University of Zululand,
P/Bag X1001, Kwa-Dlangezwa 3886, South Africa

Abstract

Gravitational collapse of radiation shells in a non self-similar higher dimensional spherically symmetric spacetime is studied. Strong curvature naked singularities form for a highly inhomogeneous collapse, violating the cosmic censorship conjecture. As a special case, self similar models can be constructed.

KEY WORDS: Gravitational collapse, naked singularity, cosmic censorship, higher dimensions.

PACS number(s): 04.20.Dw

*Author to whom all correspondence should be directed; email: sgghosh@hotmail.com

†email: sarayaka@nagpur.dot.net.in

‡email: abeesham@pan.uzulu.ac.za

1 Introduction

General relativity was formulated in spacetime with four dimensions ($4D$), of course. However, there are theoretical hints that we might live in a world with more dimensions. A generalization of general relativity to higher dimensions has been of considerable interest in recent times. It is believed that the underlying spacetime in the large energy limit of the Planck energy may have higher dimensions. At this level, all the basic forces of nature are supposed to unify and hence it would be pertinent in this context to consider solutions of the gravitational field equations in higher dimensions. Lately, there has been significant attention to studying gravitational collapse, in higher dimensions [1]. Ilha *et al* [2] have generalized the Oppenheimer-Snyder collapse model to higher dimensions.

Gravitational collapse continues to be a very important topic in gravitational research ever since Penrose [3] articulated the cosmic censorship conjecture some three decades ago. The cosmic censorship conjecture forbids the existence of naked singularities. In its strong version, the spacetime must be globally hyperbolic, whereas according to the weaker version, an event horizon must form during collapse and the singularity could be invisible to an asymptotic observer. The collapse of spherical matter in the form of dust or radiation forms shell focusing strong curvature singularities violating the cosmic censorship conjecture [4]-[11].

The purpose of this letter is to study how these features get modified with extra dimensions. We study impinging radiation in a higher dimensional spacetime. The Vaidya metric [12], describing a spherically symmetric spacetime with radiation, is an exact solution of Einstein's field equations. It has been generalized to higher dimensions [13] and we shall call it the higher dimensional Vaidya metric. We find that non self-similar higher dimensional Vaidya spacetime does admit strong curvature singularities in the sense of Tipler [14] for a sufficiently inhomogeneous collapse, providing an explicit counter example to cosmic censorship conjecture.

2 Higher Dimensional Vaidya Spacetime

The idea that spacetime should be extended from four to higher dimensions was introduced by Kaluza and Klein [15] to unify gravity and electromagnetism. Five dimensional ($5D$) spacetime is particularly more relevant because both $10D$ and $11D$ super-gravity theories yield solutions where a $5D$ spacetime results after dimensional reduction [16]. Hence, we shall confine ourselves to the $5D$ case.

The higher dimensional Vaidya spacetime which describes an implosion of radiation

shells is [13]

$$ds^2 = -\left(1 - \frac{m(v)}{r^2}\right)dv^2 + 2dvdr + r^2d\Omega^2 \quad (1)$$

where $d\Omega^2 = d\theta_1^2 + \sin^2\theta_1(d\theta_2^2 + \sin^2\theta_2d\theta_3^2)$ is the metric of the 3-sphere, where v is a null coordinate with $-\infty < v < \infty$, r is a radial coordinate with $0 \leq r < \infty$, and the arbitrary function $m(v)$ (which is restricted only by the energy conditions), represents the mass at advanced time v . The energy momentum tensor associated with eq. (1) can be written as

$$T_{ab} = \frac{3}{2r^3} \frac{dm}{dv} k_a k_b \quad (2)$$

with the null vector k_a satisfying $k_a = -\delta_a^v$ and $k_a k^a = 0$. We have used units in which $8\pi G = c = 1$. Clearly, for the weak energy condition to be satisfied, we require that dm/dv be non negative. Thus the mass function is a non-negative increasing function of v . The Kretschmann scalar ($K = R_{abcd}R^{abcd}$, R_{abcd} is the Riemann tensor) for the metric (1) takes the form

$$K = \frac{72m^2(v)}{r^8} \quad (3)$$

which diverges along $r = 0$ establishing a scalar polynomial singularity. The Weyl scalar ($C = C_{abcd}C^{abcd}$, C_{abcd} is the Weyl tensor) has the same expression as the Kretschmann scalar and thus the Weyl scalar also diverges whenever the Kretschmann scalar diverges and so the singularity is physically significant [17].

