3420

SCALING PROPERTIES OF GRAVITATIONAL CLUSTERING IN THE NON-LINEAR REGIME

R.Nityananda* and T.Padmanabhan†

*Raman Research Institute, Bangalore 560080, India
e-mail: rajaram@rri.ernet.in

†Inter-University Centre for Astronomy and Astrophysics
Post Bag 4, Ganeshkhind, Pune 411 007, India
e-mail: paddy@iucaa.ernet.in

ABSTRACT

The growth of density perturbations in an expanding universe in the non-linear regime is investigated. The underlying equations of motion are cast in a suggestive form, and motivate a conjecture that the scaled pair velocity, $h(a,x) \equiv -[v/(\dot{a}x)]$ depends on the expansion factor a and comoving coordinate x only through the density contrast $\sigma(a,x)$. This leads to the result that the true, non-linear, density contrast $< (\delta\rho/\rho)_x^2 >^{1/2} = \sigma(a,x)$ is a universal function of the density contrast $\sigma_L(a,l)$, computed in the linear theory and evaluated at a scale l where $l = x(1+\sigma^2)^{1/3}$. This universality is supported by existing numerical simulations with scale-invariant initial conditions having different power laws. We discuss a physically motivated ansatz $h(a,x) = h[\sigma^2(a,x)]$ and use it to compute the non-linear density contrast at any given scale analytically. This provides a promising method for analysing the non-linear evolution of density perturbations in the universe and for interpreting numerical simulations.

IUCAA-12/93 April 93;Submitted for Publication

1.Introduction

It is generally believed that large scale structures in the universe formed through the growth of small inhomogeneities via gravitational instability. One convenient measure of the inhomogeneities present in the universe at any time t is provided by the mean square fluctuation in the mass, $\sigma^2(t,x) =< (\delta M/M)_x^2 >$, contained within a sphere of comoving radius x. When $\sigma^2 \ll 1$, the evolution of inhomogeneities can be studied using linear perturbation theory in an expanding background. But when σ^2 grows to a value of order unity, linear theory breaks down. This phase of the evolution is usually handled by numerical simulations and analytic methods seem to have very limited validity. Our understanding of structure formation will increase significantly if the results of numerical simulations can be understood in the framework of simple physical concepts and analytic approximations. This paper discusses one possible approach to understand the evolution of density fluctuation in the nonlinear domain.

The idea behind this approach is as follows: It is known (Peebles 1980) that the differential equation governing the correlation function $\xi(t,x)$ is essentially 'driven' by the average pair velocity $v_{\mathbf{pair}}(t,x)$. We present arguments which motivate the ansatz that the ratio $[v_{\mathbf{pair}}(t,x)/-(\dot{a}x)]$ depends on t and x only through the density contrast $\sigma^2(t,x)$; *i.e.*, we can set $v_{\mathbf{pair}}(t,x) = [-\dot{a}xh(\sigma^2)]$ where $h(\sigma^2)$ is some specific function which is independent of the initial conditions chosen. With this physically motivated assumption, it is possible to derive a differential equation for σ^2 and integrate it in closed form. The solution shows that, $\sigma^2(t,x)$ can be related to the density contrast $\sigma_L^2(t,l)$ evaluated by linear theory at the scale $l = x(1+\sigma^2)^{1/3}$ in a *universal* fashion. Such universality, if established, would be a key feature of gravitational dynamics in an expanding Universe. Using this result one can compute the density contrast at highly non-linear epochs knowing the linear density contrast evolved to the same epoch.

The paper is organized as follows: In the next section we discuss Newtonian mechanics in an expanding universe and stress certain properties of the resulting equations. In particular, as is well known, they admit the possibility of similarity solutions in which the statistical properties of density fluctuations at different times are identical apart from a change of spatial scale. In section 3 we discuss the similarity solution in some more detail and motivate possible generalizations of this principle to estimate the non-linear density contrast. The simplest generalization does not work very well since it misses important features of the gravitational dynamics. These are discussed in section 4 which derives the main result of the paper, estimates σ^2 at non-linear epochs and compares the results with numerical simulations. The last section discusses the implications and possibilities for extension.

2. Newtonian Gravity in expanding coordinates

This section reviews some known aspects of clustering in a framework somewhat different from, though equivalent to, the standard approach. Consider a set of particles interacting via Newtonian gravity in an expanding universe. We confine our attention to regions with dimensions much smaller than the Hubble radius so that the equation of motion for the i^{th} particle is well approximated by:

$$\ddot{\mathbf{r}}_i = -\sum_{j\neq i} \frac{Gm}{|\mathbf{r}_{ij}|^3}; \; \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j \tag{1}$$

Here $\mathbf{r}_i$ stands for the proper coordinate related to the comoving coordinate $\mathbf{x}_i$ by $\mathbf{r}_i = a(t)\mathbf{x}_i$. Using this we can write the above equation as

$$\ddot{\mathbf{x}}_i + \frac{2\dot{a}}{a}\dot{\mathbf{x}}_i = -\sum_{j\neq i} \frac{Gm}{a^3}\frac{\mathbf{x}_{ij}}{|\mathbf{x}_{ij}|^3} - \frac{\ddot{a}}{a}\mathbf{x}_i \equiv -\nabla\Phi \tag{2}$$

