Relic Gravity Waves from Braneworld Inflation

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We discuss a scenario in which extra dimensional effects allow a scalar field with a steep potential to play the dual role of the inflaton as well as dark energy (quintessence). The post-inflationary evolution of the universe in this scenario is generically characterised by a 'kinetic regime' during which the kinetic energy of the scalar field greatly exceeds its potential energy resulting in a 'stiff' equation of state for scalar field matter $P_\phi \simeq \rho_\phi$. The kinetic regime precedes the radiation dominated epoch and introduces an important new feature into the spectrum of relic gravity waves created quantum mechanically during inflation. The amplitude of the gravity wave spectrum *increases with wavenumber* for wavelengths shorter than the comoving horizon scale at the commencement of the radiative regime. This 'blue tilt' is a generic feature of models with steep potentials and imposes strong constraints on a class of inflationary braneworld models. Prospects for detection of the gravity wave background by terrestrial and space-borne gravity wave observatories such as LIGO II and LISA are discussed.

I. INTRODUCTION

The last two decades have seen considerable effort being devoted to the construction of fundamental theories of nature in more than three spatial dimensions. In such models the four dimensional Planck scale $M_4 \equiv G^{-1/2} = 1.2 \times 10^{19}$ GeV is related to its fundamental value $M_{\rm f}$ by $M_4^2 = M_{\rm f}^{2+n}\mathcal{R}^n$ where n is the number of extra dimensions. In the original Kaluza-Klein picture the extra dimensions were compact and microscopic $R \sim 10^{-33}$ cm, hence unobservable. However it was soon realized that a theory in which one or more of the extra dimensions is macroscopic ($R \sim 1$ mm) has several interesting features. For instance a model in which two of the extra dimensions extend up to a millimeter has the considerable advantage of decreasing the fundamental Planck scale to electroweak scales $M_{\rm f} \sim 1$ TeV $\ll M_4$ thereby alleviating the hierarchy problem associated with particle physics [1]. A further development of these ideas [2,3] led to a scenario in which our universe is a three dimensional domain wall (brane) embedded in an infinite four dimensional space (bulk). The metric describing the full 4+1 dimensional space-time is non-factorizable and the small value of the true five dimensional Planck mass is related to its large effective four dimensional value by the extremely large warp of five dimensional space [3]. Since gravity and matter fields remain confined to the brane the presence of the extra (bulk) dimension does not affect Newton's law which remains inverse square.

As demonstrated in [4,5] the prospects of inflation in such a scenario improve due to the presence of an additional quadratic density term in the Einstein equations. The class of potentials which lead to inflation increases and includes potentials which are normally too steep to be associated with inflation [6]. As we demonstrate in this paper, for a suitable choice of parameter values, scalar field with steep potentials can successfully play the dual role of being the inflaton at early times and dark energy (quintessence) at late times (see also [7]). In inflationary models with steep potentials the immediate aftermath of braneworld inflation is characterised by a 'kinetic regime' during which the inflaton has the 'stiff' equation of state $p \simeq \rho$ [8].

In this paper we examine an important observational imprint of brane-inflation - the spectrum of relic gravity waves. The quantum mechanical creation of gravitons from the vacuum is an important generic feature of expanding cosmological models [9]. In models characterised by an early inflationary epoch the relic gravity wave amplitude is related to the Hubble parameter during inflation [10] while its spectrum depends upon both the inflationary and post-inflationary equation of state [10–12]. As demonstrated in [13] the gravity wave amplitude in braneworld models is considerably enhanced over that in standard inflation. Additionally, as we show in this paper, the presence of a kinetic regime soon after brane-inflation leads to a 'blue' spectrum for gravity waves, on scales smaller than the comoving horizon scale at the commencement of the radiative regime (see also [14]). This unique feature of steep braneworld inflation results in an enhanced gravity wave background which can be used to succesfully constrain braneworld models.

II. BRANEWORLD INFLATION

In the 4+1 dimensional brane scenario inspired by the Randall-Sundrum [3] model, the standard 0-0 Friedman equation is modified to [4]

$$H^2 = \frac{8\pi}{3M_4^2}\rho(1+\frac{\rho}{2\lambda_b}) + \frac{\Lambda_4}{3} + \frac{\mathcal{E}}{a^4} \tag{1}$$

where $\mathcal{E}$ is an integration constant which transmits bulk graviton influence onto the brane and λ_b is the three dimensional brane tension which provides a relationship between the four and five-dimensional Planck masses

$$M_4 = \sqrt{\frac{3}{4\pi}}(\frac{M_5^2}{\sqrt{\lambda_b}})M_5, \tag{2}$$

and also relates the four-dimensional cosmological constant Λ_4 to its five-dimensional counterpart via

$$\Lambda_4 = \frac{4\pi}{M_5^3}\left(\Lambda_5 + \frac{4\pi}{3M_5^3}\lambda_b^2\right). \tag{3}$$

Following [5] we make the additional assumption that Λ_4 is too small to play an important role in the early universe (the possible presence of a small cosmological constant today provides the lower bound $\Omega_\Lambda = M_4^2\Lambda_4/8\pi \sim 0.7$). The "dark radiation" term $\mathcal{E}/a^4$ is expected to rapidly disappear once inflation has commenced so that we effectively get [4,5]

$$H^2 = \frac{8\pi}{3M_4^2}\rho(1+\frac{\rho}{2\lambda_b}), \tag{4}$$

where $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$, if one is dealing with a universe dominated by a single minimally coupled scalar field. The equation of motion of a scalar field propogating on the brane is

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{5}$$

From (4) and (5) we find that the presence of the additional term ρ^2/λ_b increases the damping experienced by the scalar field as it rolls down its potential. This effect is reflected in the slow-roll parameters which have the form [5]

$$\epsilon = \epsilon_{\rm FRW}\frac{1+V/\lambda_b}{(1+V/2\lambda_b)^2},$$
$$\eta = \eta_{\rm FRW}(1+V/2\lambda_b)^{-1}, \tag{6}$$

where

$$\epsilon_{\rm FRW} = \frac{M_4^2}{16\pi}\left(\frac{V'}{V}\right)^2, \ \eta_{\rm FRW} = \frac{M_4^2}{8\pi}\left(\frac{V''}{V}\right) \tag{7}$$

are slow roll parameters in the absence of brane corrections. The influence of the brane term becomes important when $V/\lambda_b \gg 1$ and in this case we get

$$\epsilon \simeq 4\epsilon_{\rm FRW}(V/\lambda_b)^{-1}, \ \eta \simeq 2\eta_{\rm FRW}(V/\lambda_b)^{-1}. \tag{8}$$

Clearly slow-roll ($\epsilon, \eta \ll 1$) is easier to achieve when $V/\lambda_b \gg 1$ and on this basis one can expect inflation to occur even for relatively steep potentials, such the exponential and the inverse power-law which we discuss below.

A. Exponential potentials

The exponential potential

$$V(\phi) = V_0 e^{\tilde{\alpha}\phi/M_P} \tag{9}$$

with $\dot{\phi} < 0$ (equivalently $V(\phi) = V_0 e^{-\tilde{\alpha}\phi/M_P}$ with $\dot{\phi} > 0$) has traditionally played an important role within the inflationary framework [15] since, in the absence of matter, it gives rise to power law inflation $a \propto t^c$, $c = 2/\tilde{\alpha}^2$ provided $\tilde{\alpha} \leq \sqrt{2}$. ($M_P$ is the reduced four dimensional Planck mass $M_P = M_4/\sqrt{8\pi}$, $M_4 = 1.2 \times 10^{19}$ GeV.) For $\tilde{\alpha} > \sqrt{2}$ the potential becomes too steep to sustain inflation and for larger values $\tilde{\alpha} \geq \sqrt{6}$ the field enters a kinetic regime during which $\dot{\phi}^2 \gg V(\phi)$ and $p_\phi \simeq \rho_\phi$, resulting in a rapidly decreasing field energy density $\rho_\phi \propto a^{-6}$. Thus within the standard general relativistic framework, steep potentials which have been suggested as candidates for cold dark matter and quintessence [16–19] are not capable of sustaining inflation. However extra-dimensional effects lead to interesting new possibilities for the inflationary scenario. The increased damping of the scalar field when $V/\lambda_b \gg 1$ leads to a decrease in the value of the slow-roll parameters $\epsilon = \eta \simeq 2\tilde{\alpha}^2\lambda_b/V$, so that slow-roll ($\epsilon, \eta \ll 1$) leading to inflation now becomes possible even for large values of $\tilde{\alpha}$.

Within the framework of the braneworld scenario, the field equations (4) and (5) can be solved *exactly* in the slow-roll limit when $\rho/\lambda_b \gg 1$. In this case

$$\frac{\dot{a}(t)}{a(t)} \simeq \frac{1}{\sqrt{6M_P^2\lambda_b}}V(\phi), \tag{10}$$

which, when substituted in

$$3H\dot{\phi} \simeq -V'(\phi) \tag{11}$$

leads to $\dot{\phi} = -\tilde{\alpha}\sqrt{2\lambda_b/3}$. The equation of motion for the ϕ-field follows immediately

$$\phi(t) = \phi_i - \sqrt{\frac{2\lambda_b}{3}}\tilde{\alpha}(t-t_i). \tag{12}$$

An expression for number of inflationary e-foldings is also easy to establish

$$\mathcal{N} = \log\frac{a(t)}{a_i} = \int_{t_i}^{t} H(t')dt' \simeq \frac{V_0}{2\lambda_b\tilde{\alpha}^2}(e^{\tilde{\alpha}\phi_i} - e^{\tilde{\alpha}\phi(t)}) \tag{13}$$

$$= \frac{V_i}{2\lambda_b\tilde{\alpha}^2}\left[1 - \exp\{-\sqrt{\frac{2\lambda_b}{3M_P^2}}\tilde{\alpha}^2(t-t_i)\}\right], \tag{14}$$

where $V_i = V_0 e^{\tilde{\alpha}\phi_i}$. From Eq. (14) we find that the expansion factor passes through an inflection point at

$$t - t_i = \gamma = \sqrt{\frac{3M_P^2}{2\lambda_b}}\tilde{\alpha}^{-2}\log\left(\frac{V_i}{2\lambda_b\tilde{\alpha}^2}\right),$$

since

$$\begin{aligned} &\ddot{a} > 0 \ \text{ for } \ t - t_i < \gamma, \\ &\ddot{a} < 0 \ \text{ for } \ t - t_i > \gamma, \\ &\ddot{a} = 0 \ \text{ for } \ t - t_i = \gamma. \end{aligned} \tag{15}$$

The epoch $t_{\rm end} = t_i + \gamma$ therefore marks the end of inflation. Substitution in (12) and (9) gives

$$\phi_{\rm end} = -\frac{M_P}{\tilde{\alpha}}\log\left(\frac{V_0}{2\lambda_b\tilde{\alpha}^2}\right), \tag{16}$$

$$V_{\rm end} \equiv V_0 e^{\tilde{\alpha}\phi_{\rm end}/M_P} = 2\lambda_b\tilde{\alpha}^2. \tag{17}$$

From (13) & (17) we also find

$$\mathcal{N} = \log a/a_i = V_0\left[e^{\tilde{\alpha}\phi_i/M_P} - e^{\tilde{\alpha}\phi(t)/M_P}\right]/V_{\rm end}, \tag{18}$$

$$\text{or} \ \ V_{\rm end} = \frac{V_i}{\mathcal{N}+1}. \tag{19}$$

Eqns. (17) & (19) had earlier been obtained in [6] using a different method. The COBE normalized value for the amplitude of scalar density perturbations

$$A_s^2 \simeq \frac{8}{75}\frac{V_i^4}{M_4^4\tilde{\alpha}^2\lambda_b^3} \simeq 4\times 10^{-10} \tag{20}$$

can now be used to determine both λ_b and $V_{\rm end}$:

$$\lambda_b \simeq \frac{2.3 \times 10^{-10}}{\tilde{\alpha}^6} \left(\frac{M_4}{\mathcal{N}+1}\right)^4, \tag{21}$$

$$V_{\rm end} \simeq 4.6 \times 10^{-10} \left(\frac{M_4}{\tilde{\alpha}(\mathcal{N}+1)}\right)^4, \tag{22}$$

in agreement with the results obtained in [6]. For $\mathcal{N} \simeq 70$ we get

$$\lambda_b \simeq 9.3 \times 10^{-18} \frac{M_4^4}{\tilde{\alpha}^6} \simeq \frac{1.9 \times 10^{59}}{\tilde{\alpha}^6}\ {\rm GeV}^4, \tag{23}$$

$$V_{\rm end}^{1/4} \simeq \frac{6.5 \times 10^{-5}}{\tilde{\alpha}} M_4 \simeq \frac{7.8 \times 10^{14}}{\tilde{\alpha}}\ {\rm GeV}. \tag{24}$$

(The tensor/scalar ratio in the CMB anisotropy in braneworld inflation is small [13] and we work under the assumption that scalar density perturbations are responsible for most of the COBE signal.)

