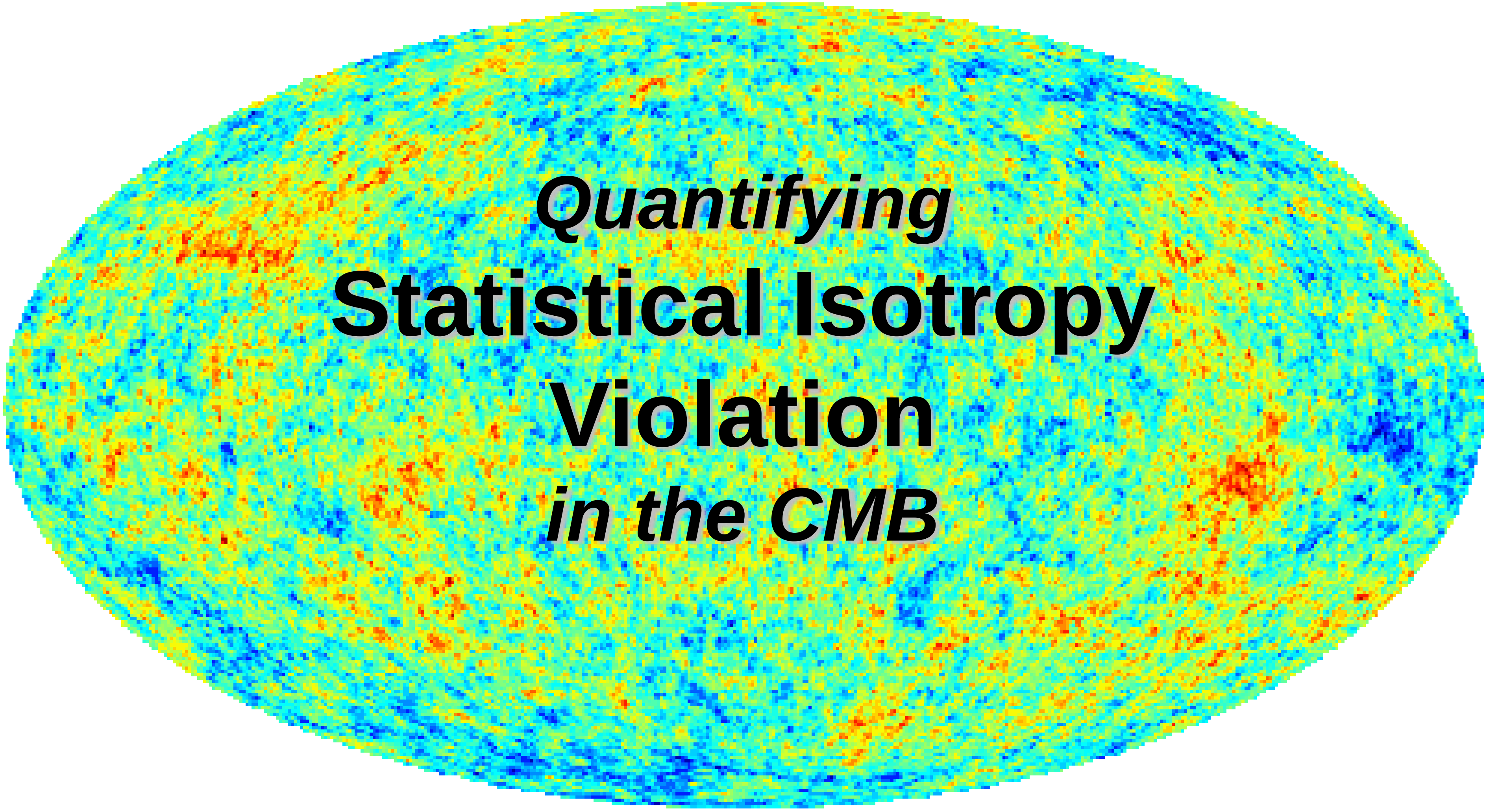


Quantifying Statistical Isotropy Violation *in the CMB*

Indo-UK Seminar

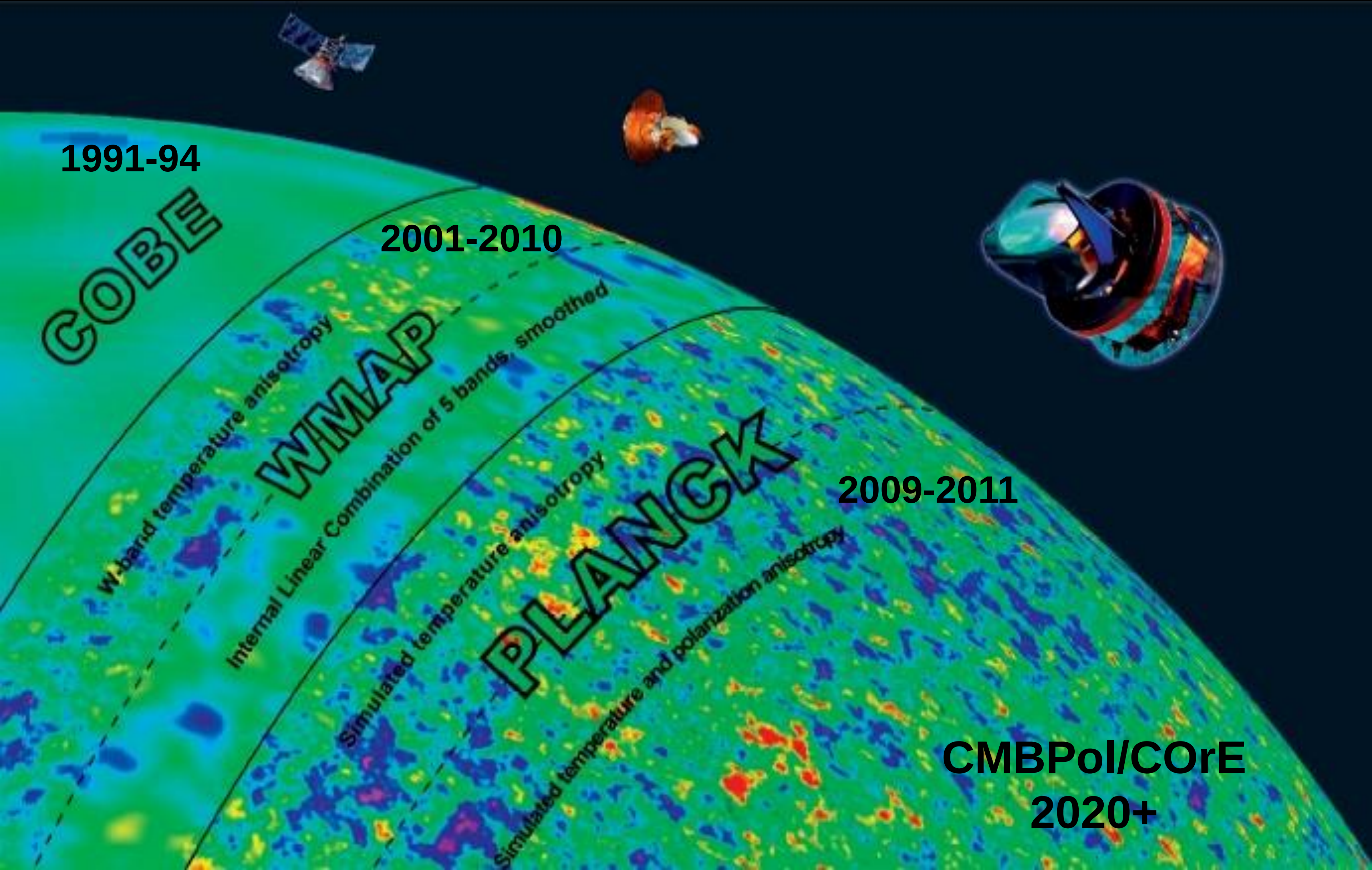
IUCAA, Pune

(Aug. 10, 2011)