The right hand side of (2) has been expressed as the gradient of a scalar potential Φ which obeys the equation

$$\nabla^2\Phi = 4\pi G\left[\sum_i \frac{m}{a^3}\delta(\mathbf{x}-\mathbf{x}_i) + \frac{3}{4\pi G}\left(\frac{\ddot{a}}{a}\right)\right] \tag{3}$$

Equations (2) and (3) govern the dynamics of the particles in the expanding background. In (3) the "mass density" is contributed by two terms, one of which is independent of $\mathbf{x}$. This fact can be used to recast the equations in a more suggestive form by choosing $a(t)$ such that it satisfies the equation :

$$\frac{\ddot{a}}{a} = -\frac{4\pi G\rho_0}{3}\left(\frac{a_0}{a}\right)^3 \equiv -\frac{4\pi G}{3}\rho_b(t). \tag{4}$$

This equation describes the evolution of the expansion factor in the matter dominated phase of a Friedman universe. Using these expressions we can write (2) and (3) in the form

$$\ddot{\mathbf{x}}_i + \frac{2\dot{a}}{a}\dot{\mathbf{x}}_i = -\nabla\Phi; \tag{5}$$

$$\nabla^2\Phi = \frac{4\pi G}{a^3}\left[\sum_j m\delta(\mathbf{x}-\mathbf{x}_j) - \rho_0 a_0^3\right] \tag{6}$$

In the above form there is explicit dependence on t due to the presence of the terms $(\dot{a}/a)$ and a^3. It is useful to transform these equations such that the time dependence disappears. Introducing new dimensionless time and space coordinates τ and $\mathbf{q}_i$ via

$$\tau \equiv \ln(t/T); \;\; \mathbf{x}_i \equiv L\mathbf{q}_i; \;\; L^3 = Gmt_0^2/a_0^3 \tag{7}$$

with an arbitrary constant T, and transforming the equations, we easily find that:

$$\frac{d^2\mathbf{q}_i}{d\tau^2} + \frac{1}{3}\frac{d\mathbf{q}_i}{d\tau} = -\nabla_q U \tag{8}$$

$$\nabla_q^2 U = 4\pi \sum_i \delta(\mathbf{q} - \mathbf{q}_j) - \frac{2}{3}. \tag{9}$$

This equation has several noteworthy features which we shall briefly comment upon.

(i) The compensating negative background density, of $(-2/3)$ in the units chosen, cancels the long range part of the gravity coming from the mean density. The usual assumption made in the analysis of Jeans instability is thus justified *in these coordinates.* One gets exponential growth in $\tau = \ln t$ *i.e.*, power law growth in t. More precisely, the analysis without the damping term would have led to modes with $e^{\gamma\tau}$ with $\gamma^2 = \frac{2}{3}$. With this term, one gets $\gamma^2 + \frac{1}{3}\gamma = \frac{2}{3}$, *i.e.*, $\gamma = \frac{2}{3}$ or -1. The damping term has ensured that the growth and decay rates of the two modes are *not* equal and one recovers the correct $t^{2/3}$ and t^{-1} factors for the growing and decaying modes.

(ii) The damping term is also needed to recover other known results. For example, two bound bodies spiral inwards with a shrinking period in the $\mathbf{q}$ and τ coordinates, this being just an alternate description of a bound system with a fixed size and period in $\mathbf{r}$ and t. It is also instructive to look at the free evolution of a test particle in an otherwise uniform universe. Writing $p_i = dq_i/d\tau$, one gets $dp_i/d\tau = -\frac{1}{3}p_i$, *i.e.*, $p_i \propto \exp(-\frac{1}{3}\tau) = t^{-1/3}$. This is the well known result that the peculiar velocity of a particle decays as the inverse of the scale factor, when expressed in terms of p.

(iii) Particles at the boundary of a void feel the strong repulsion coming from the locally uncompensated negative background term which pushes them further out.

(iv) The translation invariance in τ, *without the damping term* would just lead to a conventional interacting system of particles. The presence of the damping term reminds us that we should not be trying to apply statistical mechanics with this interaction quite apart from the well known difficulties associated with the singular short range behaviour of Newtonian gravity.

(v) The usual calculation (multiplication by $\dot{q}$ and integrating) which leads to the conservation of energy for a time independent potential will now show, because of the damping term, that the 'energy' in these coordinates always decreases, being constant only in the trivial case in which all 'velocities' and 'accelerations' are zero. This decreasing 'energy' can be viewed as the driving force behind the continued evolution with formation of bound structures and evacuation of voids. In fact, the form of behaviour closest to a steady state which is known for this system is self similar clustering (Peebles 1980). The various distribution functions (two particle, three particle, etc.,), describing the system have a particularly simple dependence on τ. Writing $dq/d\tau = p$, we can have for example a two particle distribution function of the form

$$f(q_1 - q_2, p_1, p_2; \tau) = e^{6\alpha\tau} g(e^{\alpha\tau}(q_1 - q_2), e^{\alpha\tau}p_1, e^{\alpha\tau}p_2) \tag{10}$$

Here f has been written as a function only of the coordinate difference $q_1 - q_2$ after averaging over the mean position $(q_1 + q_2)/2$. Notice that before such averaging the system is not self similar.

Finally, these equations suggest a possible way of interpreting the success of adhesion models (Bagla and Padmanabhan, 1993 a). This work is in progress and will be reported elsewhere.

