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QUANTUM UNCERTAINTY IN THE HORIZON SIZE IN AN INFLATIONARY UNIVERSE

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It is shown that uncertainty principle prevents simultaneous, exact measurement of the expansion factor $S(t)$ and its time derivative $\dot{S}(t)$ in a FRW universe. We compute the uncertainties in these parameters in the semiclassical limit. It is shown that (i) there is a minimum uncertainty Δl_H in the Hubble distance $l_H(t) = (\dot{S}/S)^{-1}$ in a semiclassical universe. (ii) this uncertainty is amplified to an unacceptably large value if there was an inflationary phase in the early universe.

1. Introduction and Summary

It was shown in the past few years that an exponential expansion in the early phase of the universe can cure cosmology of some fine-tuning problems[1] (see also Refs. 2, and 3). Unfortunately such a universe brought along with it a different set of fine-tuning problems.[4] A generic inflationary universe will produce unacceptably large density inhomogeneity. We have shown elsewhere that, for any effective potential having a single energy scale, the density inhomogeneities will be 10^6 times too large: i.e. $\left(\frac{\delta\rho}{\rho}\right) \geq 50$, unless some form of fine tuning is resorted to.[5]

In this paper we point out yet another difficulty with the idea of inflation.

In inflationary models a very small region of the universe expands exponentially to form the observable part of the present day universe. Smaller regions in today's universe have evolved from still smaller scales in the preinflationary universe. For example, a size of 10^{28} cms—which is about the dimension of the observable universe—today would correspond to a size of 10 cms at the end of inflation [assuming inflation ended around $T \sim 10^{14}$ GeV]. If inflation bloated up sizes by a factor of about $\sim 10^{30}$, then the same region would have occupied a size of about 10^{-29} cm in the preinflation-

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ary universe. This length scale is deeply entrenched in the domain of quantum fluctuations. We shall, in this paper, study these quantum fluctuations using uncertainty principle as our guide. We shall show that inflation amplifies these fluctuations to unacceptably large values.

The fluctuations mentioned above are the small scale quantum fluctuations. These fluctuations are to be distinguished from inhomogeneities usually discussed in literature. The latter arise due to the fact that the scalar field responsible for the inflation falls off from the flat potential at different times at different places. This is a dynamical effect. In contrast, the uncertainties and fluctuations that we study below are purely kinematic in origin.

We approach the problem in the following manner: We consider a scalar field ϕ as the main source for an expanding universe. Einstein's equations relate the geometrical variables (like expansion factor $S(t)$, $\dot{S}$, curvature invariants etc.) to the scalar field and its congugate momentum. Since the scalar field is subject to the quantum laws, especially uncertainty principle, the scalar field and its congugate momentum cannot be measured simultaneously with arbitrary accuracy. This, in turn, imposes restriction on the measurability of various geometrical parameters describing the universe. We compute the relative uncertainties in these parameters in various models.

The process described above essentially relates the quantum uncertainty of a source [in our case, the scalar field] to that of the field produced by the source [in our case, the gravitational field represented by the expanding universe].

It may be argued that if the *expectation values* of quantum fields are used as the source of gravity, then the fluctuation in the source field will not lead to any uncertainties in the metric. This argument, however, is incorrect. It can be shown that the expectation value of, say, T_{ik} cannot act consistently as a source of gravity. [For details of this argument—and a previous application of this technique—see Refs. 6 and 7].

We find that the quantum uncertainty in the inverse Hubble distance is disturbingly large in inflationary models. For the sake of comparison we have computed the same in models without inflation and find that it is negligibly small (see Eqs. 47 and 69). This would suggest that the inflationary process amplifies quantum uncertainties to disturbingly large values. The method of analysis which we use is a semi-classical one. It is possible to construct examples in which this method produces an uncertainty which is considerably larger than the actual quantum uncertainty. If this is the source of large uncertainty in the inflationary models, then there is nothing much to worry; however, we feel this is not the case. It is incredible that the same method when applied to inflationary and noninflationary models of the universe produces uncertainties differing by a factor like 10^{120}! It seems to us that inflation does contribute to a large extent. Nevertheless, the reader should bear this point in mind.