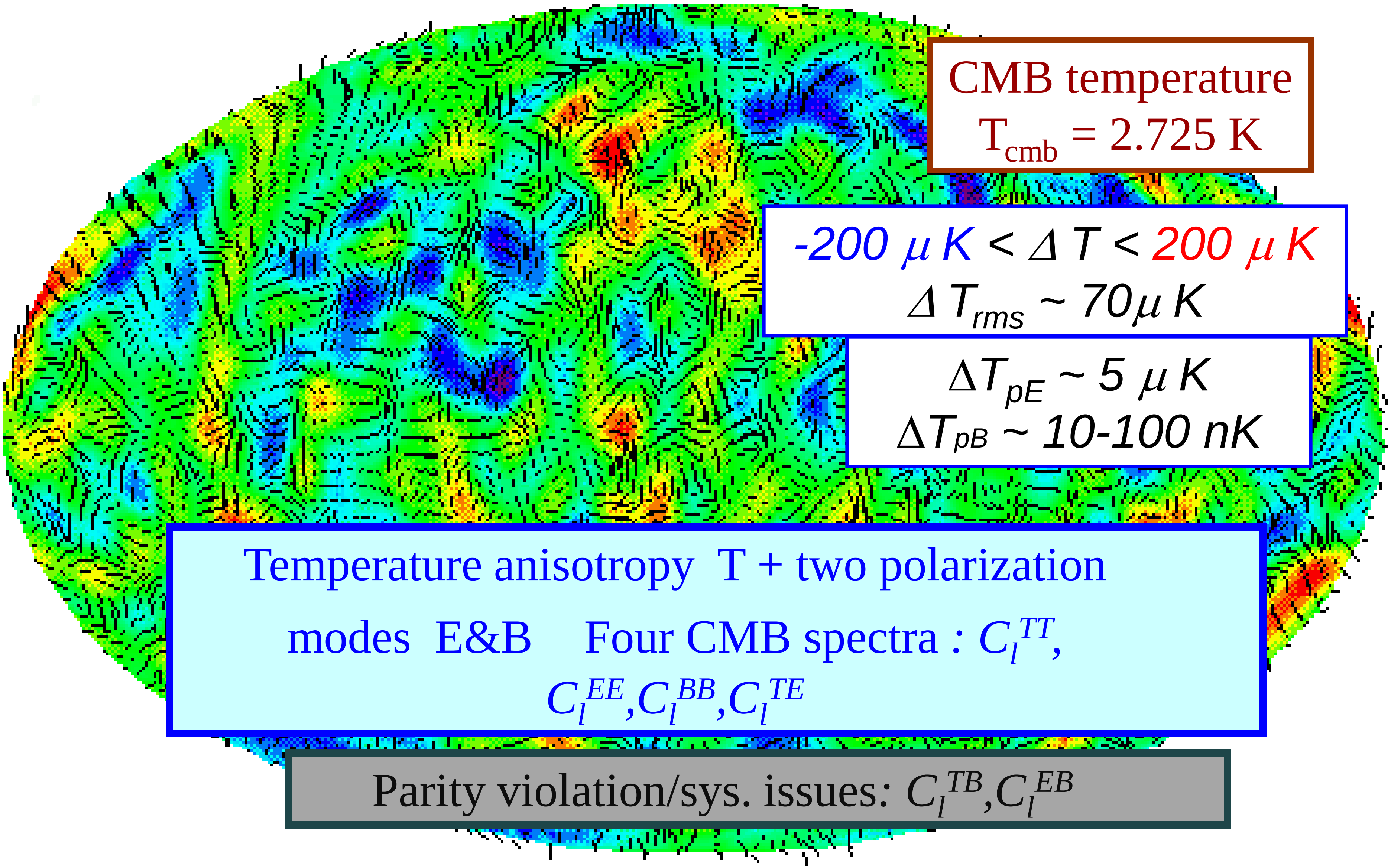
Tarun Souradeep

I.U.C.A.A, Pune, India

CMB space missions



CMB Anisotropy & Polarization



CMB temperature
$T_{cmb} = 2.725$ K

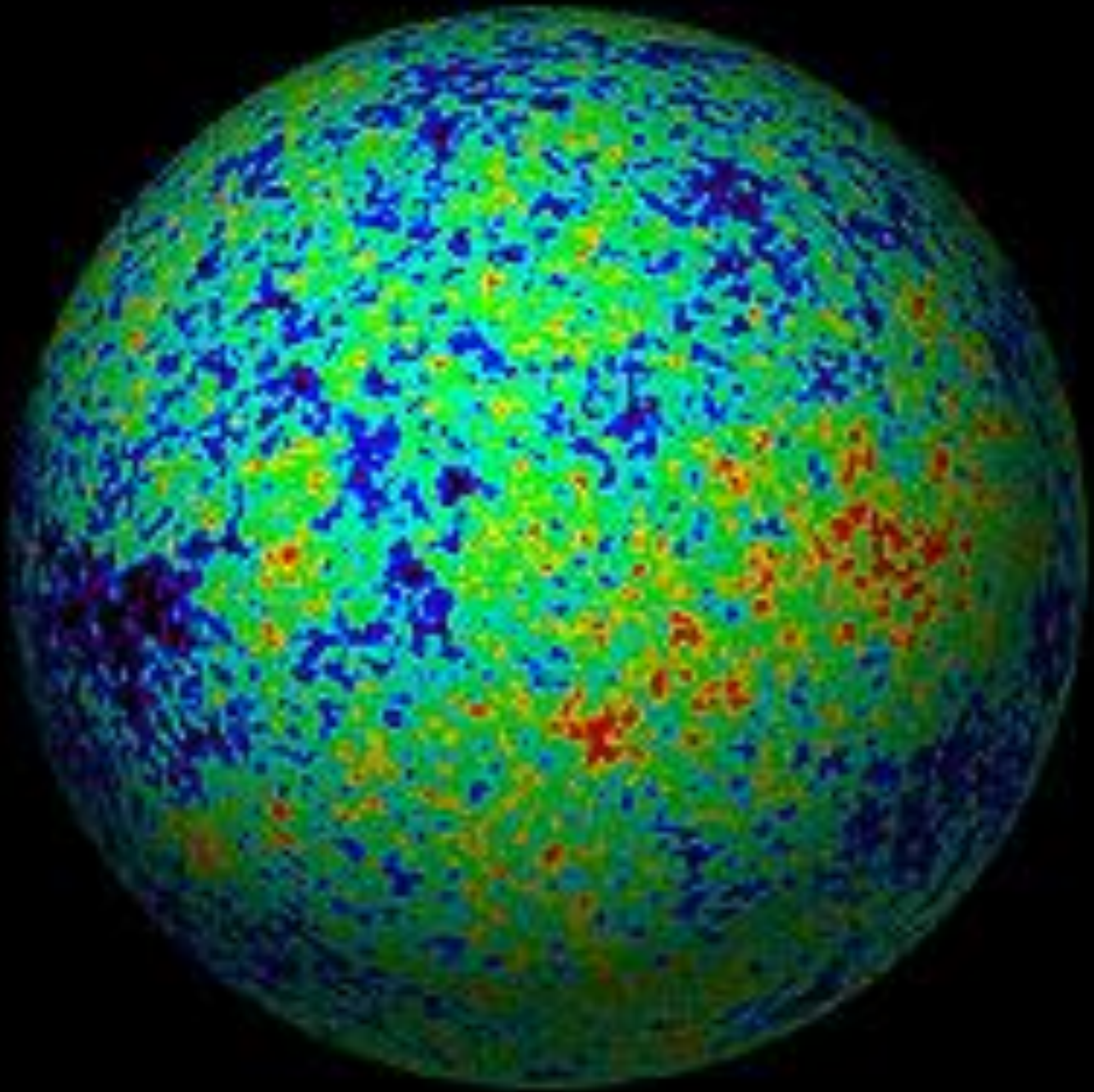
$-200\ \mu K < \Delta T < 200\ \mu K$
$\Delta T_{rms} \sim 70 \mu K$

$\Delta T_{pE} \sim 5\ \mu K$
$\Delta T_{pB} \sim$ 10-100 nK

Temperature anisotropy T + two polarization modes E&B Four CMB spectra : C_l^{TT}, C_l^{EE},C_l^{BB},C_l^{TE}