3. Connecting linear and non-linear density contrast

We shall now describe the statistically self similar situation in more detail. Let us suppose that the power spectrum at some epoch has the form $P(k) \propto k^n$, at sufficiently large scales which are linear. Then $\sigma_L^2(x) \propto k^3 P(k) \propto x^{-(3+n)}$ at the early, linear stage. Since σ grows as a in the linear theory, it follows that $\sigma_L^2(x,a) \propto a^2 x^{-(n+3)}$. This evolution is self-similar with $\sigma_L^2 \propto s^{-(n+3)}$ and $s = (x/t^\alpha)$; $\alpha = [4/3(3+n)]$. In the extreme non-linear case, bound structures with fixed proper radius $l = a(t)x$ will not participate in the cosmic expansion. At a fixed $l, \sigma_{NL}^2 = < (\delta\rho/\rho_b)^2 >$ must now grow as $(a^6/a^3) = a^3$. The a^6 factor arises from the fact that the background density ρ_b decreases as a^{-3} and $\sigma^2 \propto \rho_b^{-2}$; the a^3 factor in the denominator arises from the fact that in all samples of proper radius l used in computing the variance, only a fraction proportional to a^{-3} will contribute. Thus, in the non-linear regime $\sigma_{NL}^2(t,x) \propto a^3(t)F(a(t)x)$. The form of F can be determined by matching the linear and non-linear expressions at say, $\sigma = \sigma_c$. This will lead to the standard result (Peebles 1980) that $\sigma_{NL}^2 \propto a^3[ax]^{-\gamma}$ with $\gamma = 3(n+3)/(n+5)$. Thus we find that

$$\sigma_L^2(a,x) \propto a^2(t)x^{-(n+3)}; \quad \sigma_{NL}^2(a,x) \propto a^3[ax]^{-\gamma} \propto a^{6/(n+5)} x^{-3(n+3)/(n+5)} \qquad (11)$$

A graphical version of this argument is given in Fig.1. Self-similarity of this solution is evident from the fact that $\sigma_{NL} \propto s^{-\gamma}$. Notice that, in the range $n > -2$, the non-linear correlation function at fixed comoving scale x grows more *slowly* compared to the linear spectrum. It follows from these relations that one can write

$$\begin{aligned} \left(\frac{\sigma_{NL}}{\sigma_c}\right) &= \left(\frac{\sigma_L}{\sigma_c}\right)^{3/(n+5)} \quad (\text{for } \sigma_L \gg \sigma_c) \\ &= \left(\frac{\sigma_L}{\sigma_c}\right) \quad (\text{for } \sigma_L \ll \sigma_c), \end{aligned} \qquad (12)$$

where σ_c is the typical value around which the transition to non-linearity occurs. We shall now examine this result in greater detail.

The above result (12) is supposed to give the non-linear density contrast at a given scale provided the linear density contrast at the same scale is given. Unfortunately, this result is of very little practical utility and suffers from several limitations as it stands. Let us study these limitations since they need to be tackled in any attempt to improve upon the relation.

To begin with, this result is obtained by assuming a power law form for the spectrum. The power spectrum *e.g.*, just after recombination, is not a pure power law in any of the realistic models of structure formation. Hence it is not clear what is the value of n which one should use. One could, of course, try to provide a local definition of n by taking $n_{\rm eff} = (d \ln P / d \ln k)$. Unfortunately, the resulting σ_{NL} does not lead to even rough agreement with the numerical simulations.

Secondly, notice that the *actual* value of σ_{NL} depends on the value of σ_c chosen. It is difficult to estimate σ_c from theoretical considerations with any degree of accuracy. Even if σ_c is estimated, the above relation can be valid only for $\sigma_{NL} \gg \sigma_c$ and $\sigma_{NL} \ll \sigma_c$. In the regime with $\sigma_{NL} \simeq \sigma_c$ we do not have a reliable estimate.

Finally, notice that the above expression attempts to relate $\sigma_{NL}(x, a)$ to σ_L at the same x and a. This is somewhat unrealistic because non-linear growth at a given scale cannot be expected to be a strictly local function of the linear density contrast at the *same* comoving scale.

Let us now ask how some of the above difficulties can be circumvented by a more sophisticated approach. One possibility which suggests itself is to write a differential equation for σ_{NL} which can be generalized to the case in which the power spectrum is not a pure power law. This is fairly easy to do by noting that

$$\frac{\partial \ln \sigma}{\partial \ln a} = \frac{3}{(n+5)}, \quad \frac{\partial \ln \sigma}{\partial \ln x} = \frac{3(n+3)}{2(n+5)} \tag{13}$$

Hence, eliminating n,

$$\frac{\partial \ln \sigma}{\partial \ln a} - \frac{\partial \ln \sigma}{\partial \ln x} = \frac{2}{3}. \tag{14}$$

Such an equation implies that at each epoch and scale, one is evolving σ using the result of the self similar solution for the currently and locally applicable conditions. This equation is assumed to hold for $\sigma > \sigma_c$. For $\sigma < \sigma_c$ we use the rather trivial evolution equation of the linear theory:

$$\frac{\partial \ln \sigma}{\partial \ln a} = 1. \tag{15}$$

These equations are to be interpreted as follows. At some very early epochs (say, at $a = a_{\rm rec}$) we are given the density contrast $\sigma(x, a_{\rm rec}(x))$. We will assume that the initial epoch is chosen sufficiently early so that $\sigma_{\rm rec} \ll 1$ at all relevant scales. With this initial condition the above equations can be integrated forward for all further times thereby giving $\sigma(a, x)$. This approach clearly does not require σ to be a power law.