We would also like to comment on the use of the word "horizon" in this paper. This term is used to describe the inverse Hubble distance $\left(\frac{\dot{S}}{S}\right)^{-1}$ and may have nothing to do with the causal horizon of the space-time.

In what follows, we use units with $\hbar$ and c set to unity leaving $G = L_p^2$, where L_p is the Planck length. The signature of the metric is $(+, -, -, -)$.

2. Basic Formalism and Results

Consider a scalar field $\phi(\underset{\sim}{x}, t)$ coupled to gravity and described by the action

$$A = \frac{1}{16\pi G}\int \sqrt{-g}\, R d^4x + \int \left[\frac{1}{2}\phi_{,i}\phi^{,i} - V(\phi)\right]\sqrt{-g}\, d^4x, \tag{2.1}$$

where $V(\phi)$ is the potential for the scalar field. We confine our attention to the homogeneous mode of the field ϕ, [i.e. we take $\phi(\underset{\sim}{x}, t) = \phi(t)$] in a Robertson-Walker space-time, with the line element

$$dS^2 = dt^2 - S^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{2.2}$$

In this "mini-superspace" of $\phi(t)$, $S(t)$ the action in (2.1) can be written as

$$A = \frac{3\pi}{4G}\int dt\left[kS - \dot{S}^2 S + \frac{8\pi G}{3}S^3\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right)\right]. \tag{2.3}$$

The integration over the spatial coordinates is normalized to $(2\pi^2)$ which is the volume of the closed universe. Varying this action with respect to $S(t)$ and $\phi(t)$, we get

$$2\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + k}{S^2} = -8\pi G\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right) \tag{2.4}$$

and

$$\frac{1}{S^3}\frac{d}{dt}(\dot{\phi}S^3) = -\frac{\partial V}{\partial \phi}. \tag{2.5}$$

These equations are seen to be consistent with the constraint equation

$$\frac{\dot{S}^2 + k}{S^2} = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{2.6}$$

which represents the reparametrization invariance of the action in (2.3) under the change $t \to f(t)$. From the action in (2.3) it follows that the momenta conjugate to ϕ and S are given by

$$p_\phi = 2\pi^2 S^3\dot{\phi} \tag{2.7}$$

and

$$p_s = \frac{3\pi}{2G} S\dot{S}. \tag{2.8}$$

Classically one should be able to measure ϕ and p_ϕ (or S and p_s) simultaneously with arbitrary accuracy. On the other hand, in any proper quantum mechanical treatment the uncertainties in ϕ and p_ϕ are related by the constraint:

$$\Delta p_\phi \Delta\phi \geq \hbar. \tag{2.9}$$

Similarly, we may expect some kind of uncertainty relation between S and p_s in a quantum gravitational model for the universe. Unfortunately, such a full quantum theory, which treats both the scalar field and gravity quantum mechanically, is not yet available. Therefore we have to estimate the uncertainties in S and p_s in an indirect manner. In the following we shall use the *known* uncertainty principle connecting ϕ and p_ϕ to derive the semi-classical uncertainty relations between S and p_s. [This approach is similar to the one used previously to study quantum fluctuations in blackhole space-times; see Ref. 6.]

Equations (2.5) and (2.6) determine the classical solution $\phi(t)$ and $S(t)$. Eliminating the time variable between these two we can express S as a function of ϕ:

$$S = f(\phi). \tag{2.10}$$

Therefore,

$$\frac{dS}{dt} = \dot{S} = f'(\phi)\dot{\phi}. \tag{2.11}$$

[The overhead dot indicates differentiation with respect to time and the prime indicates differentiation with respect to argument of the function]. This relation can be re-expressed in terms of p_s and p_ϕ using (2.7), (2.8) and (2.11). We get