From (17) we find that

$$\rho_{\rm end}/2\lambda_b \simeq V_{\rm end}/2\lambda_b = \tilde{\alpha}^2, \tag{25}$$

and

$$H_{\rm end} \simeq \sqrt{\frac{8\pi V_{\rm end}}{3}} \frac{\tilde{\alpha}}{M_4}, \tag{26}$$

$$H_i \simeq (\mathcal{N}+1) H_{\rm end}. \tag{27}$$

Clearly $\rho_{\rm end}/2\lambda_b \gg 1$ if $\tilde{\alpha}^2 \gg 1$ which suggests that the brane term in (4) will continue to dominate the dynamics of the universe for some time after inflation has ended, as demonstrated in Fig. 1.

Eqs. (13) & (14) can be rewritten as

$$\mathcal{N} = \log \frac{a(t)}{a_i} = \frac{V_i}{V_{\rm end}} \left[1 - \exp \frac{\tilde{\alpha}}{M_P} \{\phi(t) - \phi_i)\}\right] \tag{28}$$

$$= \frac{V_i}{V_{\rm end}} \left[1 - \exp\left\{-\sqrt{\frac{2\lambda_b}{3M_P^2}} \tilde{\alpha}^2 (t - t_i)\right\}\right]. \tag{29}$$

Expanding the right hand side of (29) in a Taylor series we get

$$\frac{a(t)}{a_i} \simeq \exp\left[\frac{V_i}{V_{\rm end}} \frac{\tilde{\alpha}}{M_P} (\phi_i - \phi(t))\right] \tag{30}$$

$$= \exp\left[\sqrt{\frac{2\lambda_b}{3M_P^2}} \tilde{\alpha}^2 \frac{V_i}{V_{\rm end}} (t - t_{\rm end})\right] \tag{31}$$

which demonstrates that inflation proceeds at an exponential rate during early epochs.

1. Post-inflationary evolution: The kinetic regime

A short while after inflation ends, the brane-term in the field equations (4) becomes unimportant. The scalar field rolling down a steep potential is now subject to minimum damping and soon goes into a 'free fall' mode during which $\dot{\phi}^2 \gg V(\phi)$ and $\rho_\phi \propto a^{-6}$. Integrating the system of equations (4) & (5) numerically, we find that the value of the Hubble parameter at the commencement of the kinetic regime can be conveniently related to its value at the end of inflation by the fitting formula

$$\frac{H_{\rm kin}}{H_{\rm end}} = a + \frac{b}{\tilde{\alpha}^2}, \quad (\tilde{\alpha} \gtrsim 3) \tag{32}$$

where $a = 0.085$ and $b = -0.688$. In addition, a small amount of radiation is also present due to particles being produced quantum mechanically during inflation which give rise to an energy density [23,24] $\rho_R \sim 0.01 g_p H_{\rm end}^4$, where $g_p \simeq 10 - 100$ is the number of different particle species created from the vacuum. (In the case of the exponential potential, quantum mechanical particle production provides the only mechanism by means of which the universe can 'reheat'.) If the particles created quantum mechanically were to be thermalized immediately after inflation, then one would obtain for the radiation temperature the value [6,25]

$$T_{\rm end} \simeq \frac{2\times 10^{-5}}{\tilde{\alpha}} \frac{M_4}{(\mathcal{N}+1)^2} = \frac{9\times 10^{10}}{\tilde{\alpha}} \left(\frac{51}{\mathcal{N}+1}\right)^2 \text{GeV}. \tag{33}$$

It can be easily shown that in this case the density in the inflaton far exceeds the density in radiation at the end of inflation

$$\left(\frac{\rho_\phi}{\rho_r}\right)_{\rm end} \simeq 5\lambda_b^2 M_4^4/g_p V_{\rm end}^3 \sim 2\times 10^{16} \left(\frac{\mathcal{N}+1}{51}\right)^4 g_p^{-1}. \tag{34}$$

This leads to a prolonged 'kinetic regime' during which scalar matter has the 'stiff' equation of state $P_\phi \simeq \rho_\phi$ and the universe evolves as

$$\frac{a(t)}{a_{\rm kin}} = \left[\sqrt{\frac{\tilde{\alpha}^2\lambda_b}{M_P^2}}(t - t_{\rm kin}) + 1\right]^{1/3}, \tag{35}$$

while the evolution of the scalar field is described by

$$\phi(t) = \phi_{\rm kin} - \sqrt{6} M_P \log\left(\frac{a(t)}{a_{\rm kin}}\right). \tag{36}$$

Here $a_{\rm kin}, \phi_{\rm kin}$ correspond to values at the start of the kinetic regime and one can assume to a first approximation $\phi_{\rm kin} \simeq \phi_{\rm end}$, $a_{\rm kin} \simeq a_{\rm end}$, although, as we demonstrate below, the kinetic regime does not commence immediately after inflation ends but shortly afterwards. After the commencement of the kinetic regime, radiation and inflaton densities fall off at different rates $\rho_{\rm rad}/\rho_\phi \propto a^2$ and a time will come when the two will equalize

$$\left(\frac{\rho_\phi}{\rho_r}\right)_{\rm eq} = \left(\frac{\rho_\phi}{\rho_r}\right)_{\rm kin} (\frac{a_{\rm kin}}{a_{\rm eq}})^2 \simeq 1. \tag{37}$$

The value of the scalar field at this juncture is given by

$$\frac{\phi_{eq}}{M_P} = \frac{\phi_{\rm kin}}{M_P} - \frac{\sqrt{6}}{2} \log\left(\frac{\rho_\phi}{\rho_r}\right)_{\rm kin} \tag{38}$$

and the temperature of the universe can be estimated from

$$T_{\rm eq} = T_{\rm kin}\left(\frac{a_{\rm kin}}{a_{\rm eq}}\right) = T_{\rm kin}\left(\frac{\rho_r}{\rho_\phi}\right)_{\rm kin}^{1/2} \tag{39}$$

where $T_{\rm kin}$ is the temperature at the start of the kinetic regime. As demonstrated in Fig. 1 the influence of the brane term in (4) causes a short delay between the commencement of the kinetic regime and the end of inflation.

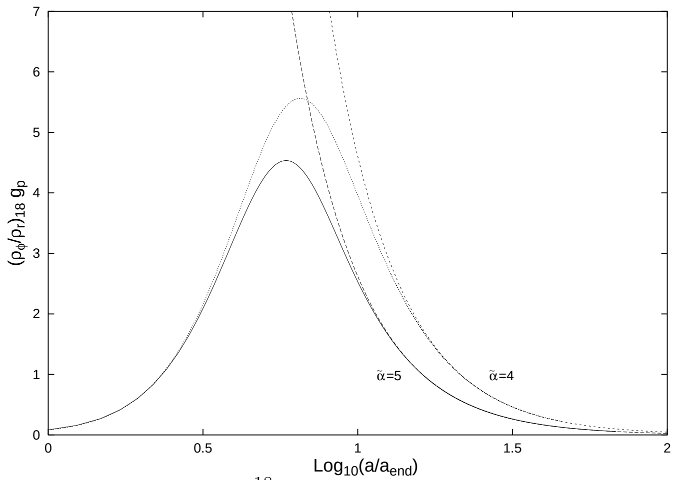

FIG. 1. The evolution of $(\rho_\phi/\rho_{\rm r})_{18} = \rho_\phi/\rho_{\rm r} \times 10^{-18}$ is shown as a function of the expansion factor shortly after inflation ends. The ratio $\rho_\phi/\rho_{\rm rad}$ first increases due to the dominance of the brane-term which causes the density in the ϕ-field to decrease much more slowly than the density in radiation. The decay law $\rho_\phi/\rho_{\rm rad} \propto a^{-2}$ marking the commencement of the kinetic regime is shown for comparison (dashed line). We see that the influence of the brane term is stronger for *smaller* value of the parameter $\tilde\alpha$.

When the kinetic regime finally commences the temperature of the universe has dropped to

$$T_{\rm kin} = T_{\rm end}\left(\frac{a_{\rm end}}{a_{\rm kin}}\right) \simeq T_{\rm end}\left(c + \frac{d}{\tilde\alpha^2}\right) \tag{40}$$

where $c \simeq 0.142,\ d = -1.057$ and $\tilde\alpha \gtrsim 3$ is assumed. From this expression and Fig. 1 we see that the influence of the brane term is *larger* for smaller values of $\tilde\alpha$. (For very small values of $\tilde\alpha$ the temperature at the onset of the kinetic regime $T_{\rm kin}$ can fall by almost two orders of magnitude relative to $T_{\rm end}$.) The reason for this can be understood by rewriting (4) as

$$H^2 = \frac{8\pi}{3M_4^2}\rho B_{\rm rane} \tag{41}$$

where $B_{\rm rane} = 1 + \rho/2\lambda_b$ is the brane-correction term to the FRW equations. Immediately after inflation the influence of the brane term is still strong so that $\rho \simeq V_0 e^{\tilde\alpha\tilde\phi}$ where $\tilde\phi = \phi/M_P$. Consequently one can rewrite $B_{\rm rane}$ as $B_{\rm rane}(\tilde\alpha) \simeq 1 + \tilde\alpha^2 e^{-\tilde\alpha(\tilde\phi_{\rm end} - \tilde\phi)}$, from where it becomes clear that as $\tilde\phi$ rolls to smaller values ($\tilde\phi < \tilde\phi_{\rm end}$) the influence of the brane term diminishes for larger values of $\tilde\alpha$. This effect is illustrated in Fig 1. From (40) one also finds for the commencement of the kinetic regime, the expression

$$\frac{a_{\rm kin}}{a_{\rm end}} = \frac{T_{\rm end}}{T_{\rm kin}} \simeq (c + \frac{d}{\tilde\alpha^2})^{-1}, \tag{42}$$

some values of $a_{\rm kin}/a_{\rm end}$ are given in Table 1.

TABLE I. Commencement of the kinetic regime.

$\tilde\alpha$	2.6	3	4	5	10	∞
$a_{\rm kin}/a_{\rm end}$	1584.9	100	14.13	11.22	7.94	7.05
$\left[(\rho_\phi/\rho_r)_{kin}\right] \times g_p \times 10^{-18}$	0.009	0.24	1.79	2.23	2.35	2.4

The equality between scalar-field matter and radiation can be estimated from (39). Integrating the equations of motion numerically, we find that for $\tilde{\alpha} \gtrsim 3$ the resulting value of $T_{\rm eq}$ is well described by

$$T_{\rm eq} \simeq \frac{T_{\rm end}}{(\rho_\phi/\rho_r)^{1/2}_{\rm end}} \left(e + \frac{f}{\tilde{\alpha}^2}\right) \tag{43}$$

$$\simeq \frac{240}{\tilde{\alpha}} \sqrt{\frac{g_p}{2}} \left(e + \frac{f}{\tilde{\alpha}^2}\right) \ \text{GeV} \tag{44}$$

where $e = 0.0265,\ f = -0.176$. For $\tilde{\alpha} > 15$ we get

$$T_{\rm eq} \simeq \frac{6}{\tilde{\alpha}} \sqrt{\frac{g_p}{2}} \ \text{GeV}. \tag{45}$$

From (44) & (45) we find that the temperature at matter-radiation equality is sensitive to the value of $\tilde{\alpha}$: for steep potentials (large $\tilde{\alpha}$) $T_{\rm eq}$ is smaller as shown in Fig. 2. An upper limit on $\tilde{\alpha}$ can be established by requiring that the density in stiff matter has dropped below the radiation density during cosmological nucleosynthesis: $T_{\rm eq} > T_{\rm nucl} \sim$ few MeV, substitution in (45) leads to the generous upper limit $\tilde{\alpha} \lesssim 10^4$.