The physical situation here is that of a radial influx of a null fluid in an initially flat and empty region of the higher dimensional spacetime. For $v < 0$ we have $m(v) = 0$, i.e., higher dimensional flat spacetime, and for $v > T$, $dm/dv = 0$, $m(v)$ is positive definite. The metric for $v = 0$ to $v = T$ is higher dimensional Vaidya, and for $v > T$ we have the higher dimensional Schwarzschild solution. The first shell arrives at $r = 0$ at time $v = 0$ and the final at $v = T$. A central singularity of growing mass is developed at $r = 0$. We shall now test whether future directed null geodesics terminate at the singularity in the past. If they do, the singularity is naked.

3 The Existence and Nature of Naked Singularities

In this section we investigate the existence of a naked singularity for the higher dimensional Vaidya spacetime. Let $K^a = dx^a/dk$ be the tangent vector to the null geodesic, where k is

an affine parameter. The geodesic equations, on using the null condition $K^a K_a = 0$, take the simple form

$$\frac{dK^v}{dk} + \frac{m(v)}{r^3} (K^v)^2 = 0 \quad (4)$$

$$\frac{dK^r}{dk} + \frac{1}{2r^2} \frac{dm}{dv} (K^v)^2 = 0 \quad (5)$$

Following [8, 9], we introduce

$$K^v = \frac{P}{r} \quad (6)$$

and, from the null condition, we obtain

$$K^r = \left(1 - \frac{m(v)}{r^2}\right) \frac{P}{2r} \quad (7)$$

The function $P(v, r)$ obeys the differential equation

$$\frac{dP}{dk} - \left(1 - \frac{3m(v)}{r^2}\right) \frac{P^2}{2r^2} = 0 \quad (8)$$

In general, Eq. (8) may not yield an analytical solution. However, for our purpose the explicit solution of eq. (8) is not necessary. Radial null geodesics of the metric (1), by virtue of Eqs. (6) and (7), satisfy

$$\frac{dr}{dv} = \frac{1}{2} \left[1 - \frac{m(v)}{r^2}\right] \quad (9)$$

Clearly, the above differential equation has a singularity at $r = 0$, $v = 0$. The nature (a naked singularity or a black hole) of the collapsing solutions can be characterized by the existence of radial null geodesics coming out from the singularity. The character of the singularity depends on the exact form of $m(v)$. For example, with $m(v) \sim \lambda v^2$ the spacetime is self-similar [18], admitting a homothetic Killing vector and singularities can be analyzed with ease. However, self-similarity is a strong geometric condition on the spacetime. It is therefore, of interest to us to examine more general forms of the function $m(v)$. Here, we construct specific examples which satisfy the weak energy condition but develop strong curvature naked singularities.

Example-I To analyze the nature of the singular point first we consider a higher dimensional analogue of Lake's [7] solution, which would require

$$m(v) = \lambda v^2 + f(v) \quad (10)$$

where λ is a constant and $f(v) = O(v^2)$ as $v \rightarrow 0$. It follows therefore that for $0 < \lambda \leq 1/27$, the singular point becomes an unstable node and the family of null geodesics meets the singularity with definite tangents. The possible values of the tangents are given by the roots of the characteristic for eq. (9)

$$\lambda\gamma_0^3 - \gamma_0^2 + 2 = 0 \quad (11)$$

where

$$\gamma_0 = \lim_{r \rightarrow 0} \lim_{v \rightarrow 0} \gamma = \lim_{r \rightarrow 0} \lim_{v \rightarrow 0} \frac{v}{r} \quad (12)$$

Thus gravitational collapse of null fluid in higher dimensions leads to a naked singularity if $\lambda \leq 1/27$, and to formation of a black hole otherwise. The two positive roots of eq. (11), $\gamma_0 = 2.21833$ and 5.69593 , correspond to $\lambda = 1/50$. For all such values, the singularity is naked. The degree of inhomogeneity of the collapse is defined as $\mu \equiv 1/\lambda$ (see [10]). We see that for a collapse sufficiently inhomogeneous, naked singularities develop. Comparison with the analogous $4D$ case shows that a naked singularity occurs for a slightly larger value of the inhomogeneity factor in higher dimensions. The global nakedness of the singularity can then be seen by making a junction onto the higher dimensional Schwarzschild space-time. The Kretschmann scalar is $K = a\lambda^2/r^4$, for some constant a . Thus as $r \rightarrow 0$ the collapse forms a scalar polynomial curvature singularity.

The critical direction associated with the node is given by

$$r = \mu v + g(v) \quad (13)$$

where $\mu = 1/\gamma_0$ and

$$f = (1 - 2\mu)g(2\mu v + g) - g'(\mu v + g)^2 \quad (14)$$

For a spacetime to be self-similar we require that $g = 0$. The radial null geodesics given by Eq. (13) is the Cauchy horizon associated with the node, which is a strong curvature singularity (see below). One can see that the weak energy condition is satisfied for

$$2\lambda v \geq (2\mu - 1)2\mu g + 2g'(3\mu + g' - 1)(\mu v + g) + g''(\mu v + g)^2 \quad (15)$$

and the Cauchy horizon expanding for

$$g' > -\mu \quad (16)$$

the function g being restricted by (15) and (16).