$$p_s = -\frac{3}{4\pi G}\frac{f'}{S^2}p_\phi. \tag{2.12}$$

Equations (2.10) and (2.12) express the gravitational variables S and p_s in terms of ϕ and p_ϕ. In the *semi-classical limit*, we can use (2.10) and (2.12) to relate the uncertainties in S and p_s (viz ΔS, Δp_s) to the uncertainties of ϕ and p_ϕ (viz $\Delta\phi$, Δp_ϕ). We get,

$$|\Delta S| = f'|\Delta\phi| \tag{2.13}$$

$$|\Delta p_s| = \frac{3}{4\pi G}\frac{f'}{S^2}|\Delta p_\phi| + \frac{3p_\phi}{4\pi G}\frac{d}{d\phi}\left(\frac{f'}{S^2}\right)|\Delta\phi|. \tag{2.14}$$

Taking the product we see that,

$$|\Delta S|\,|\Delta p_s| = \frac{3}{4\pi G}\left(\frac{f'}{S}\right)^2 (\Delta\phi)(\Delta p_\phi) + 0((\Delta\phi)^2) \tag{2.15}$$

while quantum theory does not restrict an individual measurement of $(\Delta\phi)$ in anyway, it does demand

$$\Delta\phi\,\Delta p_\phi \geq \hbar \tag{2.16}$$

leading to the following uncertainty relation for the gravitational variables:

$$\Delta S\,\Delta p_s \geq \frac{3}{4\pi G}\left[\frac{f'}{S}\right]^2 \hbar = \frac{3}{4\pi G}\left(\frac{\dot{S}}{\dot{\phi}S}\right)^2 \hbar \tag{2.17}$$

[In arriving at the last expression we have used the relation $f' = \dot{S}/\dot{\phi}$].

Because of the rather unorthodox procedure used in arriving at (2.17) it is probably worthwhile to examine the nature of the argument involved in the derivation of (2.17). To do so, consider a simpler system of two particles with masses M and m and coordinates $Q(t)$ and $q(t)$. Let us suppose that the dynamics of this 2-particle system is described by the action

$$A = \int dt\left[\frac{1}{2}M\dot{Q}^2 + \frac{1}{2}m\dot{q}^2 - V(Q,q)\right]. \tag{2.18}$$

Let us assume further that the particle described by $q(t)$ is quantum mechanical while the one described by $Q(t)$ is classical, or at least semiclassical. Given that,

$$\Delta p\,\Delta q \geq \hbar \tag{2.19}$$

we are interested in knowing whether one can set any semi-classical bounds on $\Delta P\,\Delta Q$. Such a bound can be obtained as follows:

We begin with the classical solution

$$q = q(t), \quad Q = Q(t) \tag{2.20}$$

and eliminate t between these equations. Obtaining,

$$Q = f(q) \tag{2.21}$$

similarly,

$$P = \frac{\partial L}{\partial \dot{Q}} = M\dot{Q} = Mf'\dot{q} = \frac{M}{m}f'p; \quad p = \frac{\partial L}{\partial \dot{q}}. \tag{2.22}$$

Taking differentials,

$$\Delta Q = f'\,\Delta q \tag{2.23}$$

$$\Delta P = \frac{M}{m} f'\,\Delta p + \frac{M}{m} f' p\,\Delta q\,. \tag{2.24}$$

Thus

$$\Delta P\,\Delta Q = \frac{M}{m}(f')^2\,\Delta p\,\Delta q + 0(\Delta q)^2 \geq \frac{M}{m}(f')^2 \frac{1}{2}\hbar\,. \tag{2.25}$$

To understand the meaning of this relation notice that

$$\frac{M}{m}(f')^2 = \frac{M}{m}\left(\frac{dQ}{dq}\right)^2 = \frac{M\dot{Q}^2}{m\dot{q}^2}\,. \tag{2.26}$$

Earlier we arbitrarily assumed that Q is "semi-classical". In such a case, the kinetic energy associated with Q, $\frac{1}{2}M\dot{Q}^2$, is likely to be much larger compared to the kinetic energy associated with q. The bound derived above will be clearly consistent with the more rigorous, quantum mechanical bound of $\Delta P\,\Delta Q \geq \frac{1}{2}\hbar$. The argument leading to (2.17) is very similar except that the Lagrangian is far more complicated.