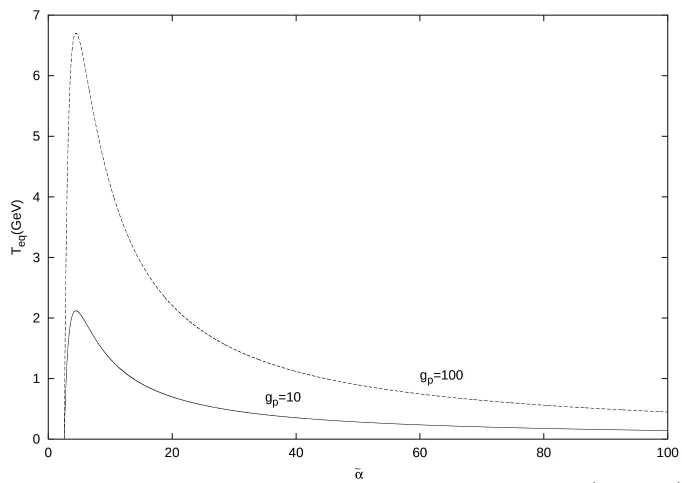

FIG. 2. The temperature of the universe at the epoch of radiation domination (in GeV) is shown as a function of the parameter $\tilde{\alpha}$ for a universe in which radiation is created due to gravitational particle production.

The universe has expanded by a factor $a_{\rm eq}/a_{\rm end} = T_{\rm end}/T_{\rm eq} \sim 10^{10} g_p^{-1/2}$ during the kinetic regime, between the end of inflation and matter-radiation equality. As the universe expands further it enters the radiation dominated regime. As we show in Sec. (III) a long kinetic regime leaves behind a unique signature in the relic gravity wave background which can be used to strongly constrain inflationary models with steep potentials.

Finally we should mention that the number of inflationary e-foldings in this model can be unambiguously determined from the following considerations. A length scale which crosses the Hubble radius during the inflationary epoch (a_i) and re-enters it today (a_0) will satisfy $k = a_i H_i = a_0 H_0$, equivalently

$$\frac{k}{a_0 H_0} = \frac{a_i}{a_{\rm end}} \frac{a_{\rm end}}{a_0} \frac{H_i}{H_0} \equiv e^{-\mathcal{N}} \frac{T_0}{T_{\rm end}} \frac{H_i}{H_0}. \tag{46}$$

Noting that $H_i \simeq (\mathcal{N}+1) H_{\rm end}$ and substituting from (19), (22) & (33) leads to $\mathcal{N} \simeq 70$.

2. Post-inflationary evolution: The radiative regime

We have demonstrated that the temperature of the universe at the commencement of radiation domination is a few GeV in exponential potential models with quantum-mechanical particle production (see also [6]). After a period during which ρ_ϕ can drop significantly below $\rho_{\rm rad}$ (overshoot), the scalar field enters a 'tracking regime' during which the ratio $\rho_\phi/\rho_{\rm rad}$ is held constant [17,18]:

$$\frac{\rho_\phi}{\rho_B + \rho_\phi} = \frac{3(1+w_B)}{\tilde{\alpha}^2} \lesssim 0.2, \tag{47}$$

equivalently

$$\frac{V''}{H^2} = \frac{9}{2}(1-w_B)^2, \tag{48}$$

($w_B = 0,\ 1/3$ respectively for dust, radiation). The inequality in (47) reflects nucleosynthesis constraints which require the value of the dimensionless parameter $\tilde{\alpha}$ to be large $\tilde{\alpha} \gtrsim 5$.

During tracking $w_\phi \simeq w_{\rm B}$, which leads to

$$\frac{1}{2}\dot{\phi}^2 = \frac{1+w_B}{1-w_B} V(\phi), \tag{49}$$

substituting in (47) we get

$$\frac{\dot{\phi}^2}{\rho_{\rm total}} = \frac{3(1+w_B)^2}{\tilde{\alpha}^2}. \tag{50}$$

Since

$$\rho_{\rm total} = \frac{4M_P^2}{3(1+w_B)^2 t^2} \tag{51}$$

we get

$$\dot{\phi}^2 = \frac{4M_P^2}{\tilde{\alpha}^2 t^2} \tag{52}$$

i.e. the evolution of $\dot{\phi}$ is independent of the value of w_B ! Integrating Eq. (52) between the epoch of radiation and matter domination we get

$$\phi_{\rm MD} = \phi_{\rm RD} - \frac{2M_P}{\tilde{\alpha}} \log\left(\frac{t_{\rm MD}}{t_{\rm RD}}\right) \tag{53}$$

where $\phi_{\rm RD}, \phi_{\rm MD}$ are scalar field values at the commencement of radiative and matter dominated regime respectively. It should be noted that the inequality in (47) prevents the exponential potential from becoming dominant during the course of cosmological evolution and hence greatly limits its contribution to the 'dark energy' during later epochs. As we shall show in the next section, a small change in the potential can give rise to a model of 'quintessential inflation' [20] in which the scalar field density can drive the accelerated expansion of the universe at the present time.

B. The cosine hyperbolic potential

The many excellent properties of the exponential potential also feature in the potential [19]

$$V(\phi) = V_0(\cosh \tilde{\alpha}\phi/M_P - 1)^p, \quad p > 0, \tag{54}$$

which has the following asymptotic forms

$$V(\phi) \simeq \frac{V_0}{2} e^{\tilde{\alpha}p\phi/M_P}, \quad \tilde{\alpha}p\phi/M_P \gg 1,\ \phi > 0, \tag{55}$$

$$V(\phi) \simeq \frac{V_0}{2} (\frac{\tilde{\alpha}\phi}{M_P})^{2p}, \quad |\tilde{\alpha}p\phi/M_P| \ll 1. \tag{56}$$

An important difference between (9) and (54) is that (54) goes over to the standard chaotic form for $p = 1$ and $\phi \lesssim M_P/\tilde{\alpha}$ allowing reheating to take place conventionally during oscillations of ϕ (provided ϕ couples to other fields in nature). Reheating for the exponential potential, which does not permit oscillations of ϕ, involves more exotic mechanisms [18,6].

By expanding eqn. (54) as $V(\phi) = V_0 \cosh \tilde{\alpha}\phi/M_P - V_0$ (for $p = 1$) one finds that the term V_0, which corresponds to a cosmological constant, has been subtracted out in our potential. Since current observations favour the presence of a small 'Λ-term' [21,22] one might consider it appropriate to include V_0 by replacing (54) by $V(\phi) = V_0 \cosh \tilde{\alpha}\phi/M_P$. From the discussion in the preceeding section it is easy to see that such a model could give rise to inflation during an early epoch. It would also give rise to dark energy today provided the value of V_0 were set to [21,22] $V_0 \simeq 10^{-47}\text{GeV}^4$.

Another means of getting a negative equation of state at late times is by noting that, for small values of ϕ, the form of the potential changes to (56) giving rise to oscillations of the scalar field. As demonstrated in [19] scalar field oscillations about $\phi = 0$ give rise to a mean equation of state given by

$$\langle w_\phi \rangle = \left\langle \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \right\rangle = \frac{p-1}{p+1}. \tag{57}$$

The resulting scalar field density and expansion factor have the form

$$\langle \rho_\phi \rangle \propto a^{-3(1+\langle w_\phi \rangle)}, \tag{58}$$

$$a \propto t^c, \;\; c = \tfrac{2}{3}(1 + \langle w_\phi \rangle)^{-1}. \tag{59}$$

We find that for $p = 1$ the scalar field behaves like pressureless (cold) dark matter. On the other hand $p < 1/2$ results in a negative mean equation of state $\langle w_\phi \rangle < -1/3$, enabling the scalar field to play the role of dark energy (quintessence). We have numerically solved for the behaviour of this model after including a radiative term (arising from inflationary paricle production discussed in the previous section) and standard cold dark matter. Our results for a particular realisation of the model with parameters $V_0 \simeq 5 \times 10^{-46}$ GeV^4, $\tilde{\alpha} = 5$ and $p = 0.2$ are shown in figures 3 & 4. We find that, due to the very large value of the scalar kinetic energy at the commencement of the radiative regime (described by the ratio $\dot{\phi}^2/V(\phi)$), the scalar field density overshoots the radiation density (see also [26]). After this, the value of ρ_ϕ stabilizes and remains relatively unchanged for a considerable length of time during which the scalar field equation of state is $w_\phi \simeq -1$. Tracking commences late into the matter dominated epoch and the universe accelerates today during rapid oscillations of the scalar field. This model provides an interesting example of 'quintessential inflation'. However as shall discuss in section III, the long duration of the kinetic regime in this model results in a large gravity wave background which could be in conflict with nucleosynthesis constraints.

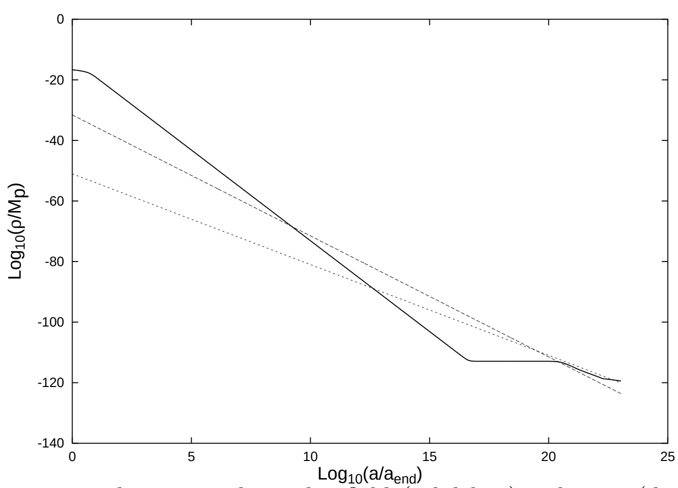

FIG. 3. The post-inflationary energy density in the scalar field (solid line) radiation (dashed line) and cold dark matter (dotted line) is shown as a function of the scale factor for the model decribed by (54) with $V_0 \simeq 5 \times 10^{-46}$ GeV^4, $\tilde{\alpha} = 5$ and $p = 0.2$. The enormously large value of the scalar field kinetic energy (relative to the potential) ensures that the scalar field density overshoots the background radiation value, after which ρ_ϕ remains approximately constant for a substantially long period of time. At late times the scalar field briefly tracks the background matter density before becoming dominant and driving the current accelerated expansion of the universe.

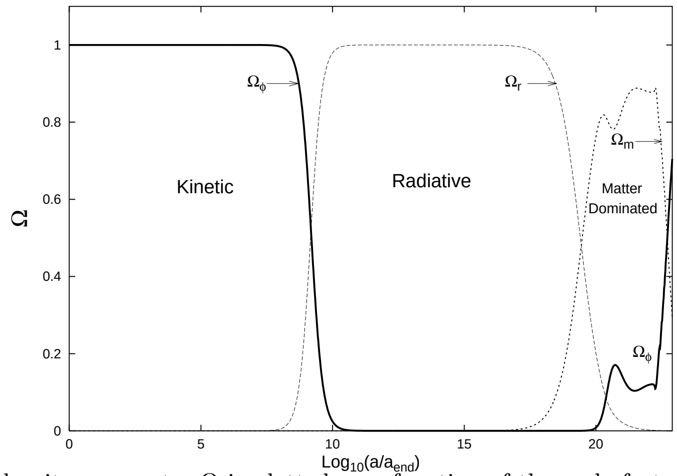

FIG. 4. The dimensionless density parameter Ω is plotted as a function of the scale factor for the model in figure 3. Late time oscillations of the scalar field ensure that the mean equation of state turns negative $\langle w_\phi \rangle \simeq -2/3$, giving rise to the current epoch of cosmic acceleration with $a(t) \propto t^2$ and present day values $\Omega_{0\phi} \simeq 0.7, \Omega_{0m} \simeq 0.3$.