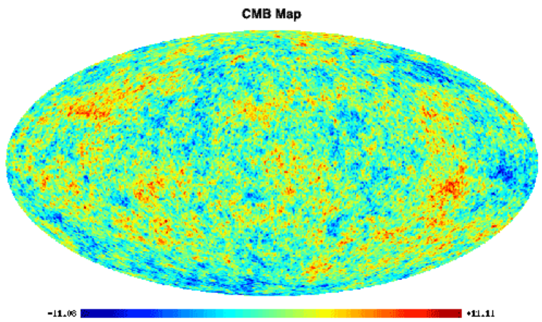
Parity violation/sys. issues: C_l^{TB},C_l^{EB}

Cosmic “Super–IMAX” theater



Statistics of CMB

CMB Anisotropy Sky map => Spherical Harmonic decomposition



$$\Delta T(\theta,\phi) = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta,\phi)$$

Gaussian CMB anisotropy completely specified by the

angular power spectrum IF

Statistical isotropy

$$\langle a_{lm} a^{*}_{l'm'} \rangle = C_l \ \delta_{ll'} \delta_{mm'}$$

=> Correlation function C(n,n’) is rotationally invariant

Current Angular power spectrum

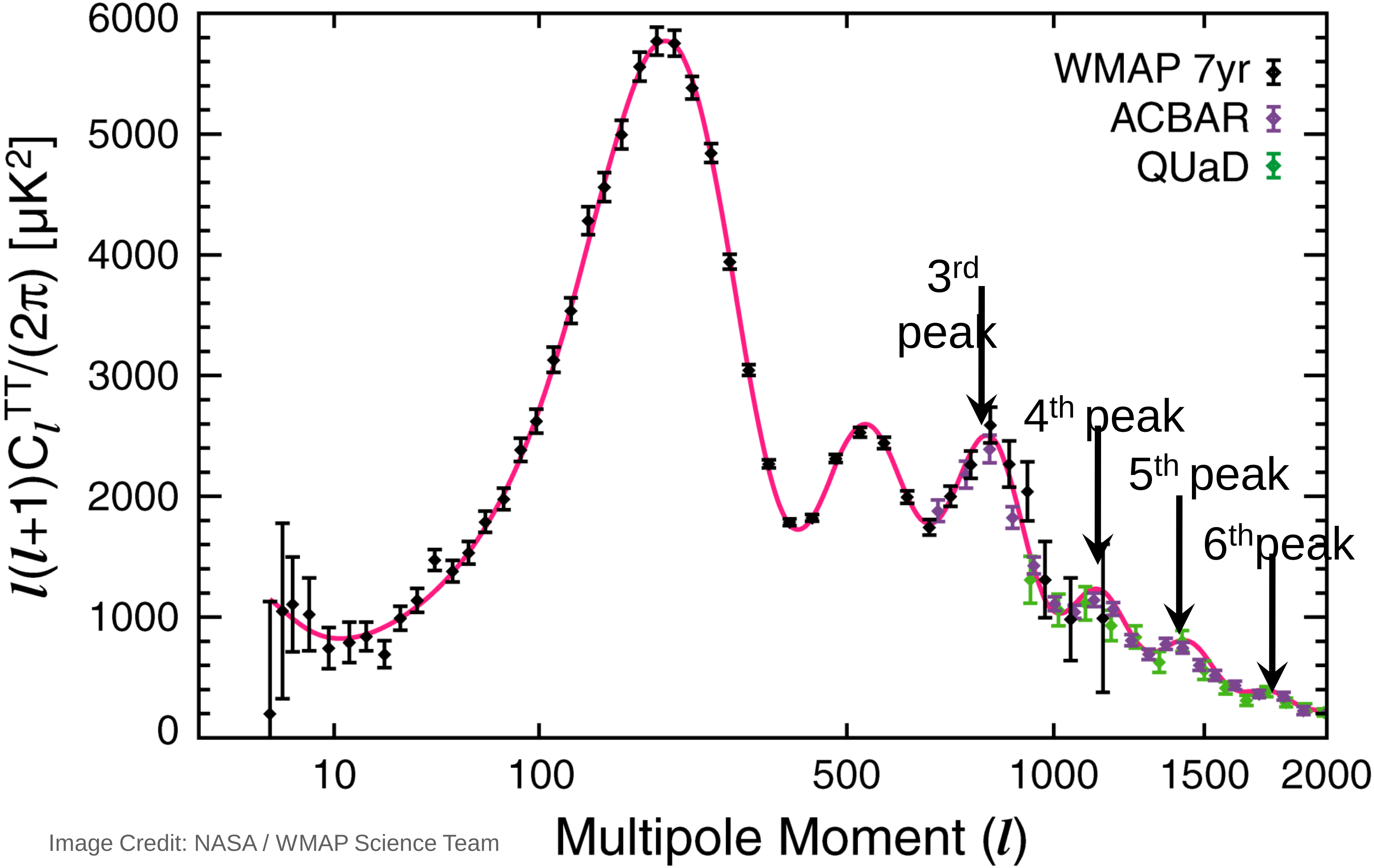


Image Credit: NASA / WMAP Science Team

Good old Cosmology, … New trend !