Because of the extreme simplicity, the evolution equation above can be integrated in a general form. The general solution for $\sigma > \sigma_c$ given by

$$\ln \sigma_{NL}(a, x) = \frac{3}{2} \ln \left(\frac{a}{a_{\rm rec}} \right) + F(ax) \tag{16}$$

where F is a function to be determined by matching σ_{NL} and σ_L at $\sigma = \sigma_c$. Since the linear density contrast evolves as $\sigma_L \propto a$, it follows that we can relate the σ_{NL} and σ_L as follows:

$$\sigma_{NL}(a, x) = \sigma_L^{3/2}(a, l); \quad l = x \sigma_{NL}^{2/3}(a, x) \quad (\text{for } \sigma_{NL} > \sigma_c) \tag{17}$$

This shows that when $\sigma > \sigma_c, \sigma_{NL}$ at a point x is determined by σ_L at a point $l = x\sigma_{NL}^{2/3}$. In other words non-linear evolution does introduce a degree of nonlocality in comoving scale x when this approximation is used.

While the above attempt is an improvement, it is still not very satisfactory. The differential equation (14) is difficult to derive directly from any quantitative dynamical considerations. It is merely the simplest equation that can be written down in which the power law index n is eliminated. That this equation is inadequate is clear from two aspects of our solution. Firstly, it does not make a smooth transition from linear to non-linear scales and secondly it does not provide us with a value for σ_c. Nevertheless the solution suggests a very simple ansatz for the evolution of the density contrast: It may be possible to take the true density contrast, $\sigma_{NL}(a, x)$, to be a universal function of $\sigma_L[a, f(x, \sigma_{NL})]$ where $f(x, \sigma_{NL}) \simeq x$ for $\sigma_{NL} \ll 1$ and $f(x, \sigma_{NL}) \simeq x\sigma_{NL}^{2/3}$ for $\sigma_{NL} \gg 1$. [In fact, a relation with this structure has recently been suggested by Hamilton et al. (1991) and used to obtain the input linear spectrum from the observed non-linear structure observed at the current cosmological epoch. Our emphasis in this paper is on exploring the physical origin, dynamical derivation and implications of such a scaling relation.] In such a more realistic theory, we expect f to vary smoothly from one limit to the other. It turns out that such a scaling relation can indeed to be obtained from the equations governing the evolution of density perturbations. This is task which we shall now take up.

4. Non-linear dynamics - a better approximation

We begin by defining the correlation function $\xi(x, t)$ as the fourier transform of the power spectrum:

$$\xi(\mathrm{x}, t) = \int \frac{d^3\mathrm{k}}{(2\pi)^3} P(k, t) e^{i\mathrm{k}.\mathrm{x}} = \int_0^\infty \frac{dk}{k} \Delta^2(k, t) \left(\frac{\sin kx}{kx} \right) \tag{18}$$

To an excellent approximation, $\xi(x, t)$ and $\sigma(x, t)$ are related to each other by

$$\sigma^2(x, t) \cong \frac{3}{x^3} \int_0^x \xi(y, t) y^2 dy \tag{19}$$

We shall now obtain an *exact* equation satisfied by $\sigma(a, x)$ by an argument which follows the treatment of (Peebles 1980) for the evolution of the correlation function ξ. Since the mean number of neighbours to any particle is given by

$$N(x, t) = (na^3) \int_0^x 4\pi y^2 dy [1 + \xi(y, t)] \tag{20}$$

when n is the comoving number density, the conservation law for pairs implies

$$\frac{\partial \xi}{\partial t} + \frac{1}{ax^2} \frac{\partial}{\partial x} [x^2 (1 + \xi) v] = 0 \tag{21}$$

where v denotes the mean relative velocity of pairs at scale x and epoch t. Using (19) , we find

$$(1+\xi) = \frac{1}{3x^2}\frac{\partial}{\partial x}[x^3(1+\sigma^2)] \tag{22}$$

Substituting this in (21) , we get

$$\frac{1}{3x^2}\frac{\partial}{\partial x}[x^3\frac{\partial}{\partial t}(1+\sigma^2)] = -\frac{1}{ax^2}\frac{\partial}{\partial x}\left[\frac{v}{3}\frac{\partial}{\partial x}[x^2(1+\sigma^2)]\right] \tag{23}$$

Integrating, we find:

$$x^3\frac{\partial}{\partial t}(1+\sigma^2) = -\frac{v}{a}\frac{\partial}{\partial x}[x^3(1+\sigma^2)] \tag{24}$$

The integration would allow the addition of an arbitrary function of t on the right hand side. We have set this function to zero so as to reproduce the correct limiting behaviour (see below). It is now convenient to change the variables from t to a, thereby getting an equation for σ^2:

$$a\frac{\partial}{\partial a}[1+\sigma^2(a,x)] = \left(\frac{v}{-\dot{a}x}\right)\frac{1}{x^2}\frac{\partial}{\partial x}[x^3(1+\sigma^2(a,x))] \tag{25}$$