C. Power law potentials

The calculations presented in section II A can easily be extended to include other steep potentials such as

$$V(\phi) = \frac{(\mu M_4)^4}{(\phi/M_4)^{\tilde{\alpha}}}, \quad \tilde{\alpha} \gg 1. \tag{60}$$

The slow-roll parameter in this case has the simple form

$$\epsilon \simeq \frac{\tilde{\alpha}^2}{4\pi} \left(\frac{\lambda_b}{\mu^4 M_4^4}\right) \left(\frac{\phi}{M_4}\right)^{\tilde{\alpha}-2} \tag{61}$$

and one immediately notices that a larger value of the parameter $\tilde{\alpha}$ assists slow roll when $\phi/M_4 \ll 1$ provided $\tilde{\alpha} > 2$. This behavior, which is completely at odds with general relativistic results, is clearly due to the presence of the brane correction term λ_b/V in (8).

Inflation ends when $\epsilon \sim 1$ which leads to the useful result

$$\frac{V_{\rm end}}{\lambda_b} \left(\frac{\phi_{\rm end}}{M_4}\right)^2 \simeq \frac{\tilde{\alpha}^2}{4\pi}. \tag{62}$$

The equation of motion in the slow roll regime can be solved exactly resulting in

$$\phi^2(t) = \phi_i^2 + \frac{M_4 \tilde{\alpha}}{\sqrt{3\pi\lambda_b}}(t - t_i). \tag{63}$$

Next consider the expression for the number of inflationary e-foldings

$$\mathcal{N} \simeq -\frac{8\pi}{M_4^2} \int_{\phi_i}^{\phi_{\rm end}} \frac{V}{V'} \left(\frac{V}{2\lambda_b}\right) d\phi, \tag{64}$$

for the power law potential (60) one gets

$$\mathcal{N} \simeq \frac{4\pi}{\lambda_b} \frac{1}{\tilde{\alpha}(\tilde{\alpha}-1)} \left[V_i \left(\frac{\phi_i}{M_4}\right)^2 - V_{\rm end} \left(\frac{\phi_{\rm end}}{M_4}\right)^2\right]. \tag{65}$$

Equations (62) & (65) lead to the following useful relation

$$\frac{V_i}{\lambda_b} \left(\frac{\phi_i}{M_4}\right)^2 \simeq \frac{1}{4\pi}[\mathcal{N}\tilde{\alpha}(\tilde{\alpha}-2) + \tilde{\alpha}^2], \tag{66}$$

where ϕ_i & V_i refer to the value of the scalar field and its potential at the commencement of inflation. The amplitude of scalar perturbations in this model is [5,6]

$$A_S^2 = \frac{64\pi}{75M_4^2}\left(\frac{V_i}{V'}\right)^2\left(\frac{V_i}{\lambda_b}\right)^3\left(\frac{V_i}{M_4^4}\right) \tag{67}$$

$$= \frac{64\pi}{75M_4^2}\left(\frac{\phi_i}{\tilde{\alpha}}\right)^2\left(\frac{V_i}{\lambda_b}\right)^3\left(\frac{V_i}{M_4^4}\right), \tag{68}$$

which, after the substitution

$$\left(\frac{\phi_i}{M_4}\right)^2 = \frac{1}{4\pi}\frac{\lambda_b}{V_i}[\mathcal{N}\tilde{\alpha}(\tilde{\alpha}-2)+\tilde{\alpha}^2] \tag{69}$$

acquires the form

$$A_s^2 \simeq \frac{16}{75}\left(\frac{V_i}{\lambda_b}\right)^2\left(\frac{V_i}{M_4^4}\right)[\mathcal{N}\frac{\tilde{\alpha}(\tilde{\alpha}-2)}{\tilde{\alpha}^2}+1]. \tag{70}$$

Using the COBE normalized value $A_s^2 \simeq 4\times10^{-10}$ we get, for $\tilde{\alpha} \gg 1$

$$V_i \simeq (\mathcal{N}+1)V_{\rm end}, \tag{71}$$

$$\lambda_b \simeq \mu^{\frac{24}{\tilde{\alpha}+4}} M_4^4 \left(\frac{2\times10^{-9}}{(\mathcal{N}+1)^4}\right)^{\frac{\tilde{\alpha}-2}{\tilde{\alpha}+4}}\left(\frac{4\pi}{\tilde{\alpha}^2}\right)^{\frac{3\tilde{\alpha}}{\tilde{\alpha}+4}}, \tag{72}$$

$$V_{\rm end} \simeq \mu^{\frac{16}{\tilde{\alpha}+4}} M_4^4 \left(\frac{2\times10^{-9}}{(\mathcal{N}+1)^4}\right)^{\frac{\tilde{\alpha}}{\tilde{\alpha}+4}}\left(\frac{4\pi}{\tilde{\alpha}^2}\right)^{\frac{2\tilde{\alpha}}{\tilde{\alpha}+4}}, \tag{73}$$

$$\phi_{\rm end} \simeq \mu^{\frac{4}{\tilde{\alpha}+4}} M_4 \left(\frac{(\mathcal{N}+1)^4}{2\times10^{-9}}\right)^{\frac{1}{\tilde{\alpha}+4}}\left(\frac{\tilde{\alpha}^2}{4\pi}\right)^{\frac{2}{\tilde{\alpha}+4}}. \tag{74}$$

(We note that in the limit $\tilde{\alpha} \gg 1$, $V_{\rm end}/2\lambda_b \simeq \tilde{\alpha}^2/4\pi$, in agreement with our earlier results for the exponential potential.) The radiation density at this epoch can be obtained by substituting for $V_{\rm end}, \lambda_b$ in

$$(\rho_r)_{\rm end} \simeq g_p T_{\rm end}^4 \simeq \frac{1}{5} g_p \frac{V_{\rm end}^4}{\lambda_b^2 M_4^4} \tag{75}$$

which gives

$$T_{\rm end} \simeq \left(\frac{1}{5}\right)^{1/5} \mu^{\frac{4}{\tilde{\alpha}+4}} \left(\frac{2\times10^{-9}}{(\mathcal{N}+1)^4}\right)^{\frac{\tilde{\alpha}+2}{2(\tilde{\alpha}+4)}}\left(\frac{4\pi}{\tilde{\alpha}^2}\right)^{\frac{\tilde{\alpha}}{2(\tilde{\alpha}+4)}} M_4. \tag{76}$$

We also get

$$\left(\frac{\rho_\phi}{\rho_r}\right)_{\rm end} \simeq \frac{5\lambda_b^2 M_4^4}{g_p V_{\rm end}^3} \simeq \frac{5}{2}\times10^9 g_p^{-1}(\mathcal{N}+1)^4 \tag{77}$$

a result which is independent of both $\tilde{\alpha}$ and μ !

Consequently

$$T_{\rm end} \simeq \frac{2.6\times10^{11}}{\tilde{\alpha}}\mu^{4/(\tilde{\alpha}+4)}\ {\rm GeV}\ \ \text{for } \mathcal{N}\sim70,\ \tilde{\alpha}\gg1 \tag{78}$$

$$\left(\frac{\rho_\phi}{\rho_r}\right)_{\rm end} \simeq 6.4\times10^{16} g_p^{-1}\ \ \text{for } \mathcal{N}\sim70. \tag{79}$$

Shortly after inflation has ended the kinetic regime commences and the universe follows the expansion law (35).

Inverse power law potentials provide excellent models for quintessence [16] and it is important to investigate whether, within the braneworld framework, such potentials can describe *both* quintessence and inflation (see also [6,7]). To achieve this one must ensure that the scalar field remains in the kinetic regime for an appreciable length of time

$(a/a_{\rm end} \sim 10^{10})$ long enough for the ratio ρ_ϕ/ρ_r to drop below unity and for radiation domination to commence. During the kinetic regime the scalar field evolves as

$$\frac{\phi(t)}{M_4} = \frac{\phi_{\rm kin}}{M_4} + \sqrt{\frac{3}{4\pi}}\log\left(\frac{a(t)}{a_{\rm kin}}\right)$$
$$= \frac{\phi_{\rm kin}}{M_4} + \sqrt{\frac{1}{12\pi}}\log\left(\frac{t}{t_{\rm kin}}\right) \tag{80}$$

where $\phi_{\rm kin} > \phi_{\rm end}$ and from (77) we see that $\phi_{\rm end}/M_4 \gtrsim 1$. The equalization of the density in stiff-matter and radiation marks the commencement of the radiative regime and the value of the scalar field at this juncture is given by

$$\frac{\phi_{\rm eq}}{M_4} = \frac{\phi_{\rm kin}}{M_4} + \sqrt{\frac{3}{16\pi}}\log\left(\frac{\rho_\phi}{\rho_r}\right)_{\rm kin}. \tag{81}$$

A lower limit on $(\rho_\phi/\rho_r)_{\rm kin}$ can be derived by assuming that the ratio ρ_ϕ/ρ_r decreases as a^{-2} between the end of inflation and the commencement of the kinetic regime (in fact it decreases slower), in addition if we make the conservative assumption $T_{\rm kin} \simeq 0.01 T_{\rm end}$ we get

$$\left(\frac{\rho_\phi}{\rho_r}\right)_{\rm kin} = \left(\frac{T_{\rm end}}{T_{\rm kin}}\right)^{-2}\left(\frac{\rho_\phi}{\rho_r}\right)_{\rm end} \gtrsim 6\times 10^{12} g_p^{-1}. \tag{82}$$

Eqn. (81) can now be used to obtain a lower bound on the scalar field value at the commencement of the radiative regime

$$\frac{\phi_{\rm eq}}{M_4} \gtrsim \frac{\phi_{\rm kin}}{M_4} + \sqrt{\frac{3}{16\pi}}\log\left(6\times 10^{12} g_p^{-1}\right). \tag{83}$$

A large value of $\phi_{\rm eq}$ indicates that the field has rolled down to regions where the potential is less steep. During tracking flat potentials can give rise to inflation, therefore one must ensure that inflation does not recur before the universe becomes radiation dominated. Imposing the requirement $\epsilon_{\rm FRW} > 1$ where

$$\epsilon_{\rm FRW} = \frac{M_4^2}{16\pi}\left(\frac{V'}{V}\right)^2 = \frac{\tilde\alpha^2}{16\pi}\left(\frac{\phi}{M_4}\right)^{-2} \tag{84}$$

is the slow roll parameter, we get

$$\frac{\phi}{M_4} \lesssim \frac{\phi_*}{M_4} \equiv \frac{\tilde\alpha}{\sqrt{16\pi}}. \tag{85}$$

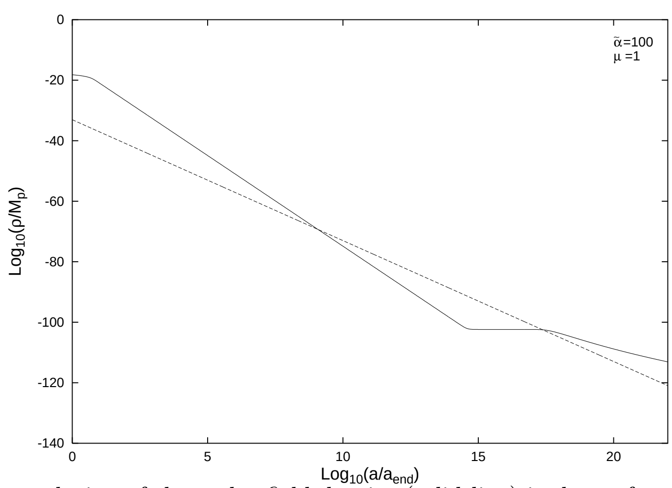

FIG. 5. The post-inflationary evolution of the scalar field density (solid line) is shown for the potential (60) with $\mu = 1$ and $\tilde\alpha = 100$. The radiation density is also shown (dashed line). We see that the scalar field energy density dominates the expansion dynamics of the universe during both early and late times. This model re-inflates much too soon, resulting in an unacceptably large value of the dark energy today.