The strength of a singularity is an important issue because there have been attempts to relate it to stability [11]. A singularity is termed gravitationally strong or simply strong, if it destroys by crushing or stretching any object which falls into it. Along a null geodesic affinely parameterized by k , let $\psi = R_{ab}K^aK^b$ where R_{ab} is the Ricci tensor and where the geodesic terminates at $\lambda = 0$. We consider the following condition:

$$\lim_{k \rightarrow 0} k^2 \psi > 0 \quad (17)$$

which is equivalent to the termination of a geodesic in a strong-curvature singularity in the sense of Tipler (cf. [19]). Eq. (17), with the help of eqs. (2), (10) and (6), can be expressed as

$$\lim_{k \rightarrow 0} k^2 \psi = \lim_{k \rightarrow 0} \frac{3}{2r} \frac{dm}{dv} \left(\frac{kP}{r^2} \right)^2 \quad (18)$$

Our purpose here is to investigate the above condition along future directed null geodesics coming out from the singularity. Using the fact that, as the singularity is approached, $k \rightarrow 0$ and $r \rightarrow 0$ and using L'Hôpital's rule, we first observe that $\lim_{k \rightarrow 0} kP/r^2 = 2/1 + \lambda\gamma_0^2$ for $P_0 = \infty$ and $\lim_{k \rightarrow 0} kP/r^2 = 1/1 - \lambda\gamma_0^2$ for $P_0 \neq \infty$ ($P_0 = \lim_{k \rightarrow 0} P$) and hence eq. (18) gives

$$\lim_{k \rightarrow 0} k^2 \psi = \frac{12\lambda\gamma_0}{(1 + \lambda\gamma_0^2)^2} > 0 \quad \text{for } P_0 = \infty \quad (19)$$

and

$$\lim_{k \rightarrow 0} k^2 \psi = \frac{3\lambda\gamma_0}{(1 - \lambda\gamma_0^2)^2} > 0 \quad \text{for } P_0 \neq \infty \quad (20)$$

where $\gamma_0 \neq 1/\sqrt{\lambda}$. Thus along radial null geodesics, the strong curvature condition is satisfied.

Recently, Nolan [20] gave an alternative approach to check the nature of singularities without having to integrate the geodesics equations. It was shown in [20] that a radial null geodesic which runs into $r = 0$ terminates in a gravitationally weak singularity if and only if \dot{r} is finite in the limit as the singularity is approached (this occurs at $k = 0$), the over-dot here indicates differentiation along the geodesics. So assuming a weak singularity, we have

$$\dot{r} \sim d_0 \quad r \sim d_0 k \quad (21)$$

Using the asymptotic relationship above and that $m(v) \sim \lambda v^2$ (as $k \rightarrow 0$), the geodesic equations yield

$$\frac{d^2v}{dk^2} \sim \delta k^{-1} \quad (22)$$

where $\delta = -\lambda\gamma_0^4 d_0$ = a non-zero const., which is inconsistent with $\dot{v} \sim d_0\gamma_0$, which is finite. Since the coefficient δ of k^{-1} is non-zero, the singularity is gravitationally strong [20].

Example-II We now construct a Joshi and Dwivedi [8] type solution for our higher dimensional spacetime. We choose the mass function as:

$$m(v) = \beta^2 v^{2\alpha} (1 - 2\beta\alpha v^{\alpha-1}) \quad (23)$$

where $\alpha > 1$ and $\beta > 0$ are constants. This is a representative class of a more general problem $m(v) \sim v^n$ ($n > 2$). Clearly, $\alpha = 1$ corresponds to a self-similar model. The null radiation shells start imploding at $v = 0$ and the final shell arrives at $v = T$. To ensure that the weak energy condition is satisfied, i.e., $dm/dv \geq 0$, T must satisfy,

$$T^{\alpha-1} < \frac{1}{\beta(3\alpha - 1)} \quad (24)$$

which also guarantees that $m(v) > 0$. Using (9), an outgoing radial null geodesic for the mass function (23) meeting the singularity $v = 0$, $r = 0$ in the past is given by

$$r = \beta v^\alpha \quad (25)$$

This integral curve meets the singularity with tangent $r = 0$ and the singularity is naked. Since $dr/dv > 0$ with increasing v , the null geodesics (25) escape to infinity and the singularity is globally naked. It is seen that the condition $r^2 > m(v)$ is satisfied along the trajectories.

It is easy to see that the Kretschmann scalar diverges along singular null geodesics meeting the singularity in the past in the approach to the singularity, establishing the presence of a scalar polynomial singularity. However, unless $m(v) \sim v^2$ in the approach to the singularity, the strong curvature condition is not satisfied along radial null geodesics. In the analogous situation in $4D$, it was found that the singularities are strong-curvature only for mass functions which are initially linear functions of v [6].