This equation shows that the behaviour of $\sigma^2(a,x)$ is essentially decided by the dimensionless ratio $h(a,x) \equiv [v(a,x)/(-\dot{a}x)]$ between the mean relative velocity v and the Hubble velocity $\dot{a}x = (\dot{a}/a)x_{\mathbf{prop}}$. To understand the behaviour of this equation let us consider its solutions in two limiting cases. In the non-linear limit, peculiar motion exactly compensates for the Hubble expansion to form bound structure; hence $(v/-\dot{a}x) = 1$ giving,

$$a\frac{\partial}{\partial a}(1+\sigma^2) - x\frac{\partial}{\partial x}(1+\sigma^2) = 3(1+\sigma^2) \tag{26}$$

This equation has the general solution $(1+\sigma^2) = a^3F(ax)$. In the linear limit, we know that $\sigma^2 = Aa^2x^{-(n+3)} \ll 1$, if the power spectrum is $P(k) \propto k^n$. Ignoring σ^2 compared to unity on the right hand side of (25), we get

$$a\frac{\partial}{\partial a}[Aa^2x^{-(n+3)}] \cong 3\left(\frac{v}{-\dot{a}x}\right) = 2Aa^2x^{-(n+3)} \tag{27}$$

Therefore,

$$\left(\frac{v}{-\dot{a}x}\right) = (2/3)Aa^2x^{-(n+3)} = (2/3)\sigma^2 \propto [t^{-4/3(n+3)}x]^{-(n+3)} \propto s^{-(n+3)} \tag{28}$$

where $s = (x/t^\alpha)$ and $\alpha = [4/3(n+3)]$. The solution in the non-linear limit, $(1+\sigma^2) = a^3F(ax)$ will be a function of s alone only if $F(ax) = (ax)^\gamma$ with $\gamma = 6/(3\alpha+2)$; in that case $a^{3-\gamma}x^{-\gamma} \propto s^{-6/(3\alpha+2)}$. This recovers the 'standard results' and justifies the boundary conditions chosen earlier. Also note that, in the linear limit, $(v/-\dot{a}x) = (2/3)\sigma^2$. This result in the linear regime is true for any spectrum, not for just a power law (Peebles 1980).

It is now clear that any general relation between σ_{NL} and σ_L will translate itself into a similar relation for h in the two limits. Conversely, the behaviour of $h(a,x)$ will allow us to determine σ_{NL} in terms of σ_L. The results obtained above suggest the hypothesis that the quantity $h(a,x) = [v(x,a)/-\dot{a}x]$ is purely a function of σ^2.

$$h(a,x) = h[\sigma^2(x)] \tag{29}$$

In the linear limit, when $\sigma^2 \ll 1$, we saw that $h(a,x) \cong (2/3)\sigma^2$; in the extreme non-linear limit ($\sigma^2 \gg 1$) bound structures would have formed in which v will balance out Hubble expansion; that is, $v = -\dot{a}x$ implying $h \cong 1$. It seems reasonable to explore the assumption that h depends on a and x only through some universal function $h = h(\sigma^2)$ which has the asymptotic behaviour derived above. Of course, this assumption has to be ultimately checked by comparing the results with the numerical simulations [A word of caution on our notation is in order. We are using the same symbol h to denote two functional forms $h(a,x)$ and $h(\sigma^2)$. A more pedandic notation will be $h(a,x) = H(\sigma^2)$, introducing another symbol H. We will not do this since it should be clear from the context what we mean by h].

When $h(a,x) = h[\sigma\ (a,x)]$, it is possible to find a useful solution to (25). We are interested in the solution to this equation which reduces to the form $\sigma^2 \propto a^2 x^{-(n+3)}$ for $\sigma^2 \ll 1$. This can be obtained as follows: We rewrite the equation in the form

$$a\frac{\partial D}{\partial a} - h(D)x\frac{\partial D}{\partial x} = 3hD; \tag{30}$$

where $D = (1+\sigma^2)$. Suppose we can find a function $F(D)$ such that

$$F(D) = \alpha \ln a + \beta \ln x \tag{31}$$

with α, β, being constants. If this is true, then

$$F' dD = \frac{\alpha}{a}da + \frac{\beta}{x}dx \tag{32}$$

giving $a(\partial D/\partial a) = (\alpha/F')$ and $x(\partial D/\partial x) = (\beta/F')$. Substituting this into (30), we find that $F(D)$ must satisfy the relation

$$F'(D) = \frac{\alpha - \beta h}{3hD} \tag{33}$$

Or, equivalently,

$$F = \int \left(\frac{\alpha - \beta h}{3hD}\right) dD = \int \frac{(\alpha - \beta h)}{3h}\frac{d\sigma^2}{(1+\sigma^2)} \tag{34}$$

The solution $\sigma^2 = \sigma^2(a,x)$ is obtained by combining (31) and (34)We can easily find that

$$\int \frac{d\sigma^2}{h(\sigma^2)(1+\sigma^2)} = \ln\left[a^3\left\{x^3(1+\sigma^2)\right\}^m\right] \tag{35}$$

where $m \equiv (\beta/\alpha)$ is a constant. When $h(\sigma^2) \simeq (2/3)\sigma^2$ and $\sigma^2 \ll 1$, we can ignore σ^2 compared to $\ln \sigma^2$ and this relation gives the linear theory result: $\sigma_L^2 \propto a^2 x^{2m}$ allowing us to relate m to the index n of the power spectrum. Since we expect $\sigma_L^2 \propto a^2 x^{-(n+3)}$ if $P(k) \propto k^n$, we get $m = -(n+3)/2$. This is the solution with the correct boundary condition which we need. It is also easy to verify that when $h \simeq 1$, $\sigma^2 \gg 1$, we get

$$\sigma^2 \propto (a^3 x^{3m})^{1/(1-m)} \propto a^3 (ax)^{3m/(1-m)} \propto a^3 (ax)^{-\gamma} \tag{36}$$

where $\gamma = 3m/(m-1) = 3(n+3)/(n+5)$. Thus the limiting forms derived earlier arise from the limiting behaviour of $h(\sigma^2)$.