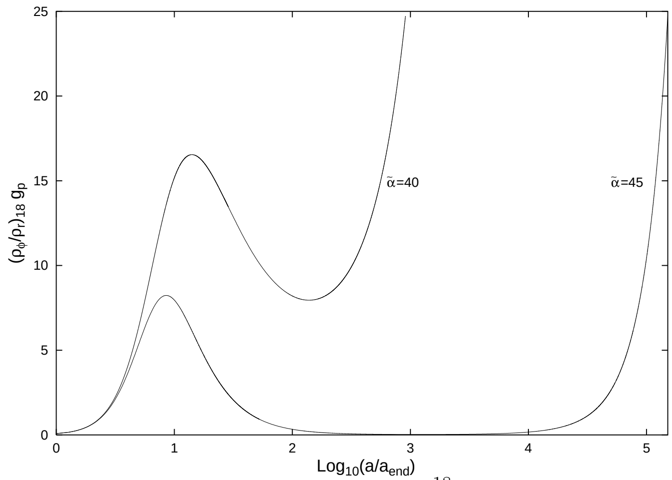

FIG. 6. Post-inflationary evolution of $(\rho_\phi/\rho_r)_{18} = (\rho_\phi/\rho_r) \times 10^{-18}$ is shown for the inverse power law potential (60) $V(\phi) \propto \phi^{-\tilde{\alpha}}$ ($\mu = 1$). The initial increase in ρ_ϕ/ρ_r is due to brane corrections immediately after inflation. During the kinetic regime ρ_ϕ/ρ_r initially decreases as a^{-2} but increases again during a second epoch of inflation which commences at $a/a_{end} \sim 10^2, 10^4$ for $\tilde{\alpha} = 40, 45$ respectively. In both cases the post-inflationary expansion of the universe is *always* scalar field dominated and the universe *never* enters the radiative regime.

The estimate given by (85) is valid provided the scalar field enters the tracking regime soon after radiation domination. Numerical estimates show that this is not always the case. Indeed, as demonstrated in Figure 5, the scalar field can remain in the kinetic regime for a considerable period of time after the universe becomes radiation dominated. For large values of $\mu \sim O(1)$ and $\tilde{\alpha} \lesssim 75$ we find that the scalar field can cause the universe to re-inflate either before or soon after the commencement of the radiative regime (see figures 6 & 5). As a result, the matter dominated regime is *never* reached. However, as demonstrated in [7], for a narrow range of parameter values the universe can be made to inflate at the present epoch providing a possible model for 'quintessential inflation'. In figure 7 we show the evolution of the universe filled with matter, radiation and a scalar field potential with parameters $\tilde{\alpha} = 20$, $\mu \simeq 10^{-26}$. The scalar field in this case successfully plays the dual role of being the inflaton at an early epoch and dark energy today, lending support to the analysis of [7] (see also [20,27,28]). From (77) it is easy to show that the duration of the kinetic regime, described by the ratio $a_{\rm eq}/a_{\rm kin} = T_{\rm kin}/T_{\rm eq} \sim 10^9 g_p^{-1/2}$, depends very weakly upon the values of μ and $\tilde{\alpha}$. For the cosmologically relevant values $\tilde{\alpha} = 20, \mu \simeq 10^{-26}$ we find $T_{\rm kin} \sim 10^6{\rm GeV}, T_{\rm eq} \sim 10{\rm MeV}$ and $a_{\rm eq}/a_{\rm kin} \sim 10^8$. As we show in the next section, relic gravity waves created during inflation impose strong constraints on braneworld inflationary models having kinetic regimes of such long duration.

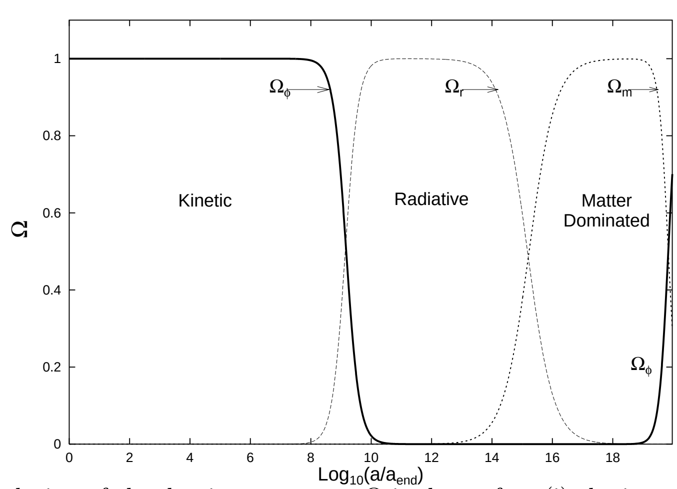

FIG. 7. Post-inflationary evolution of the density parameter Ω is shown for: (i) the inverse power law potential (60) with $\tilde{\alpha} = 20, \mu \simeq 10^{-26}$ (solid line) (ii) radiation (dashed line) and (iii) matter (dotted line). In this model of 'quintessential inflation' the scalar field dominates the energy density of the universe twice, initially during inflation and finally during the current 'dark energy' dominated epoch, giving present day values $\Omega_{0\phi} \simeq 0.7, \Omega_{0m} \simeq 0.3$.

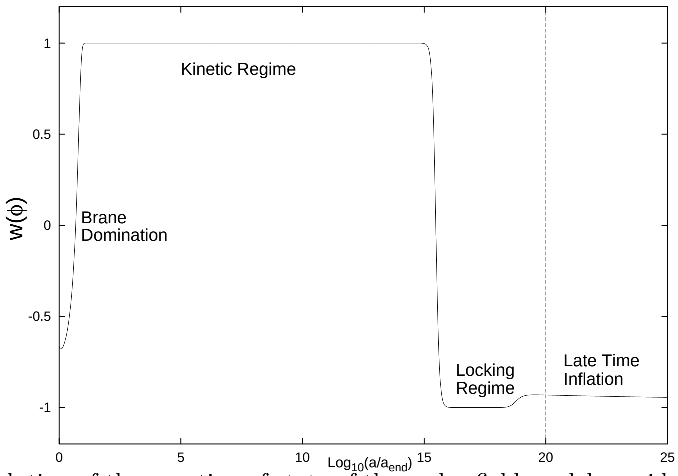

FIG. 8. Post-inflationary evolution of the equation of state of the scalar field model considered in figure 7. The evolution of $w(\phi)$ follows four distinct stages: (i) *Post-inflationary brane domination.* Diminishing post-inflationary brane effects cause the equation of state in the ϕ-field to gradually increase from $w(\phi) \simeq -2/3$ to $w(\phi) \simeq 1$. (ii) *The kinetic regime.* During this regime the kinetic energy of the scalar field exceeds its potential energy leading to $w(\phi) \simeq 1$. (iii) *Locking.* During the radiative regime the energy density in the scalar field overshoots the energy density in radiation resulting in an extensive period during which the scalar field equation of state locks to $w(\phi) \simeq -1$. The scalar field during this regime behaves like an effective cosmological constant since its energy density ρ_ϕ does not change appreciably with time. (iv) *Late time inflation.* During the late matter dominated regime the ratio of the scalar field density to that in matter grows until the scalar field dominates the energy density of the universe. This leads to the current epoch of accelerated expansion ('late time inflation') during which $w(\phi) \lesssim -0.9$. The vertical dashed line corresponds to the present epoch.

III. GRAVITY WAVE BACKGROUND

An important feature of inflationary scenarios of the very early universe is the quantum mechanical production of relic gravity waves [10–12] which create fluctuations in the cosmic microwave background and whose stochastic signature presents a challenge to gravity wave observatories such as LIGO and LISA. Gravity waves in a spatially homogeneous and isotropic background geometry satisfy the minimally coupled Klein-Gordon equation $\Box h_{ik} = 0$ [9], which, after a separation of variables $h_{ij} = \phi_k(\tau) e^{-i\mathbf{kx}} e_{ij}$ (e_{ij} is the polarization tensor) reduces to

$$\ddot{\phi}_k + 2\frac{\dot{a}}{a}\dot{\phi}_k + k^2\phi_k = 0 \tag{86}$$

where $\tau = \int dt/a(t)$ is the conformal time coordinate and $k = 2\pi a/\lambda$ is the comoving wavenumber. Since brane driven inflation is near-exponential we can write $a = \tau_0/\tau$ ($|\tau| < |\tau_0|$), in this case normalized positive frequency solutions of (86) corresponding to the adiabatic vacuum in the 'in state' are given by [29,13]

$$\phi^+_{\rm in}(k,\tau) = \left(\frac{\pi\tau_0}{4}\right)^{1/2}\left(\frac{\tau}{\tau_0}\right)^{3/2} H^{(2)}_{3/2}(k\tau) F(H_{\rm in}/\tilde{\mu}) \tag{87}$$

where $\tilde{\mu} = M_5^3/M_4^2$ and $H_{\rm in} \equiv -1/\tau_0$ is the inflationary Hubble parameter. The term

$$F(x) = \left(\sqrt{1+x^2} - x^2 \log\{\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}\}\right)^{-1/2} \tag{88}$$

is responsible for the increased gravity wave amplitude in braneworld inflation [13]. At low energies ($H_{\rm in}/\tilde{\mu} \ll 1 \Rightarrow \rho/\lambda_b \ll 1$) $F \simeq 1$, while at high energies ($\rho/\lambda_b \gg 1$)

$$F \simeq \sqrt{\frac{3H_{\rm in}}{2\tilde{\mu}}} \simeq \sqrt{\frac{3\rho_{\rm in}}{2\lambda_b}}. \tag{89}$$

The corresponding dimensionless amplitude of gravity waves is given by

$$h_{\rm GW}^2 = \left(\frac{H_{\rm in}}{M_4}\right)^2 F^2 \simeq \left(\frac{H_{\rm in}}{M_4}\right)^2 \left(\frac{3\rho}{2\lambda_b}\right). \tag{90}$$

For the scalar field models discussed earlier, Eqn. (90) reduces to

$$h_{\rm GW}^2 \simeq 2\pi \frac{V_{\rm HC}^3}{\lambda_b^2 M_4^4}, \tag{91}$$

where $V_{\rm HC}$ corresponds to the value of the potential when the given gravity wave mode left the Hubble radius during inflation. From (70) & (91) we find a simple relation between $h_{\rm GW}$ and A_s

$$h_{\rm GW}^2 = \frac{75\pi}{8} A_s^2 \left[\mathcal{N}\frac{(\tilde\alpha - 2)}{\tilde\alpha} + 1\right]^{-1} \tag{92}$$

which reduces to

$$h_{\rm GW}^2 \simeq \frac{75\pi}{8}\frac{A_s^2}{\mathcal{N}+1} \tag{93}$$

for $\tilde\alpha \gg 1$. It is easy to show that (93) is also the correct expression for the gravity wave amplitude in inflationary models with exponential potentials for *arbitrary values* of the parameter $\tilde\alpha$ defined in (9). Finally, using the COBE-normalized values $A_s \simeq 2 \times 10^{-5}$ and assuming $\mathcal{N}_{\rm total} \simeq 70$ we obtain

$$h_{\rm GW}^2 \simeq 1.7 \times 10^{-10} \tag{94}$$

for the exponential and inverse power law potentials discussed in the preceding sections. (In deriving (94) we have used the result that most of the COBE signal is produced by scalar density perturbations [13,6].)