It should be noted that the result (2.17) is invariant under the re-labelling of our basic variables: i.e., under transformations like $S \to g(S)$. Thus we could have used any variable like S^2 or $\log S$ instead of S in deriving the above bound. We shall come back to this point later.

The fact that S and p_s (or equivalent by S and $\dot{S}$) cannot be measured simultaneously to arbitrary accuracy implies that many geometrical variables characterizing the space-time geometry cannot have definite values. Consider, for example, the Hubble distance which limits the region over which causal processes can operate.

$$l_H = H^{-1} = \left(\frac{\dot{S}}{S}\right)^{-1} = -\frac{3\pi}{2G}\frac{S^2}{p_s}\,. \tag{2.27}$$

This distance l_H depends on p_s (related to $\dot{S}$) as well as on S. Exact measurement of such a quantity will require a simultaneous knowledge of p_s and S. Since this is forbidden by (2.17), there will be a minimum, inherent uncertainty in the measurement of variables like l_H. Using Eqs. (2.17) and (2.27) we can obtain the minimum uncertainty in the horizon distance in a straightforward manner. Omitting the simple calculation, we give the final result: [we have re-introduced $\hbar$ to emphasize the quantum mechanical nature of the result]

$$\left(\frac{\Delta l_H}{l_H}\right)^2 \geq \frac{4}{\pi^2}\frac{\dot{S}}{\dot{\phi}^2 S^4}\hbar\,. \tag{2.28}$$

As a second example, we shall compute the uncertainty in the scalar curvature R, which is given by

$$R = 6\left[\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + k}{S^2}\right]. \tag{2.29}$$

Using Eqs. (2.4) and (2.6) we can re-write (2.29) in the form

$$R = -8\pi G[\dot{\phi}^2 - 4V(\phi)]. \tag{2.30}$$

Computing $\dfrac{\Delta R}{R}$ in terms of $\Delta\phi$ and Δp_ϕ and using (2.16) we can get after a straightforward calculation:

$$\left(\frac{\Delta R}{R}\right)^2 \geq \frac{24}{\pi^2}\frac{\dot{S}}{\dot{\phi}^2 S^4}\frac{\hbar}{\left[1 - 4\dfrac{V(\phi)}{\dot{\phi}^2}\right]^2}\left[1 + \frac{2}{3}\frac{S}{\dot{\phi}\dot{S}}\left(\frac{\partial V}{\partial\phi}\right)\right]. \tag{2.31}$$

Equations (2.28) and (2.22) are the main results of this section. The rest of the paper studies these relations in various contexts. We emphasize that these bounds are expected to be valid only in the semi-classical limit. We assume that the quantum state of the universe is such that the expectation values of S and p_s are closely peaked around their classical evolution. In a way, the bound on single observables like l_H or R is of very different nature from the usual uncertainty relations connecting a pair of observables. For example, in a one-dimensional quantum system, one can easily construct states which are eigenstates of operator K like, say

$$K = f(q)p + \text{H.c.} \tag{2.32}$$

In those states, measurement of K will yield definite answers with no uncertainty. On the other hand, if the system is in a semi-classical, minimum uncertainty state [with $\Delta q\,\Delta p = \frac{1}{2}\hbar$] then K will not have a definite value in that state. The spread ΔK can then be estimated by minimizing

$$\Delta K = f\Delta p + f'(q)\Delta q \tag{2.33}$$

subject to the condition $\Delta p\,\Delta q = \frac{1}{2}\hbar$. The results in (2.28) and (2.22) have to be interpreted in the above manner.