The solution we have found also exhibits the universality we noticed in the earlier approximations. To see this, note that the right hand side of (35) is a function of $\sigma_L^2(a, l = x(1+\sigma^2)^{1/3}) \propto a^2 x^{2m} \, (1+\sigma^2)^{2m/3}$, which is the density contrast $\sigma_L(a,l)$ in the linear theory *evaluated at* the scale $l = x(1+\sigma^2)^{1/3}$. Thus we can write

$$\sigma_L^2[a,l] = \exp \frac{2}{3} \int \frac{d\sigma^2}{h(\sigma^2)(1+\sigma^2)} \tag{37}$$

Since the original power law index has disappeared, this relation suggests that $\sigma_L^2(a,l)$ is a universal function of the correct $\sigma^2(a,x)$. This is precisely the kind of ansatz suggested by our earlier approximation. The function $f(x,\sigma) = x(1+\sigma^2)^{1/3}$ has the limiting form, $f \simeq x\sigma^{2/3}$ for $\sigma \gg 1, and f \simeq x$ for $\sigma \ll 1$.

The above relation between σ_L^2 and σ^2 is indeed the correct solution to equation (30) even when σ_L^2 is not a power law. This can be seen by direct substitution of the solution (37) into (30) or – more formally – by the follwoing argument. Let $A = \ln a, X = \ln x$ and $D(X,A)$ be the solution to (30) . We define curves in the (X,A) such that

$$\frac{dX}{dA}\Big|_c = -h(D[x,A]) \tag{38}$$

i.e. the tangent to the curve at any point (X,A) has the value determined by h at that point. Quite clearly, along the curve, the left hand side of (30) is a total derivative allowing us to write

$$\left(\frac{\partial D}{\partial A} - h(D)\frac{\partial D}{\partial X}\right)_c = \left(\frac{\partial D}{\partial A} + \frac{\partial D}{\partial X}\frac{dX}{dA}\right)_c = \frac{dD}{dA}\Big|_c = 3hD \tag{39}$$

Integrating

$$\exp \frac{1}{3} \int_c \frac{dD}{Dh(D)} = \exp A = a \tag{40}$$

Or, Equivalently,

$$\exp \frac{2}{3} \int_c \frac{d\sigma^2}{h(\sigma^2)(1+\sigma^2)} = a^2 \propto \sigma_L^2\big|_l \tag{41}$$

We not only need to determine the form of the curves to fix the scale l. The equation to the curve can now be written as

$$\frac{dX}{dA} = -h = -\frac{1}{3D}\frac{dD}{dA} \tag{42}$$

giving

$$3X + \ln D = \ln\left[x^3(1+\sigma^2)\right] = \text{constant} \tag{43}$$

This shows that σ_L^2 should be evaluated at fixed $l = x(1+\sigma^2)^{1/3}$, giving us the result of (37).

The actual relation between σ_{NL} and σ_L depends on the forms of $h(\sigma^2)$. From the limiting behaviour discussed before, we know that $h(\sigma^2) \simeq (2/3)\sigma^2$ for small σ and $h \simeq 1$ for $\sigma \gg 1$. Our results for the nonlinear density contrast depends crucially on the manner in which $h(\sigma)$ reaches unity. We will expect an overdense region to expand more slowly compared to background, reach a maximum radius, collapse and (vivialise) to form a bound structure. During the collapse phase the average velocity will (in general) overshoot the Hubble expansion velocity. Only after varialization is complete will the pair correlation velocity approach the asymptotic value of $(-\dot{a}x)$. In fact numerical simulations show that $h(\sigma^2)$ has a single maximum (and hence overshoots $|v| = \dot{a}x$ before falling back) at about $\sigma^2 \simeq (8-15)$ with $h_{max} \simeq (1.5-2)$. The simplest model for such a function with correct asymptotic behaviour will be:

$$h(\sigma^2) = \frac{2}{3}\sigma^2 \frac{(1+\lambda\sigma^2)}{(1+(2/3)\lambda\sigma^4)} \tag{44}$$

which has just one free parameter λ. Substituting this in (37)and integrating, we find that

$$\sigma_L^2(a,l) = \sigma^2(1+\lambda\sigma^2)^{(3\lambda+2)/3(1-\lambda)}(1+\sigma^2)^{-(3+2\lambda)/3(1-\lambda)} \tag{45}$$

where $\sigma^2 = \sigma^2(a,x)$. Best agreement with the numerical results is achieved for $\lambda \simeq 0.36$ for which

$$\sigma_L^2(a,l) = \sigma^2(1+0.36\sigma^2)^{1.604}(1+\sigma^2)^{-1.937}; l = x(1+\sigma^2)^{1/3} \tag{46}$$

This result gives the true $\sigma^2(a,x)$ as a function of the linear density contrast $\sigma_L^2(a,l)$ evaluated at $l = x(1+\sigma^2)^{1/3}$.