The 'out state' is described by a linear superposition of positive and negative frequency solutions to (86). For power law expansion $a = (t/t_0)^p \equiv (\tau/\tau_0)^{1/2-\mu}$, we have

$$\phi_{\rm out}(k,\tau) = \alpha\phi_{\rm out}^{(+)}(k\tau) + \beta\phi_{\rm out}^{(-)}(k\tau) \tag{95}$$

where

$$\phi_{\rm out}^{(+,-)}(k\tau) = \left(\frac{\pi\tau_0}{4}\right)^{1/2}\left(\frac{\tau}{\tau_0}\right)^{\mu} H_{|\mu|}^{(2,1)}(k\tau), \tag{96}$$

and μ is related to the post-inflationary equation of state

$$\mu = \frac{3}{2}\left(\frac{w-1}{3w+1}\right). \tag{97}$$

The Wronskian normalization condition $W_\tau(\phi_{\rm out}, \phi_{\rm out}^*) = i/a^2$ ensures that $|\alpha^2| - |\beta^2| = 1$. The energy density of relic gravity waves is given by [30,31]

$$\epsilon_{\rm g} = \langle T_0^0\rangle = \frac{1}{2\pi^2 a^2}\int dk k^2 \left(|\dot\phi_{\rm out}|^2 + k^2|\phi_{\rm out}|^2\right), \tag{98}$$

which reduces to

$$\epsilon_{\rm g} = \frac{1}{\pi^2 a^4}\int dk k^3 |\beta|^2, \tag{99}$$

if modes within the horizon provide the dominant contribution to the energy density. The corresponding spectral energy density is simply

$$\tilde\epsilon_{\rm g}(k) \equiv \frac{d}{d\log k}\epsilon_{\rm g} = \frac{1}{\pi^2 a^4} k^4 |\beta(k)|^2. \tag{100}$$

The Bogoliubov coefficients α & β can be determined after imposing suitable junction conditions on $\phi_{\rm in}$ and $\phi_{\rm out}$ as discussed in detail in [9–12]. For instance the small argument limit of the Hankel function

$$H_\mu^{(2,1)}(k\tau) \underset{k\tau\ll 2\pi}{=} \frac{(\frac{k\tau}{2})^\mu}{\Gamma(1+\mu)} \pm \frac{i}{\pi}\Gamma(\mu)\left(\frac{k\tau}{2}\right)^{-\mu} \quad (\mu \neq 0) \tag{101}$$

$$H_0^{(2,1)}(k\tau) \underset{k\tau\ll 2\pi}{=} 1 \mp \frac{2i}{\pi}\log(k\tau) \tag{102}$$

combined with the observation that the amplitude of a general solution to the wave equation (86) freezes to a constant value on scales larger than the cosmological horizon

$$\phi(k\tau) \underset{k\tau\ll 2\pi}{=} A(k) + B(k)\int \frac{d(\tau/\tau_0)}{a^2} \tag{103}$$

allows us to match the 'in' and 'out' modes (87) & (95) and determine Bogoliubov coefficients α & β.

Following this prescription (which has been described in detail in [12]) we proceed to evaluate the spectral density of gravity waves created in inflationary braneworld models. Before commencing on a detailed analysis of the problem let us first consider the simple but illustrative case of a universe which, after inflating, enters a non-inflationary regime with equation of state $0 \le w < 1$ $(-3/2 \le \mu < 0)$. The Bogoliubov coefficients in this case have been found to be [12]

$$\begin{aligned}
\alpha &= \frac{i}{2}\left[\gamma\left(\frac{k\tau_0}{2}\right)^{-(3/2+|\mu|)} + \gamma^{-1}\left(\frac{k\tau_0}{2}\right)^{3/2+|\mu|}\right] \\
\beta &= \frac{i}{2}\left[\gamma\left(\frac{k\tau_0}{2}\right)^{-(3/2+|\mu|)} - \gamma^{-1}\left(\frac{k\tau_0}{2}\right)^{3/2+|\mu|}\right],
\end{aligned} \tag{104}$$

for $k\tau_0 < 2\pi$, here $\gamma = \Gamma(1+|\mu|)/2\sqrt{\pi}$. (On smaller than horizon scales ($k\tau > 2\pi$) the adiabatic theorem gives $\alpha \simeq 1, \beta \simeq 0$ [32,11].)

From (99) & (104) it is easy to show that

$$\begin{aligned}
&\epsilon_{\rm g} \propto a^{-4} \text{ for } w > 1/3 \\
&\epsilon_{\rm g} \propto \epsilon_{\rm B} \text{ for } w < 1/3,
\end{aligned} \tag{105}$$

where $\epsilon_{\rm B}$ is the background matter density. In other words the gravity wave energy density scales as radiation if the equation of state of background matter driving the expansion of the universe is $P_{\rm B} > \epsilon_{\rm B}/3$. In the reverse case when $P_{\rm B} < \epsilon_{\rm B}/3$, the gravity wave equation of state mimicks that of the background, so that $\epsilon_{\rm g}/\epsilon_{\rm B} \simeq$ constant. This 'tracker-like' behavior of the gravity wave energy density was first discovered in [11,12]. (For $P_{\rm B} = \epsilon_{\rm B}/3$ $\epsilon_{\rm g} \propto a^{-4}\log(\tau/\tau_0)$.)

From (100) we also find that [12]

$$\tilde{\epsilon}_{\rm g} \propto k^{1-2|\mu|}, \tag{106}$$

as a result, gravity waves will have have a blue spectrum for equations of state stiffer than radiation ($w_\phi > 1/3, |\mu| < 1/2$) and a red spectrum if $w_\phi < 1/3$, $|\mu| > 1/2$. Our current cosmological model passes through three post-inflationary expansion epochs during which its equation of state is succesively: stiff ($w \simeq 0$), radiative ($w \simeq 1/3$), matter-dominated ($w \simeq 0$). Gravity waves created during these separate epochs are therefore likely to have a scale-dependent tilt which will vary from being 'blue' during the kinetic regime to 'white' ($\equiv$ flat) during radiation domination to 'red' during matter-domination [12,14]. We shall now demonstrate this explicitly. According to a general result which allows us to determine Bogoliubov coefficients in a multi-component universe, the Bogoliubov coefficients $\mathcal{B}_n$ describing particle production during the 'n-th' successive epoch can be determined from [11,12]

$$\mathcal{B}_n = \mathcal{B}_{1\to 2}\mathcal{B}_{2\to 3}\cdots\mathcal{B}_{n-1\to n} \tag{107}$$

where

$$\mathcal{B} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}. \tag{108}$$

In the present cosmological model the universe passes through four expansion stages: Inflation (1) $\rightarrow$ kinetic regime (2) $\rightarrow$ radiative regime (3) $\rightarrow$ matter-dominated regime (4). Bogoliubov coefficients corresponding to each of these regimes are obtained below assuming that the transition from one regime to the next is instantaneous.

1. *Kinetic regime.* Bogoliubov coefficients corresponding to modes created in the course of the 'Inflation → kinetic' $(1 \to 2)$ transition are given by [12]

$$\alpha_{1\to 2} \equiv \alpha_{\rm kin} = \frac{i}{2}\left[\frac{1}{2\sqrt{\pi}}\left(\frac{k\tau_{\rm kin}}{2}\right)^{-3/2} + 2\sqrt{\pi}\left\{1+\frac{2i}{\pi}\log(k\tau_{\rm kin})\right\}\left(\frac{k\tau_{\rm kin}}{2}\right)^{3/2}\right]$$
$$\beta_{1\to 2} \equiv \beta_{\rm kin} = \frac{i}{2}\left[\frac{1}{2\sqrt{\pi}}\left(\frac{k\tau_{\rm kin}}{2}\right)^{-3/2} - 2\sqrt{\pi}\left\{1-\frac{2i}{\pi}\log(k\tau_{\rm kin})\right\}\left(\frac{k\tau_{\rm kin}}{2}\right)^{3/2}\right], \qquad (109)$$

where $\tau_{\rm kin}$ corresponds to the commencement of the kinetic regime.

2. *Radiative regime.* Bogoliubov coefficients describing modes created during this regime arise due to successive transitions: Inflation → kinetic $(1 \to 2)$ and kinetic → radiative $(2 \to 3)$. As a result

$$\mathcal{B}_{\rm RD} = \mathcal{B}_{1\to 2}\cdot\mathcal{B}_{2\to 3} \qquad (110)$$

where the Bogoliubov coefficients $\mathcal{B}_{1\to 2}$ were derived in (109). Particle production during the 'kinetic → radiative' transition is described by

$$\alpha_{2\to 3} = \frac{1}{2}\left[\frac{\sqrt{\pi}}{2}\left(\frac{k\tau_{\rm RD}}{2}\right)^{-1/2} + \frac{2}{\sqrt{\pi}}\left(\frac{k\tau_{\rm RD}}{2}\right)^{1/2}\right]$$
$$\beta_{2\to 3} = \frac{1}{2}\left[\frac{\sqrt{\pi}}{2}\left(\frac{k\tau_{\rm RD}}{2}\right)^{-1/2} - \frac{2}{\sqrt{\pi}}\left(\frac{k\tau_{\rm RD}}{2}\right)^{1/2}\right], \qquad (111)$$

where $\tau_{\rm RD}$ corresponds to the commencement of the radiative regime. From (110), (109) & (111) we finally obtain

$$\alpha_{\rm RD} = \frac{i}{2}\left[\frac{1}{k^2\tau_{\rm kin}^{3/2}\tau_{\rm RD}^{1/2}} + k^2\tau_{\rm kin}^{3/2}\tau_{\rm RD}^{1/2} + ik\frac{\tau_{\rm kin}^{3/2}}{\tau_{\rm RD}^{1/2}}\log\left(k\tau_{\rm kin}\right)\right]$$
$$\beta_{\rm RD} = \frac{i}{2}\left[\frac{1}{k^2\tau_{\rm kin}^{3/2}\tau_{\rm RD}^{1/2}} - k^2\tau_{\rm kin}^{3/2}\tau_{\rm RD}^{1/2} + ik\frac{\tau_{\rm kin}^{3/2}}{\tau_{\rm RD}^{1/2}}\log\left(k\tau_{\rm kin}\right)\right]. \qquad (112)$$

3. *Matter dominated regime.* Gravity waves created during the matter-dominated regime bear the imprint of three successive transitions: Inflation (1) → kinetic (2), kinetic (2) → radiative (3), radiative (3) → matter-dominated (4). The Bogoliubov coefficients are therefore determined from

$$\mathcal{B}_{\rm MD} = \mathcal{B}_{1\to 2}\cdot\mathcal{B}_{2\to 3}\cdot\mathcal{B}_{3\to 4} \qquad (113)$$

where

$$\alpha_{3\to 4} = \frac{1}{2}\left[\frac{3}{k\tau_{\rm MD}} + \frac{k\tau_{\rm MD}}{3}\right]$$
$$\beta_{3\to 4} = \frac{1}{2}\left[\frac{3}{k\tau_{\rm MD}} - \frac{k\tau_{\rm MD}}{3}\right] \qquad (114)$$

and $\tau_{\rm MD}$ refers to the commencement of the current matter -dominated regime. From (113), (109), (111) & (114) we finally obtain

$$\alpha_{\rm MD} = \frac{i}{2}\left[\frac{3}{k^3\tau_{\rm MD}\tau_{\rm RD}^{1/2}\tau_{\rm kin}^{3/2}} + 3i\frac{\tau_{\rm kin}^{3/2}}{\tau_{\rm MD}\tau_{\rm RD}^{1/2}}\log\left(k\tau_{\rm kin}\right) + \frac{k^3\tau_{\rm MD}\tau_{\rm RD}^{1/2}\tau_{\rm kin}^{3/2}}{3}\right]$$
$$\beta_{\rm MD} = \frac{i}{2}\left[\frac{3}{k^3\tau_{\rm MD}\tau_{\rm RD}^{1/2}\tau_{\rm kin}^{3/2}} + 3i\frac{\tau_{\rm kin}^{3/2}}{\tau_{\rm MD}\tau_{\rm RD}^{1/2}}\log\left(k\tau_{\rm kin}\right) - \frac{k^3\tau_{\rm MD}\tau_{\rm RD}^{1/2}\tau_{\rm kin}^{3/2}}{3}\right]. \qquad (115)$$

It is easy to show that the Bogoliubov coefficients evaluated in (109), (112) & (115) satisfy $|\alpha_{\rm kin}|^2 - |\beta_{\rm kin}|^2 = |\alpha_{\rm RD}|^2 - |\beta_{\rm RD}|^2 = |\alpha_{\rm MD}|^2 - |\beta_{\rm MD}|^2 = 1$. Inspecting α & β in (109), (112) & (115) one finds that the dominant role in each of these expressions is played by the first term, so that effectively

$$|\beta_{\rm kin}|^2 \simeq \frac{1}{2\pi}(k\tau_{\rm kin})^{-3}, \quad 2\pi\tau_{\rm kin}^{-1} < k \leq k_{\rm RD}, \tag{116}$$

$$|\beta_{\rm RD}|^2 \simeq \frac{1}{4}k^{-4}\tau_{\rm RD}^{-1}\tau_{\rm kin}^{-3}, \quad k_{\rm MD} \leq k < k_{\rm RD}, \tag{117}$$

$$|\beta_{\rm MD}|^2 \simeq \frac{9}{4}k^{-6}\tau_{\rm MD}^{-2}\tau_{\rm RD}^{-1}\tau_{\rm kin}^{-3}, \quad 2\pi\tau^{-1} \lesssim k < k_{\rm MD} \tag{118}$$

where $k\tau = 2\pi$ describes the comoving horizon scale and $k_{\rm RD} = \pi/2\tau_{\rm RD}$, $k_{\rm MD} = 3/\tau_{\rm MD}$ have been obtained using junction conditions.