It may appear, at first sight, that the uncertainty in $(\dot{S}/S)$ arises because we choose S to be the variable. It may seem that by using, say, $x = \log S$ this result can be bypassed. *This is not the case.* Any re-definition of the form $S \rightarrow f(S)$ will also change the form of $p_s(S, \dot{S})$. When $(\dot{S}/S)$ is re-expressed in terms of x and corresponding p_x, the results will follow.

3. Quantum Uncertainty in Specific Cosmological Models

In this section we will compute the fluctuations in the horizon size l_H and in the scalar curvature R in two different models for the universe:

(i) Flat and closed models without inflation, and

(ii) Flat models with inflation.

We shall first demonstrate that our equations (2.28) and (2.31) does exhibit a reasonable and intuitively expected behavior in a universe which does *not undergo inflation.* We shall show that the uncertainties are negligible during the classical epoch and are significant only at very early phases. This demonstration reassures us of the validity of the arguments which lead to (2.28) and (2.31).

In Sec. (3.2) we shall perform the same computation for an inflationary universe, and will demonstrate the following result: The quantum uncertainties are amplified by inflation *and remain disturbingly large even at the post-inflationary, classical phase of the universe.*

Since no such trouble arises in a noninflationary model, we may confidently attribute the large uncertainties to the inflationary process.

3.1. *Noninflationary model*

Consider a $k = 0$ universe dominated by a free (i.e. $V(\phi) = 0$) scalar field. We shall set the scale for the universe by assuming that the size of the universe $S(t)$ is about 10^{28} cm at $t = 1.5 \times 10^{10}$ years.

Solving the equation of motion, (2.5), for the scalar field we get,

$$\dot{\phi} = \frac{\alpha}{S^3}, \tag{3.1}$$

where α is a constant of integration. Substituting this expression for $\dot{\phi}$ in Eq. (2.6) we find that S^3 varies linearly with time:

$$S^3 = \sqrt{12\pi\alpha L_p t}\,. \tag{3.2}$$

The value of α can be found by setting $S = 10^{28}$ cm at $t = 1.5 \times 10^{10}$ years. We get

$$\alpha = 10^{89}\ \text{cm}. \tag{3.3}$$

Thus our model is described by the functions $S(t)$ and $\phi(t)$ with

$$S(t) = 10^{28}\ \text{cm}\left(\frac{t}{3.2 \times 10^{17} s}\right)^{1/3} \tag{3.4}$$

and

$$\dot{\phi} = 10^{5}\ \text{cm}^{-2}\left(\frac{t}{3.2 \times 10^{17} s}\right)^{-1}. \tag{3.5}$$

We can now compute the various uncertainties using previously derived formulae.

Using the uncertainty relation between s and p_s (Eq. (2.17)) and Eqs. (2.29) and (3.5) we get

$$\Delta p_s \Delta S \geq \hbar \tag{3.6}$$

which is completely understandable. With the help of Eqs. (2.28), (3.4) and (3.5) the uncertainty in the horizon size can be computed to be

$$\left(\frac{\Delta l_H}{l_H}\right)^2 \geq \frac{L_p}{\alpha}\left(\frac{64}{3\pi^3}\right)^{1/2} \simeq 0.83\frac{L_p}{\alpha}, \tag{3.7}$$

or, using $\alpha = 10^{89}$ cm,

$$\left(\frac{\Delta l_H}{l_H}\right)^2_{\min} = 0.83 \times 10^{-122}. \tag{3.8}$$

Similarly, we obtain

$$\left(\frac{\Delta R}{R}\right)^2 \geq \frac{24}{\pi^2}\frac{\dot{S}}{\dot{\phi}^2 S^4} \simeq 4.9 \times 10^{-122}. \tag{3.9}$$

We can see that the uncertainties in the horizon size and the curvature scalar are too small to be of any significance. They are also independent of t in the case of the $k = 0$ model.