4 Conclusions

A rigorous formulation and proof for either version of the cosmic censorship conjecture is not available. Hence, examples showing the occurrence of naked singularities remain important and may be valuable if one attempts to formulate the notion of the conjecture in precise mathematical form. The Vaidya metric in the $4D$ case has been extensively used to study the formation of naked singularities in spherical gravitational collapse [5]. We have extended this study to a higher dimensional Vaidya metric, and found that strong curvature naked singularities do arise for slightly higher values of the inhomogeneity parameter and only for functions which are initially quadratic functions of the advanced time, i.e., $m(v) \sim v^2$. We have checked for naked singularities to be gravitationally strong by the method in [19] and by an alternative approach proposed by Nolan [20] as well, and both are in agreement. The models constructed here are not self-similar in general, and as a special case self-similar models arise. Also, it is straight forward to extend the above analysis for non radial causal curves. Whereas we confined our analysis to $5D$, there is no reason why it cannot be extended to spacetime of any dimensions ($n \geq 4$).

In conclusion, this offers a counter example to the cosmic censorship conjecture.

Acknowledgment: Two of the authors (SGG and RSV) would like to thank IUCAA, Pune (INDIA) for hospitality. Thanks are also due to Brien C. Nolan for helpful correspondence.

References

- [1] J. Soda and K. Hirata, *Phys. Lett.* **B387**, 271 (1999); A. V. Frolov, *Class. Quantum Grav.* **16**, 407 (1999); D. Garfinkle, C. Cutler and G. C. Duncan gr-qc/9908044, *preprint* (1999); J. F. V. Rocha and Anzhong Wang, gr-qc/9910109, *preprint* (1999).
- [2] A. Ilha and J. P. S. Lemos, *Phys. Rev. D* **55**, 1788 (1997); A. Ilha, A. Kleber and J. P. S. Lemos, *J. Math. Phys.* **40**, 3509 (1999).
- [3] R. Penrose, *Riv. del Nuovo Cim.* **1**, 252 (1969).
- [4] A. Papapetrou, in *A Random Walk in Relativity and cosmology*, edited by N. Dadhich, J. K. Rao, J. V. Narlikar and C. V. Vishveshwara, (Wiley, New York, 1985); Y. Kuroda, *Prog. Theor. Phys.* **72**, 63 (1984).
- [5] P. S. Joshi, *Global Aspects in Gravitation and Cosmology* (Clarendon Press, Oxford, 1993)

- [6] K. Lake *Phys. Rev. D* **43**, 1416 (1991).
- [7] K. Rajagopal and K. Lake *Phys. Rev. D* **35**, 1025 (1987).
- [8] P. S. Joshi and I. H. Dwivedi, *Gen. Relativ. Grav.* **24**, 129 (1992); *Phys. Rev. D* **45**, 2147; *J. Math. Phys.* **32**, 2167 (1991).
- [9] S. G. Ghosh and A. Beesham (2000). *Phys. Rev. D* **61**, 067502.
- [10] J. P. S. Lemos, *Phys. Rev. Lett.* **68**, 1447 (1992); J. P. S. Lemos, *Phys. Rev. D* **59**, 044020 (1999).
- [11] S. S. Deshingkar, P. S. Joshi and I. H. Dwivedi *Phys. Rev. D* **59**, 044018 (1999).
- [12] P. C. Vaidya, *Proc. Indian Acad. Sci.* **A33**, 264 (1951); Reprinted, *Gen. Rel. Grav.* **31**, 119 (1999).
- [13] B. R. Iyer and C. V. Vishveshwara, *Pramana-J. Phys.* **32**, 749 (1989); L. K. Patel and N. Dadhich, gr-qc/9909068 , *preprint* (1999).
- [14] F. J. Tipler, C. J. S. Clarke, and G. F. R. Ellis, in *General Relativity and Gravitation*, edited by A. Held (Plenum, New York, 1980).
- [15] T. Kaluza, *Sitz Preuss. Akad. Wiss.* **D 33**, 966 (1921); O. Klein, *Z. Phys.*, 895 (1926).
- [16] J. J. Schwarz, *Nucl. Phys.* **B226**, 269 (1983).
- [17] S. Barve and T. P. Singh, *Mod. Phys. Lett.* **A12**, 2415 (1997).
- [18] A spherical symmetric space-time is self similar if $g_{tt}(ct, cr) = g_{tt}(t, r)$ and $g_{rr}(ct, cr) = g_{rr}(t, r)$ for every $c > 0$.
- [19] C. J. S. Clarke and A. Królak, *J. Geom. Phys.* **2**, 127 (1986).
- [20] B. C. Nolan, *Phys. Rev. D* **60**, 024014 (1999).