We are now in a position to evaluate the spectral energy density of relic gravity waves given in (100). From (100) and Eqns. (116) – (118) we find that the spectral density of relic gravity waves is very sensitive to the equation of state of matter driving the expansion of the universe. Our brane-inspired cosmological model has three distinct & lengthy post-inflationary epochs during which the effective equation of state is successively: (i) stiff ($w = 1$), (ii) radiation dominated ($w = 1/3$) and (iii) matter dominated ($w = 0$). Reflecting this, the relic gravity wave spectrum will have three main components: $\tilde{\epsilon}_{\rm g} \propto \lambda^{-1}$ for $\lambda < \lambda_{\rm RD}$, $\tilde{\epsilon}_{\rm g} =$ constant for $\lambda_{\rm RD} \lesssim \lambda < \lambda_{\rm MD}$ and $\tilde{\epsilon}_{\rm g} \propto \lambda^{-2}$ for $\lambda_{\rm MD}^{\rm h} \lesssim \lambda < \lambda_0^{\rm h}$; $\lambda_{\rm RD}^{\rm h}$ & $\lambda_{\rm MD}^{\rm h}$ are related to the comoving hubble radius at the commencement of the radiation dominated and matter dominated regimes respectively (precise values of these quantities will be given later) while $\lambda_0^{\rm h} \simeq 10^{28}$ cm is the present distance to the cosmological horizon.

From (100) & (116) we find that a long kinetic regime strongly affects the gravity wave spectrum giving it considerable power on short wavelength scales $\tilde{\epsilon}_g(k) \propto \lambda^{-1}$. Since gravity waves corresponding to this part of the spectrum were created prior to the radiative regime we must ensure that the integrated gravity wave energy density satisfies the stringent constraints set by cosmological nucleosynthesis. To resolve this issue we examine Eq. (99) which after the substitution $\beta \to \beta_{\rm kin}$ reduces to

$$\epsilon_{\rm g} = \frac{32}{3\pi}h_{\rm GW}^2\epsilon_{\rm B}\,(\tau/\tau_{\rm kin}) \tag{119}$$

where $\epsilon_{\rm B}$ is the background density of stiff scalar matter driving the expansion of the universe. The radiative regime is reached when $\tau = \tau_{\rm eq}$ and $\epsilon_{\rm B} = \epsilon_{\rm stiff} + \epsilon_{\rm rad} \simeq 2\epsilon_{\rm rad}$, since $(\tau_{\rm eq}/\tau_{\rm kin}) = (T_{\rm kin}/T_{\rm eq})^2$, we arrive at the relationship

$$\epsilon_{\rm g}(\tau = \tau_{\rm eq}) = \frac{64}{3\pi}h_{\rm GW}^2\epsilon_{\rm rad}(\tau = \tau_{\rm eq})\left(\frac{T_{\rm kin}}{T_{\rm eq}}\right)^2. \tag{120}$$

Substituting from (40), (44) & (94) and allowing for the fact that the Hubble crossing gravity wave amplitude is somewhat smaller at $\tau_{\rm eq}$ than it is at present, we get

$$\epsilon_{\rm g} \sim 2 \times 10^8 g_p^{-1}\epsilon_{\rm rad} \tag{121}$$

i.e. unless $g_p > 10^9$ the energy density in gravity waves will exceed the radiation density by a very substantial amount violating nucleosynthesis constraints which demand $\epsilon_{\rm g} \lesssim 0.2\epsilon_{\rm rad}$. (Eqn (121) was derived under the assumption that $\tilde{\alpha} = 3$, larger values of $\tilde{\alpha}$ will increase $\epsilon_{\rm g}$ and further exacerbate the situation.) From (119) it also follows that the gravity wave energy density will begin to dominate scalar field matter when the universe has expanded by a factor $a_{\rm eq}/a_{\rm kin} \sim 10^5$. The backreaction of gravity waves will at this point effectively end the kinetic regime and the expansion of the universe will change from $a \propto \tau^{1/2}$ characteristic of the stiff-matter regime, to $a \propto \tau$ characteristic of the radiative regime (see also [14]). A similar result is obtained for the inverse power law potentials discussed in the previous section.

It is interesting that the equations (34), (91) & (119) can also be used to obtain a *model independent* constraint on brane-induced quintessential inflation in which reheating arises solely on account of inflationary particle production. Indeed, since $(\tau_{\rm eq}/\tau_{\rm kin}) = (a_{\rm eq}/a_{\rm kin})^2 = (\rho_\phi/\rho_{\rm rad})_{\rm kin}$ we find, using the relations (91) & (119) and the inequality $(\rho_\phi/\rho_{\rm rad})_{\rm kin} \gtrsim (\rho_\phi/\rho_{\rm rad})_{\rm end}$,

$$\frac{\epsilon_g}{\epsilon_{\rm rad}}(\tau = \tau_{\rm eq}) \gtrsim \frac{640}{3}\left(\frac{V_{\rm HC}}{V_{\rm end}}\right)^3 g_p^{-1}. \tag{122}$$

Since $V_{\rm HC}(\tau_{\rm eq})/V_{\rm end} >$ few, we obtain the conservative bound

$$0.2 \gtrsim \frac{\epsilon_g}{\epsilon_{\rm rad}}(\tau = \tau_{\rm eq}) > 2 \times 10^4 g_p^{-1}. \tag{123}$$

The requirement that gravity waves are subdominant during nucleosynthesis therefore translates into the following model independent constraint on the number of particle species $g_p > 10^5$. We therefore conclude that brane-inspired quintessential inflation (with inflationary particle production) can only be accomodated if the number of particle species through which the universe reheats is very large.

This situation can be remedied either by having a shorter kinetic regime or if the inflaton were to have a post-inflationary equation of state which is softer than $P_\phi = \rho_\phi$. Both these features are accommodated by the potential

$$V(\phi) = V_0(\cosh \tilde{\alpha}\phi/M_P - 1), \tag{124}$$

earlier discussed in section II B. While brane-driven inflation proceeds along lines identical to those examined earlier for exponential inflation, the post-inflationary evolution in this model is radically different since reheating takes place not due to gravitational particle production (as in the case of the exponential potential) but via scalar field oscillations which commence when $m^2 \equiv V'' = 8\pi V_0\tilde{\alpha}^2/M_4^2 = H^2$. Prior to this epoch, the expansion rate of the universe after the end of inflation and before reheating, has the form $a \propto t^{2/3(1+w_\phi)}$ where

$$\begin{aligned} w_\phi &= \frac{\tilde{\alpha}^2}{3} - 1, \quad \tilde{\alpha} \lesssim \sqrt{6}, \\ w_\phi &= 1, \qquad\qquad \tilde{\alpha} \gtrsim \sqrt{6}. \end{aligned} \tag{125}$$

Restricting ourselves for the moment to the stiff equation of state $w_\phi = 1$ we find for the energy density

$$\epsilon_{\rm g} = \frac{32}{3\pi} h_{\rm GW}^2 \epsilon_{\rm rad} \left(\frac{H_{\rm kin}}{H_{\rm rh}}\right)^{2/3} \tag{126}$$

where $H_{\rm kin}$ marks the commencement of the kinetic regime and $H_{\rm rh}^2 \simeq V''$ is the square of the Hubble parameter at the time of reheating. The requirement that the gravity wave density satisfy nucleosynthesis constraints leads to $\epsilon_{\rm g} \lesssim 0.2\epsilon_{\rm rad}$ and results in a firm lower bound on the value of $H_{\rm rh}$:

$$H_{\rm rh} \gtrsim \frac{0.02}{\tilde{\alpha}}(0.085 - \frac{0.69}{\tilde{\alpha}^2})\,{\rm GeV} \tag{127}$$

where we have used the relationships (32), (24) & (25) to establish (127). The temperature at reheating may be estimated from

$$T_{\rm rh}^2 \simeq \sqrt{\frac{3}{8\pi}}\zeta M_4 H_{\rm rh} \tag{128}$$

which results in the lower bound

$$T_{\rm rh} = (3V_0\tilde{\alpha}^2)^{1/4} \gtrsim \frac{3\times 10^8}{\tilde{\alpha}^{1/2}}\left(0.085 - \frac{0.69}{\tilde{\alpha}^2}\right)^{1/2} {\rm GeV}, \tag{129}$$

where $\zeta < 1$ is an indicator of the 'efficiency' of reheating and characterizes the fraction of the inflaton density converted into radiation when the universe reheats, we assume $\zeta \simeq 1$ for simplicity.

From (116) - (118), (94) & (100) we find that the spectral energy density of the relic gravity wave background is determined by the following set of equations (see Fig. 5)

$$\Omega_{\rm g}^{\rm (MD)}(\lambda) = \frac{3}{8\pi^3}h_{\rm GW}^2\Omega_{0m}\left(\frac{\lambda}{\lambda_h}\right)^2, \ \lambda_{MD} < \lambda \leq \lambda_h \tag{130}$$

$$\Omega_{\rm g}^{\rm (RD)}(\lambda) = \frac{1}{6\pi}h_{\rm GW}^2\Omega_{0r}, \ \lambda_{RD} < \lambda \leq \lambda_{MD} \tag{131}$$

$$\Omega_{\rm g}^{\rm (kin)}(\lambda) = \Omega_{\rm g}^{\rm (RD)}\left(\frac{\lambda_{RD}}{\lambda}\right), \ \lambda_{\rm kin} < \lambda \leq \lambda_{RD} \tag{132}$$

the superscripts 'kin', 'RD', 'MD' refer to the epoch when gravity waves in the given wavelength band were created, the wavelength bands themselves are determined by

$$\lambda_h = 2cH_0^{-1} \simeq 1.8 \times 10^{28} h^{-1}\text{cm}, \tag{133}$$

$$\lambda_{MD} = \frac{2\pi}{3}\lambda_h \left(\frac{\Omega_{0r}}{\Omega_{0m}}\right)^{1/2}, \tag{134}$$

$$\lambda_{RD} = 4\ \lambda_h \left(\frac{\Omega_{0r}}{\Omega_{0m}}\right)^{1/2} \frac{T_{\rm MD}}{T_{\rm rh}} = \frac{3.6 \times 10^{18}}{T_{\rm rh}(\text{GeV})}\text{cm}, \tag{135}$$

$$\lambda_{\rm kin} = cH_{\rm kin}^{-1}\left(\frac{T_{\rm rh}}{T_0}\right)\left(\frac{H_{\rm kin}}{H_{\rm rh}}\right)^{1/3} = \left(\frac{\tilde{\alpha}}{0.085 - 0.69/\tilde{\alpha}^2}\right)^{2/3} 0.004\, T_{\rm rh}^{1/3}(\text{GeV})\ \text{cm}., \tag{136}$$

The length scales λ_{MD}, λ_{RD} and $\lambda_{\rm kin}$ are related to the comoving Hubble radius at the start of the matter-dominated, radiative and kinetic regimes respectively. (We have used the relationship (32) in establishing $\lambda_{\rm kin}$. We have also assumed $\Omega_{0m} \simeq 1$, $\Omega_{0r} \simeq 2.48 \times 10^{-5}\text{h}^{-2}$, an analysis which includes the presence of a late-time accelerating stage will be discussed in a companion paper.)