To illustrate these aspects further consider a $k = +1$ universe; $\partial V/\partial\phi$ is again zero. Hence Eq. (3.1) remains valid,

$$\dot{\phi} = \frac{\alpha}{S^3} \tag{3.10}$$

but $S(t)$ is now different. It is given by:

$$S = S_{\max} \sin^{1/2}(2\tau), \tag{3.11}$$

where

$$\tau = \int^t \frac{dt}{S} \tag{3.12}$$

and

$$S_{\max} = \left(\frac{4\pi G\alpha^2}{3}\right)^{1/4}. \tag{3.13}$$

We will again take $S_{\max}$ to be $\sim 10^{28}$ cm, so that α again turns out to be $\sim 10^{89}$ cm. The uncertainties are now given by [using Eqs. (2.28), (2.31), (3.4) and (3.5)],

$$\left(\frac{\Delta l_H}{l_H}\right)^2 \geq 0.83 \times 10^{-122}\left\{1 - \left(\frac{S}{S_{\max}}\right)^4\right\}^{1/2} \tag{3.14}$$

$$\left(\frac{\Delta R}{R}\right)^2 \geq 4.9 \times 10^{-122}\left\{1 - \left(\frac{S}{S_{\max}}\right)^4\right\}^{1/2}. \tag{3.15}$$

The essential conclusions based on $k = 0$ model remain now as well. In the early universe ($S \to 0$ limit) the results are identical [This is to be expected because the curvature terms is unimportant in the early universe]. Hereafter, we shall confine our attention to $k = 0$ model.

3.2. *Universe with an inflationary phase*

In the inflationary scenario, the universe is believed to have gone through a phase of rapid exponential expansion. In order to achieve such an expansion in the simplest possible manner we shall introduce a potential for the scalar field:

$$\begin{aligned} V(\phi) &\simeq V_0 \quad 0 < \phi < \phi_0 \\ &= 0 \qquad \phi_0 < \phi . \end{aligned} \tag{3.16}$$

We take $\phi(t) < \phi_0$ for $t < t_f$, where t_f characterizes the end of inflation. The field $\phi(t)$ is supposed to "roll down" the potential for $t < t_f$ and "fall" at $t = t_f$. Clearly for $t > t_f$, $\phi > \phi_0$ and hence the field does not experience any potential. We shall assume that the universe was dominated by the scalar field from time t_f until today ($t \sim 1.5 \times 10^{10}$ years). This is certainly an oversimplified model for inflation; nevertheless, it mimics the essential features of inflation adequately.

The potential is flat for $\phi < \phi_0$ as well as for $\phi > \phi_0$. From Eq. (2.5), we see that ϕ is again given by

$$\dot{\phi} = \frac{\alpha}{S^3}. \tag{3.17}$$

The energy density T_0^0 during the inflationary phase (i.e. for $t < t_f$) is given by

$$T_0^0 = \frac{1}{2}\dot{\phi}^2 + V_0 \quad (t < t_f). \tag{3.18}$$

Substituting the expression for T_0^0 and using Eq. (3.17) in the constraint equation (2.6) we get the equation governing the evolution of S as

$$\frac{\dot{S}^2}{S^2} = \frac{4\pi G}{3}\left(\frac{\alpha^2}{S^6} + 2V_0\right) \tag{3.19}$$

which is solved by:

$$S^3 = \frac{\alpha}{\sqrt{2V_0}} \sinh(3Ht) \quad (t < t_f) \tag{3.20}$$

with

$$H^2 = \frac{8\pi G V_0}{3}. \tag{3.21}$$

For times $t > t_f$, $V(\phi) = 0$ and hence [see Eqs. (2.5) and (2.6)]

$$S^3 \propto t. \tag{3.22}$$

The constant of proportionality is fixed by demanding continuity of S at $t = t_f$. We find:

$$S^3 = \frac{\alpha}{\sqrt{2V_0}} \sinh(3Ht_f) \frac{t}{t_f} \quad (t > t_f). \tag{3.23}$$