This result does not suffer from the limitations of the earlier approximations. It gives the exact density contrast σ^2 in terms of the density contrast of the linear theory σ_L^2 by a function which makes a smooth transition from $\sigma^2 \ll 1$ regime to $\sigma^2 \gg 1$ regime. The non locality of behaviour seen in our earlier approximations is also preserved in a smooth manner.

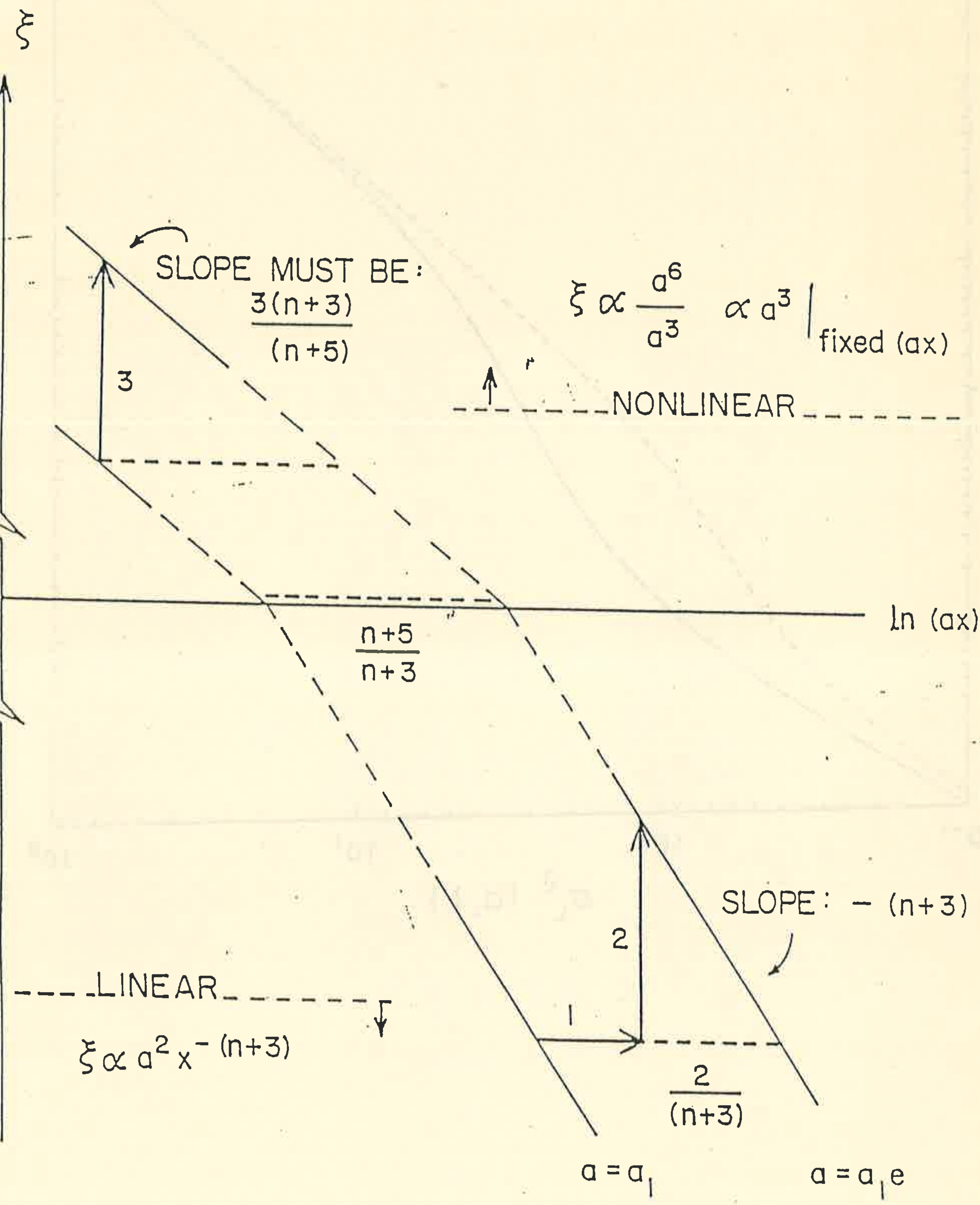
The ultimate test of any such formula, of course, is based on the agreement with numerical simulations. It turns out that our formula does remarkably well on this count. To begin with, the results of the numerical simulations with scale invariant power spectra does show the kind of universality obtained above: The correct density contrast $\sigma(a,x)$ is indeed expressible as a universal function of the linear density contrast $\sigma_L(a,l)$ where $l = x(1+\sigma^2)^{1/3}$. The best fitting function to the numerical data is the multiparameter fit.