To summarize, we find that each of the three post-inflationary expansion epochs leaves behind a distinct imprint on the relic gravity wave background. As a result $\Omega_g(\lambda)$ (i) *decreases with wavenumber* for modes created during matter domination, (ii) *remains constant* for modes created during radiation domination and (iii) *increases with wavenumber* for modes created during the kinetic regime. The increase in amplitude of the gravity wave spectrum for wavelengths shorter than $\lambda_{\rm RD}$ cm is expected to be a generic feature of inflationary models in which brane-induced damping is solely responsible for driving inflation, since these models enter the kinetic regime once the inflaton density has dropped below the brane tension value λ_b. In models without steep potentials such as braneworld models of chaotic inflation, stage (iii) will be absent, and the gravity wave spectrum will remain flat during the entire radiative stage which will now extend from $\lambda \simeq \lambda_{\rm MD}$ until the shortest wavelengths.

We can also evaluate the total relic gravity wave energy density which is given by

$$\epsilon_{\rm g} = \epsilon_{\rm g}^{(\rm kin)} + \epsilon_{\rm g}^{(\rm RD)} + \epsilon_{\rm g}^{(\rm MD)}, \quad \text{where} \tag{137}$$

$$\epsilon_g^{(\rm kin)} = \frac{1}{\pi^2 a^4}\int_{k_{\rm RD}}^{2\pi/\tau_{\rm kin}} dk k^3 |\beta_{\rm kin}|^2, \tag{138}$$

$$\epsilon_g^{(\rm RD)} = \frac{1}{\pi^2 a^4}\int_{k_{\rm MD}}^{k_{\rm RD}} dk k^3 |\beta_{\rm RD}|^2, \tag{139}$$

$$\epsilon_g^{(\rm MD)} = \frac{1}{\pi^2 a^4}\int_{2\pi/\tau}^{k_{\rm MD}} dk k^3 |\beta_{\rm MD}|^2, \tag{140}$$

substituting from (116) – (118) we obtain

$$\epsilon_{\rm g}^{(\rm kin)} = \frac{32}{3\pi} h_{\rm GW}^2 \epsilon_{\rm rad}\left(\frac{H_{\rm kin}}{H_{\rm rh}}\right)^{2/3}, \tag{141}$$

$$\epsilon_{\rm g}^{(\rm RD)} = \frac{2}{3\pi} h_{\rm GW}^2 \epsilon_{\rm rad} \log\left(\frac{\pi\tau_{\rm MD}}{6\tau_{\rm RD}}\right), \tag{142}$$

$$\epsilon_{\rm g}^{(\rm MD)} = \frac{1}{8\pi^3} h_{\rm GW}^2 \epsilon_{\rm m}. \tag{143}$$

The dominant contribution to the gravity wave energy density clearly comes from the first term in (137) so that $\epsilon_{\rm g} \simeq \epsilon_{\rm g}^{(\rm kin)}$. The nucleosynthesis bound $\epsilon_{\rm g}^{(\rm kin)} \lesssim 0.2\epsilon_{\rm rad}$ can now be used to get the upper limit $\Omega_{\rm g} \lesssim 10^{-5}$. We therefore find that the net gravity wave background in brane-world inflationary models can be significantly larger than that predicted in conventional inflationary models ($\Omega_{\rm g} \sim 10^{-12}$) although it is probably still too small to have significant cosmological implications.

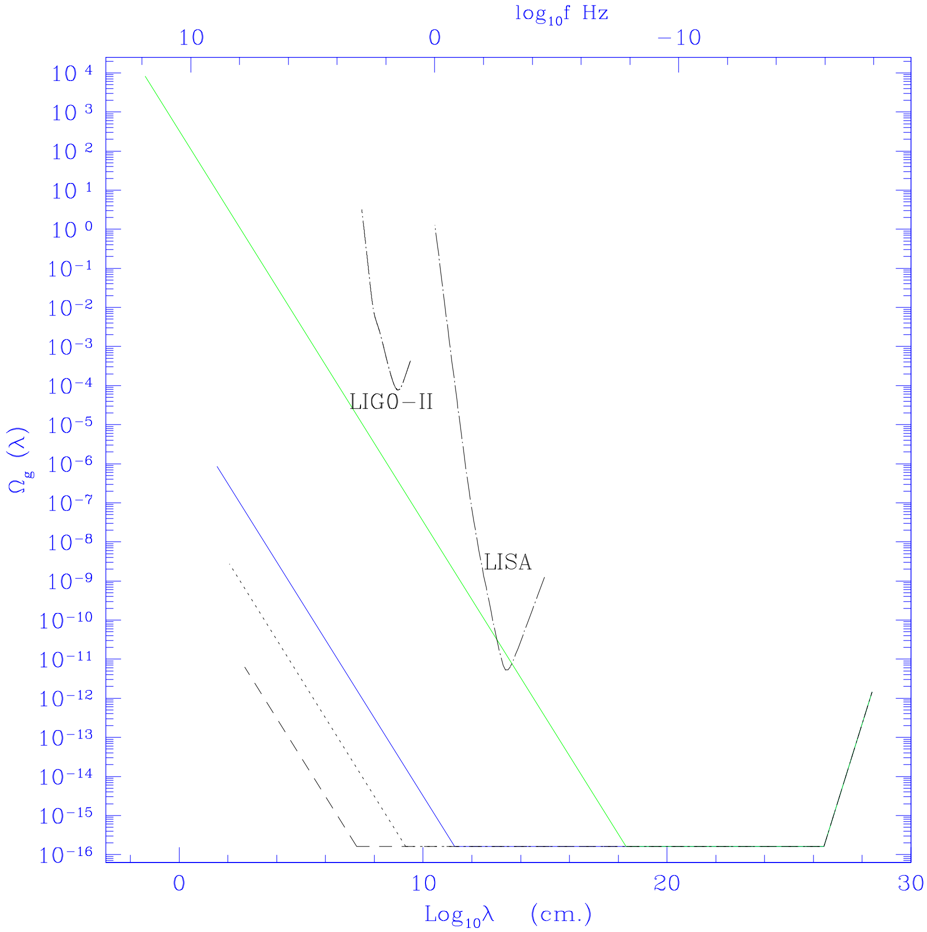

FIG. 9. The COBE-normalized gravity wave spectrum is shown for the cosine hyperbolic potential with $\{\lambda_{\rm RD}, \lambda_{\rm kin}\} = \{2 \times 10^{11}{\rm cm}, 36{\rm cm}\}, \{2 \times 10^{9}{\rm cm}, 114{\rm cm}\}$ and $\{2 \times 10^{7}{\rm cm}, 488{\rm cm}\}$ these values correspond to reheating temperatures: $T_{\rm rh} \simeq 2 \times 10^{7}$ GeV, 2×10^{9} GeV, 2×10^{11} GeV and $\tilde{\alpha} = 16, 8, 4$ respectively. The dark solid line, dotted, and short dashed line correspond to the three cases with $\tilde{\alpha} = 16, 8$ & 4, respectively. For comparison we also show the gravity wave background for the exponential potential (gray solid line). In this model radiation is created quantum mechanically during inflation, resulting in a low temperature at the commencement of the radiative stage $T_{\rm eq} \simeq 0.3 g_p^{1/2}$ GeV for $\tilde{\alpha} = 16$ (see Sec. II A). The corresponding value of the (comoving) Hubble radius at the commencement of the radiative epoch is $\lambda_{\rm RD} \simeq 2 \times 10^{18}$ cm assuming $g_p \simeq 50$ and the (comoving) Hubble radius at the commencement of the kinetic epoch is $\lambda_{\rm kin} = cH_{\rm kin}^{-1}\left(\frac{T_{\rm kin}}{T_0}\right) \simeq 0.04$ cm. The large gravity wave energy density predicted by this model appears to be in serious conflict with primordial nucleosynthesis constraints. The GW spectra assume the present value of the Hubble constant $H_0 = 70$km/s/Mpc. The chained lines marked LISA and LIGO-II are the expected sensitivity curves of the proposed Laser Interferometer Space Antenna and second phase of the Laser Interferometric gravity wave observatory.

IV. DISCUSSION/CONCLUSIONS

An important feature of braneworld models based on the Randall-Sundram ansatz is the increased rate of expansion of the universe which assists slow roll and thereby considerably enlarges the family of potentials which can contribute to inflation.

In the present paper we have made a detailed analysis of a family of steep inflationary potentials which includes: (i) $V(\phi) \propto e^{\tilde{\alpha}\phi/M_P}$, (ii) $V(\phi) \propto [\cosh(\tilde{\alpha}\phi/M_P) - 1]^p$, (iii) $V(\phi) \propto \phi^{-\tilde{\alpha}}$. In all three cases we find that the universe succesfully inflates. In the case of (ii) & (iii) the scalar field, in a limited region of parameter space, can play the dual role of being both the inflaton and dark energy. A generic feature of inflationary models with steep potentials is that the post-inflationary epoch is characterised by a prolonged kinetic regime during which the effective equation of state is $w_\phi \simeq 1$ and the universe expands as $a(t) \propto t^{1/3}$. This regime lasts until the energy density of radiation (created quantum mechanically during inflation) becomes equal to the matter density.

For models (ii) and (iii) a long duration kinetic regime leads to the possibility that inflation may resume prior to reheating ! Viable models in which this does not occur and in which the universe begins to accelerate close to the present cosmological epoch (neither much earlier nor much later) can also be constructed, but for a very limited region of parameter space. In these models *both* inflation and dark energy are generated by the very same scalar field.

Inflationary models leave behind a distinct imprint on the stochastic gravity wave background. Since the gravity wave spectrum is acutely sensitive to the post-inflationary equation of state of matter, a long-duration kinetic regime results in a gravity wave spectral density which increases with wavenumber $\tilde{\epsilon}_{\rm g} \propto k$ for wavelengths shorter than the comoving Hubble radius at the commencement of the radiative regime. For models (i) – (iii), with reheating generated *solely* on account of inflationary particle production, the kinetic regime is excessively long and the resulting 'blue tilted' energy density of gravity waves can exceed the radiation density violating cosmological nucleosynthesis considerations. In models allowing conventional reheating such as (ii), the duration of the kinetic regime is related to parameters in the potential. Such models leave behind a distinct signature on the relic gravity wave background without necessarily violating nucleosynthesis constraints. The only potential capable of being both the inflaton and quintessence without generating an unacceptably large gravity wave background is $V(\phi) = V_0 \cosh \tilde{\alpha}\phi/M_P$, provided the parameters $V_0\tilde{\alpha}^2 \simeq H_{\rm rh}^2$ satisfy the nucleosynthesis constraints (127). However this model involves considerable fine tuning since the value of the parameter V_0 must be set equal to the current value of the cosmological constant $V_0 \simeq 10^{-47}{\rm GeV}^4$.

To summarize, we have shown that models of 'quintessential inflation' are possible to construct within the context of braneworld cosmology. However, most models of quintessential inflation also generate a relic gravity wave background which can be several orders of magnitude larger than in conventional models. In the case of inflationary models with steep potentials, the relic gravity wave background is an extremely potent probe which can be used both to rule out models as well to (indirectly) confirm the reality of extra dimensions.

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