Thus the complete evolution of our model can be described by

$$S^3(t) = \left(\frac{\alpha^2}{2V_0}\right)^{1/2} \sinh(3Ht) \quad (t \leq t_f) \tag{3.24}$$

$$= \left(\frac{\alpha^2}{2V_0}\right)^{1/2} \left(\frac{t}{t_f}\right) \sinh(3Ht_f) \quad (t \geq t_f) \tag{3.25}$$

where

$$H^2 = \frac{8\pi}{3} G V_0. \tag{3.26}$$

Following standard inflationary models, we shall take $V_0 \sim (10^{14}$ GeV$)^4$ so that $H \sim (2 \times 10^9$ GeV). Sufficient inflation from a size $\cong L_p$ (at Planck time) to the present size ($\sim 10^{28}$ cm) at $t = 1.5 \times 10^{10}$ years can be achieved by the following choice of parameters:

$$\alpha = L_p, \quad Ht_f \simeq 103. \tag{3.27}$$

Using (3.27) in (3.24) and (3.7) it is easy to derive the asymptotic forms of $S(t)$

$$S(t) = [(12\pi)^{1/6} L_p^{2/3}] t^{1/3}, \quad t \ll H^{-1} \tag{3.28}$$

$$= \left[\left(\frac{\pi}{3}\right)^{1/6} \frac{L_P^{2/3}}{H^{1/3}}\right] e^{Ht}, \quad t \to t_f = 103H^{-1} \gg H^{-1} \tag{3.29}$$

$$= \left[\left(\frac{\pi}{3}\right)^{1/6} \frac{Lp^{2/3}}{(Ht_f)^{1/3}} e^{Ht_f}\right] t^{1/3}. \tag{3.30}$$

We shall now compute the fluctuation in the horizon size and in the scalar curvature at two different epochs

(a) during the inflationary phase, and

(b) in the present day post-inflationary universe.

During the $t < t_f$ phase the fluctuations in the horizon size and the curvature are given by:

$$\left(\frac{\Delta l_H}{l_H}\right)^2 \geq \frac{4}{\pi^2}\sqrt{\frac{4\pi}{3}}\cosh(3Ht) \tag{3.31}$$

$$\left(\frac{\Delta R}{R}\right)^2 \geq \frac{24}{\pi^2}\sqrt{\frac{4\pi}{3}}\cosh(3Ht)\left[1 - 4\frac{V_0}{\dot{\phi}^2}\right]^{-2}. \tag{3.32}$$

For the very early times, i.e. when $t \ll H^{-1}$ $\left(\to \frac{V_0}{\dot{\phi}^2} \ll 1\right)$, the effect of inflation is not significant. In this limit, the fluctuations are constant and are of the order unity.

$$\left(\frac{\Delta l_H}{l_H}\right) \geq \left(\frac{64}{3\pi^3}\right)^{1/4} \simeq 0.9 \tag{3.33}$$

$$\left(\frac{\Delta R}{R}\right) \geq 6^{1/2}\left(\frac{64}{3\pi^3}\right)^{1/4} \cong 2.2 \tag{3.34}$$

In fact, this is just about the limit at which we expect our semi-classical description to become unreliable.

The disturbing fact is that inflation bloats up the above fluctuations in $\Delta l/l$. This can be seen by taking the limit of $Ht \gg 1$ in (3.24). In this limit we get

$$\frac{V_0}{\dot{\phi}^2} \gg 1 \tag{3.35}$$

giving

$$\frac{\Delta l_H}{l_H} \geq \left(\frac{4}{\pi^2}\right)^{1/2}\left(\frac{8\pi V_0}{3\dot{\phi}^2}\right)^{1/4} \simeq \exp\left(\frac{3}{2}Ht\right) \tag{3.36}$$

and

$$\frac{\Delta R}{R} \geq \left(\frac{384}{\pi^2}\right)^{1/2} \left(\frac{8\pi}{3}\right)^{1/4} \left(\frac{V_0}{\phi^2}\right)^{-3/4} \simeq e^{-9/2Ht}. \tag{3.37}$$

Clearly, the uncertainty in the horizon size grows exponentially with time, though the uncertainty in the scalar curvature decreases exponentially.