$\Omega_{tot} = 1.02^{+0.02}_{-0.02}$

$w < -0.78$ (95% CL)

$\Omega_\Lambda = 0.73^{+0.04}_{-0.04}$

$\Omega_b h^2 = 0.0224^{+0.0009}_{-0.0009}$

$\Omega_b = 0.044^{+0.004}_{-0.004}$

$n_b = 2.5 \times 10$ … $^{+0.1\times10^{-7}}_{-0.1\times10^{-7}}$ cm^{-3}

… $031^{+0.016}_{-0.018}$

… CL)

$\Delta z_{dec} = 195^{+2}_{-2}$

… % CL)

Total energy density

Dark energy density

Baryonic matter densi…

'Standard' cosmological model:

Flat, ΛCDM (with nearly Power Law primordial pow…)

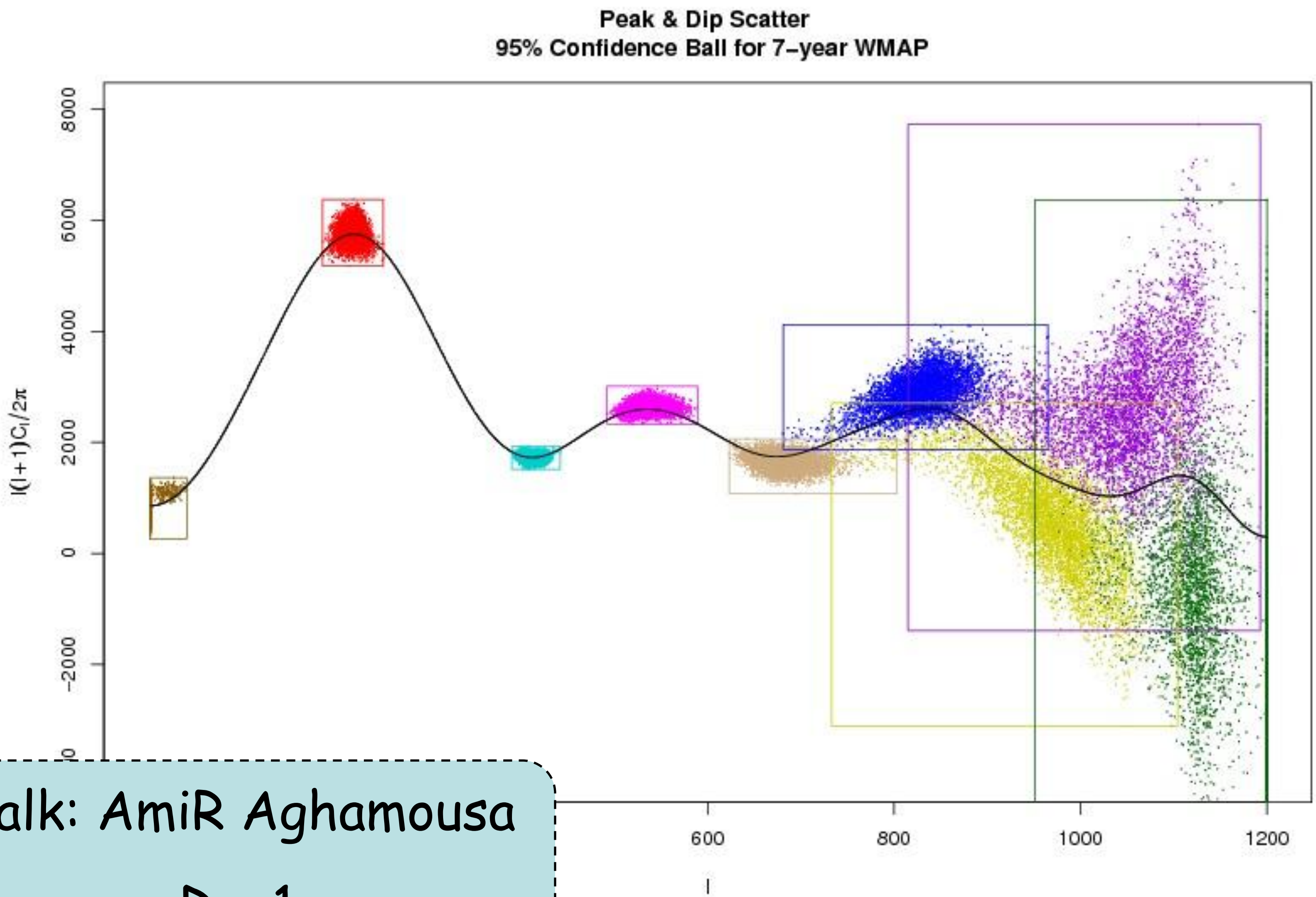
Talk: Dhiraj Hazra

PPS with features

NASA/WMAP scie…

Non-Parametric: Peak Location

(Amir Aghamousa, Mihir Arjunwadkar, TS *arXiv:1107.0516)*



Talk: AmiR Aghamousa

Day1

Beyond C_l :

Detecting patterns in CMB

Universe on Ultra-Large scales:

- Global topology
- Global anisotropy/rotation
- Breakdown of global syms, Magnetic field,…

Deflection fields

Observational artifacts:

- Foreground residuals
- Inhomogeneous noise, coverage
- Non-circular beams (eg., Hanson et al. 2010)

Statistics of CMB

$$C(\hat{n}_1, \hat{n}_2) \neq C(\hat{n}_1 \bullet \hat{n}_2)$$

Possibilities:

- Statistically Isotropic, Gaussian models
- Statistically Isotropic, ***non***-Gaussian models
- Statistically ***An***-isotropic, Gaussian models
- Statistically ***An***-isotropic, ***non***-Gaussian models