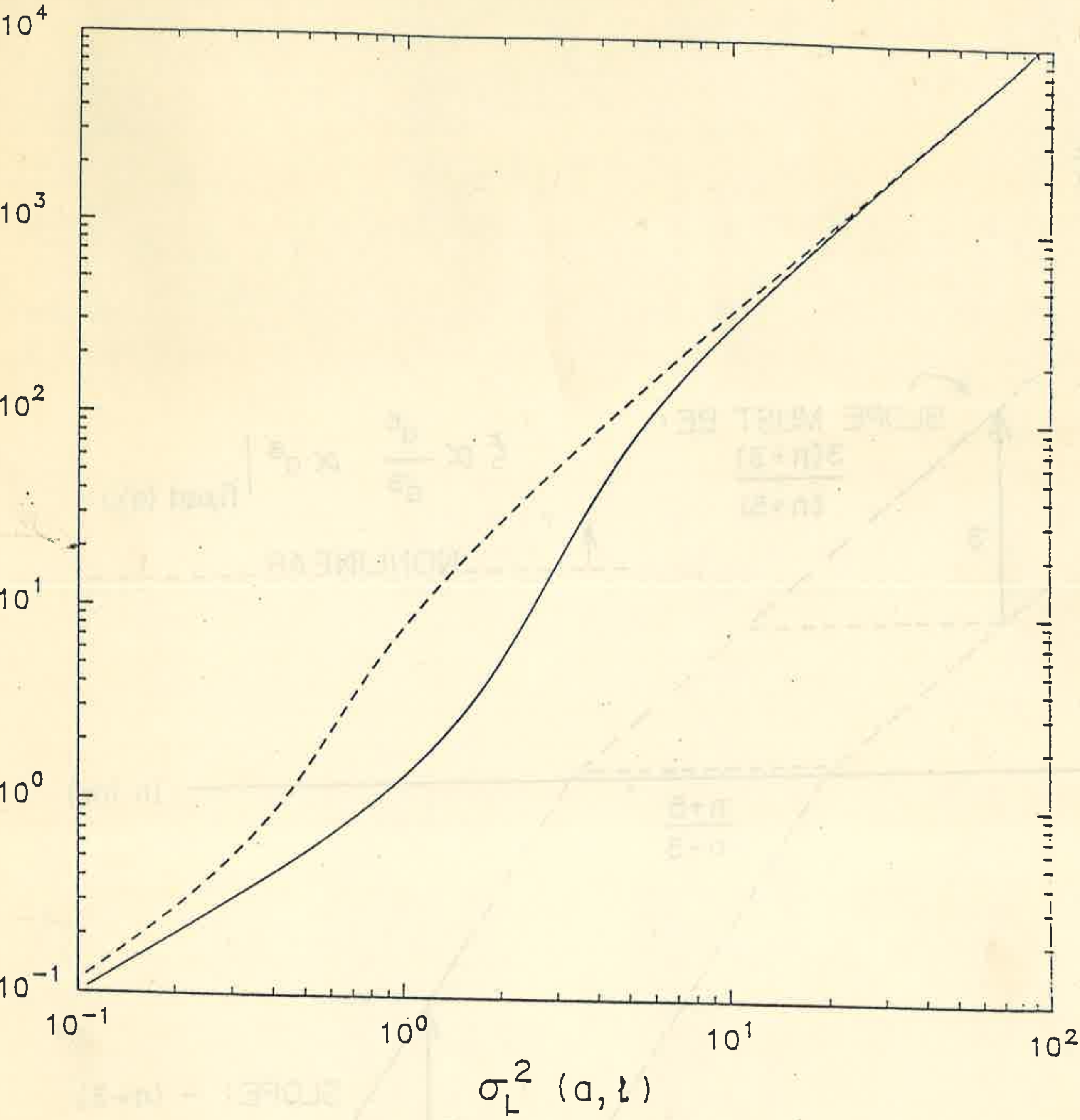
$$\sigma_L^2 = \sigma^2\left[\frac{1+0.0158\sigma^4+0.000115\sigma^2}{1+0.926\sigma^4-0.0743\sigma^6+0.0156\sigma^8}\right]^{1/3} \tag{47}$$

Figure Captions

Fig. 1 A schematic diagram showing the geometrical origin of Peebles scaling rules discussed in the text. The correlation function is plotted against the proper length in a logarithmic scale. The two curves indicate the function $\xi(ax)$ at two different epochs $a = a_1$ and at $a = a_1 e$. At large scales, in which the evaluation is linear, the correlation function changes through two processes. The scale is stretched due to expansion and the linear density evolves as $\delta \propto a$. At very small scales, where the evaluation is non-linear, the length scale does not change due to expansion, but the correlation function should increase as the cube of the expansion factor . Elementary geometry now shows that if the slope of the line at the linear end is $[-(n+3)]$ then the slope at the non-linear end must be $\left[\frac{3(n+3)}{(n+5)}\right]$. For a more detailed discussion see text.

Fig. 2 The true density contrast σ^2 is plotted against the linear density contrast. Note that the two contrasts are calculated at two different scales as discussed in the text. The unbroken curve is an exact fit to the numerical simulations while the broken curve gives the result of this paper. There is fair agreement between the analytic approximation and numerical simulations for density contrasts of the order of 10 and above.


ξ
SLOPE MUST BE:
3(n+3) / (n+5)
3
ξ ∝ a^6 / a^3 ∝ a^3 | fixed (ax)
NONLINEAR
ln (ax)
n+5 / n+3
SLOPE: − (n+3)
2
1
LINEAR
ξ ∝ a^2 x^−(n+3)
2 / (n+3)
a = a_1
a = a_1 e



10^4
10^3
10^2
10^1
10^0
10^{-1}
10^{-1}
10^0
10^1
10^2
$\sigma_L^2\ (a, l)$