The fact that $(\Delta R/R)$ decays during the inflation is consistent with known behavior of classical matter perturbation in the de Sitter space-time. It is known that the density contrast $(\delta\rho/\rho)$ in the classical limit, decays in the de Sitter phase ["no hair conjecture"]. Since $(\delta\rho/\rho)$ is directly related to $(\Delta R/R)$, the latter also decays in the de Sitter space-time. Equation (3.37) extends this behavior to quantum fluctuations in R; they are also seen to decay in the de Sitter phase.

The fluctuation in the size of the horizon are, however, disturbing. They grow by an enormous factor during inflation. *In other words, the concept of unique $l_H(t)$ is not meaningful during inflation.* This is a strange and disturbing result; as far as we know, fluctuation in the horizon size has not been discussed in literature before.

The uncertainty $(\Delta l_H/l_H)$, once produced cannot "go away". We expect it to be present in the post-inflationary era and indeed it does. Computing the uncertainties in the post-inflationary epoch, we get

$$\left(\frac{\Delta l_H}{l_H}\right)^2 \geq 0.83 \frac{\sinh 3Ht_f}{3Ht_f}. \tag{3.38}$$

Since $Ht_f \sim 102$ these uncertainties are large. As we saw before (see Sec. 2) these uncertainties are independent of time.

4. Conclusions and Speculations

The large uncertainty in l_H derived in the previous section raises several questions.

To begin with, it is clear that the result is *entirely due to inflation.* We worked out the fluctuations in the noninflationary universe in Sec. 3.1 and found that they are negligible. Thus, if the above result is accepted to be disturbing then the concept of inflation must be abandoned. To make sure that the result is independent of the details of inflation, we have used a remarkably simple model for $V(\phi)$. Any realistic model producing exponential expansion cannot escape from our conclusions.

Secondly, we are actually estimating the uncertainties in the gravitational field variables, S and $\dot{S}$. Since a quantum field acts as a source for g_{ik}, the uncertainties in the quantum field generate corresponding uncertainties in g_{ik}. Since these are actually quantum gravitational fluctuations, the interpretation of the uncertainties in ΔS or $\Delta\phi_s$ is wrought with conceptual problems. In simple quantum mechanics one may interpret the uncertainty in (say) position, δq, as follows: An ensemble of systems is prepared in the same quantum state $|\psi\rangle$. If the position q is measured for all elements of the ensemble, then the results will be clustered around a mean value $\bar{q} = \langle\psi/q/\psi\rangle$ with a spread $\delta q = \langle(q - \bar{q})^2\rangle^{1/2}$. It is not possible to operationally measure δq for a single element of the ensemble. To carry such an interpretation to the present case, one may have to think of an "ensemble of universes" in which "measurement of l_H" is carried out! If the average behavior of such an ensemble is chosen to have an inflationary phase, then our results imply a large $(\delta l_H/l_H)$. It is very tempting to say that δl_H has no relevance in a single universe in which "we happen to be". Such a temptation should be resisted until a consistent interpretation for uncertainty principle in a single universe is evolved. If all values of l_H in a range $(l_H - \Delta l_H,\ l_H + \Delta l_H)$ are almost equally probable, then $(\Delta l_H/l_H)$ should be small for reasonable semi-classical interpretation. One cannot get away from the above simple fact.

It has become fashionable nowadays to talk about the quantum cosmological epoch, quantum creation of the universe, wave function for the universe etc., etc. The mathematical details involved in some of these attempts may have little relevance to reality, unless simple interpretational questions—like the ones raised above—in the semi-classical domain can be answered.

Lastly, we point out that our analysis may offer a solution to horizon problem (without resorting to inflation) in the following manner: We saw that $(\Delta l_H/l_H)$ was small and constant in a noninflationary model, while exponentially large in an inflationary model. It may be possible to construct a model universe in which $(\Delta l_H/l_H)$ is large in the early phase but asymptotically approaches a constant, small value at late times. Such a model will be effectively free of horizons in the early epochs.

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