Ferreira & Magueijo 1997,
Bunn & Scott 2000,
Bond, Pogosyan & TS 1998, 2000

Iso-contours of correlation around a point

$$f(\hat{n}) \equiv C(\hat{n}, \hat{z})$$

Radical breakdown of SI

disjoint iso-contours

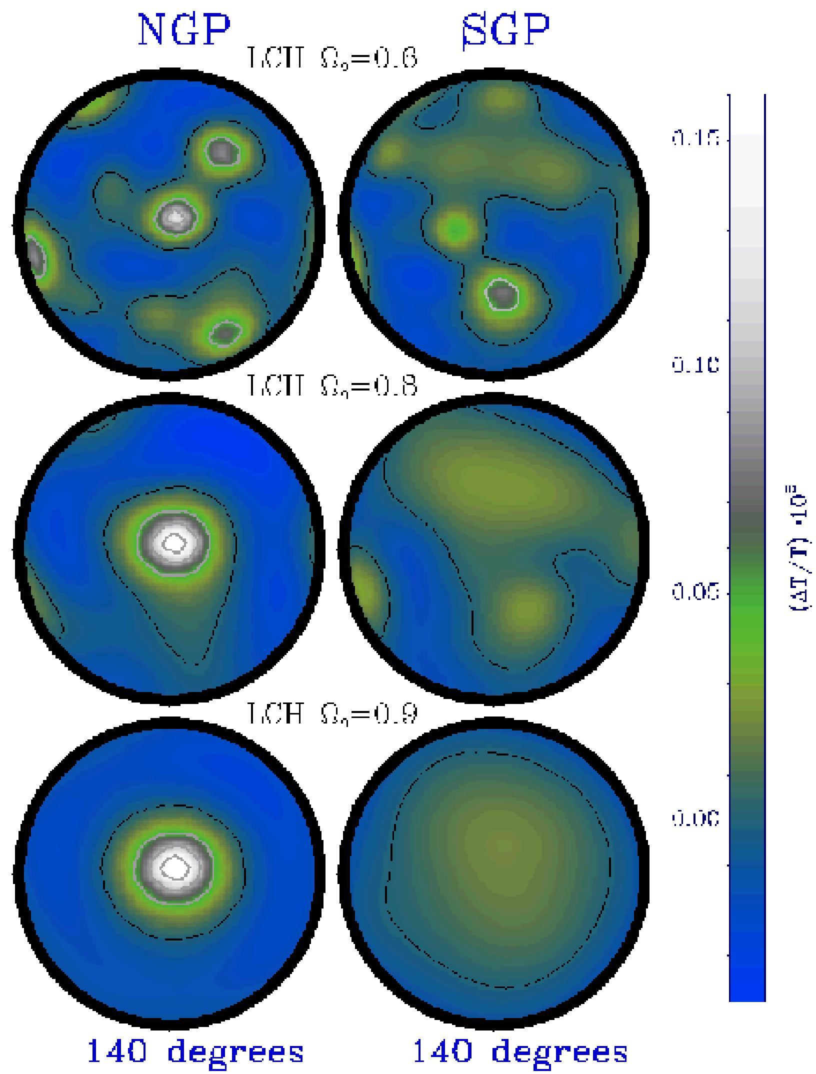
multiple imaging

Mild breakdown of SI

Distorted iso-contours

Statistically isotropic (SI)

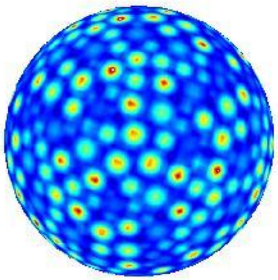
Circular iso-contours



E.g.. Compact hyperbolic Universe .

(Bond, Pogosyan & Souradeep 1998, 2002)

SI violation, or ... Correlation patterns



Beautiful Correlation patterns could underlie the CMB tapestry

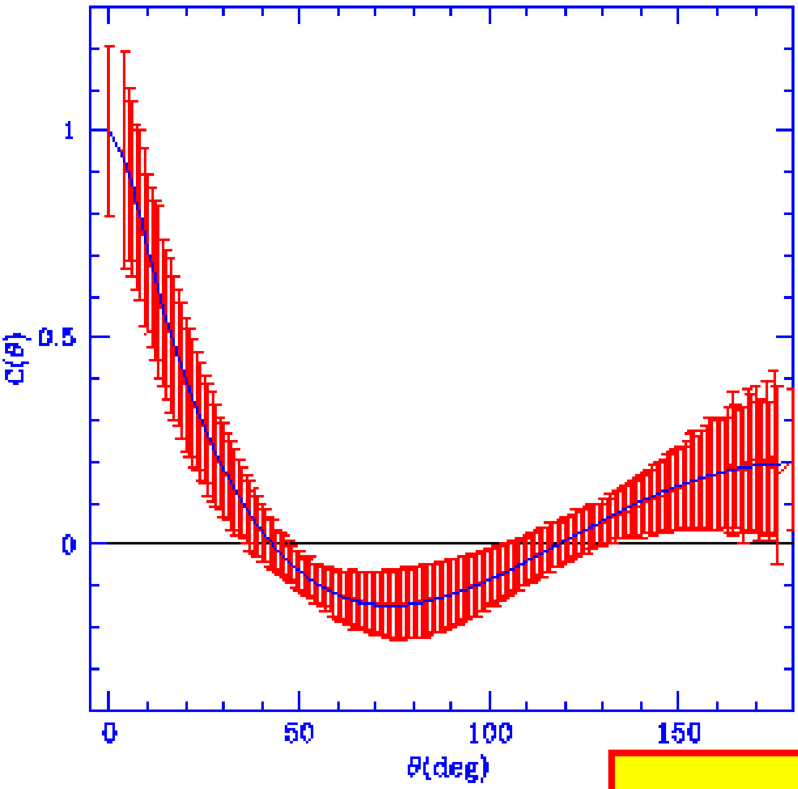
Figs. J. Levin

Can we measure correlation patterns?

the *COSMIC CATCH* is

there is only one CMB sky !

Measuring the SI correlation



Statistical isotropy

$C(\theta)$ can be well estimated by averaging over the temperature product between all pixel pairs separated by an angle θ.

$$\tilde{C}(\theta) = \sum_{\hat{n}_1}\sum_{\hat{n}_2} \Delta T(\hat{n}_1)\Delta T(\hat{n}_2)\delta(\hat{n}_1 \cdot \hat{n}_2 - \cos\theta)$$

$$C(\hat{n}_1 \bullet \hat{n}_2) = \frac{1}{8\pi^2}\int d\Re \;\; C(\Re\hat{n}_1, \Re\hat{n}_2)$$

Measuring the non-SI correlation

In the absence of statistical isotropy

Estimate of the correlation function from a sky map given by a single temperature product $\tilde{C}(\hat{n}_1, \hat{n}_2) = \Delta T(\hat{n}_1)\Delta T(\hat{n}_2)$

is poorly determined!!

(unless it is a KNOWN pattern)

- **Matched circles statistics** **(Cornish, Starkman, Spergel '98)**
- **Anticorrelated ISW circle centers** **(Bond, Pogosyan,TS '98,'02)**
- **Planar reflective symmetries** **(de OliveiraCosta, Smoot Starobinsky '96)**

Bipolar Power spectrum (BiPS) :

A Generic Measure of Statistical Anisotropy

$$\text{Recall}: C(\hat{n}_1 \bullet \hat{n}_2) = \frac{1}{8\pi^2}\int d\Re \;\; C(\Re\hat{n}_1, \Re\hat{n}_2)$$

Bipolar multipole index

$$\kappa^{\ell} = \int d\Omega_{n_1} \int d\Omega_{n_2} \left[\frac{1}{8\pi^2}\int d\Re \;\; \chi^{\ell}(\Re) \, C(\Re\hat{n}_1, \Re\hat{n}_2) \right]^2$$

A **weighted average** of the correlation function over all rotations

$$\chi^{\ell}(\Re) = \sum_{m=-\ell}^{\ell} D^{\ell}_{mm}(\Re)$$

Characteristic function

Wigner rotation matrix

Statistical Isotropy $\Rightarrow \kappa^{\ell} = \kappa^{0}\delta_{\ell 0}$

Correlation is invariant under rotations

$$C(\Re\hat{n}_1, \Re\hat{n}_2) = C(\hat{n}_1, \hat{n}_2)$$

$$\kappa^{\ell} = (2\ell + 1)^2 \int d\Omega_{n_1} \int d\Omega_{n_2} C^2(\hat{n}_1, \hat{n}_2)[\frac{1}{8\pi^2} \int d\Re\chi^{\ell}(\Re)]^2$$

$$\int d\Re\chi^{\ell}(\Re) = \delta_{\ell 0}$$

Bipolar Power spectrum (BiPS) :

A Generic Measure of Statistical Anisotropy

- Correlation is a *two point function* on a sphere

$$C(\hat{n}_1,\hat{n}_2) = \sum_{l_1 l_2 LM} A^{LM}_{l_1 l_2} \{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{LM}$$

BiPoSH

Bipolar spherical harmonics.

$$C(n_1 \bullet n_2) = \sum \frac{2l+1}{4\pi} C_l P_l(n_1 \bullet n_2)$$

$$\{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{LM} = \sum_{m_1 m_2} C^{LM}_{l_1 l_2 m_1 m_2} Y_{l_1 m_1}(\hat{n}_1) Y_{l_2 m_2}(\hat{n}_2)$$

Clebsch-Gordan

- Inverse-transform

$$A^{LM}_{l_1 l_2} = \int d\Omega_{n_1} \int d\Omega_{n_2} C(\hat{n}_1,\hat{n}_2) \{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}^*_{LM}$$

$$= \sum_{m_1 m_2} \langle a_{l_1 m_1} a_{l_2 m_2} \rangle \; C^{LM}_{l_1 m_1 l_2 m_2}$$

Linear combination of off-diagonal elements

Recall: Coupling of angular momentum states

$\langle l_1 m_1 l_2 m_2 \mid \ell M \rangle \qquad |l_1 - \ell| \le l_2 \le l_1 + \ell, \quad m_1 + m_2 + M = 0$

BiPoSH coefficients :

$$A_{l_1 l_2}^{\ell M} = \sum_{m_1} \left\langle a_{l_1 m_1} a^{*}_{l_2\; M+m_1} \right\rangle \; C^{\ell M}_{l_1 m_1 l_2\; M+m_1}$$

- Complete,Independent linear combinations of off-diagonal correlations.
- Encompasses other specific measures of off-diagonal terms, such as
  - Durrer et al. '98 :
  - Prunet et al. '04 :

$$D_l \equiv \left\langle a_{lm} a_{l+2\; m} \right\rangle = \sum_{\ell M} A_{ll'}^{\ell M} C^{\ell M}_{l+2\; m\; l\; m}$$

$$D_l^{(i)} \equiv \left\langle a_{lm} a_{l+1\; m+i} \right\rangle = \sum_{\ell M} A_{ll'}^{\ell M} C^{\ell M}_{l+1\; m+i\; l\; m}$$

BiPS:

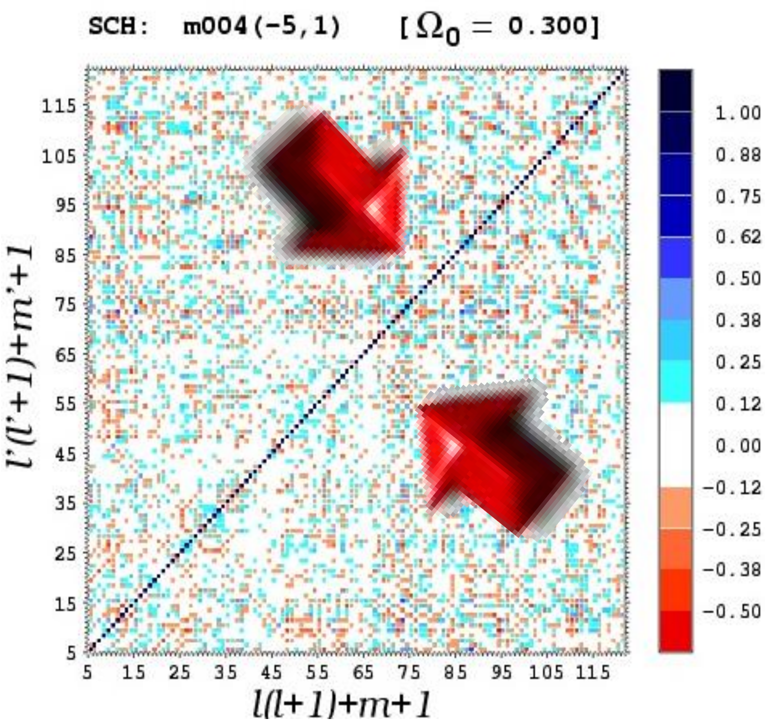
rotationally invariant

$$\kappa^{\ell} \equiv \sum_{M, l_1, l_2} | A_{l_1 l_2}^{\ell M} |^2 \ge 0$$

Understanding BiPoSH coefficients

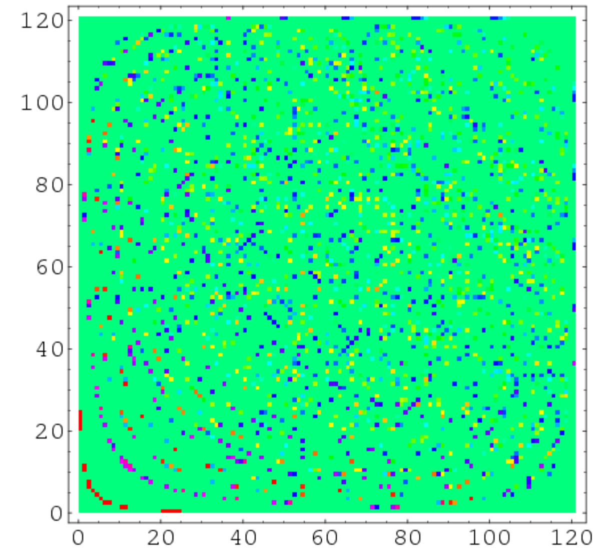
SI violation:

$$\langle a_{lm}\, a^{*}_{l'm'} \rangle \neq C_l \;\delta_{ll'}\delta_{mm'}$$

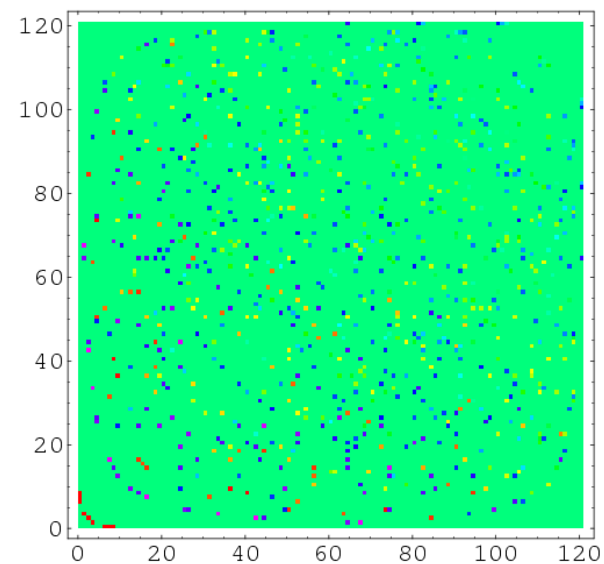


$$A^{LM}_{ll'} = \sum_{mm'} \langle a_{lm} a_{l'm'} \rangle \; C^{LM}_{lml'm'}$$

Measure cross correlation in a_{lm}



$A^{4M}_{ll'}$



$A^{2M}_{ll'}$

Spherical harmonics	Bipolar spherical harmonics
a_{lm}	$A^{\ell M}_{ll'}$
Spherical Harmonic coefficents	BiPoSH coefficents
C_l	κ^{ℓ}
Angular power spectrum	BiPS

Bipolar Power spectrum (BiPS) :

A Generic Measure of Statistical Anisotropy

Spherical harmonics	**Bipolar spherical harmonics**
a_{lm}	$A^{\ell M}_{ll'}$
Spherical Harmonic Transforms	BipoSH Transforms
C_l	κ^{ℓ}
Angular power spectrum	**BiPS**

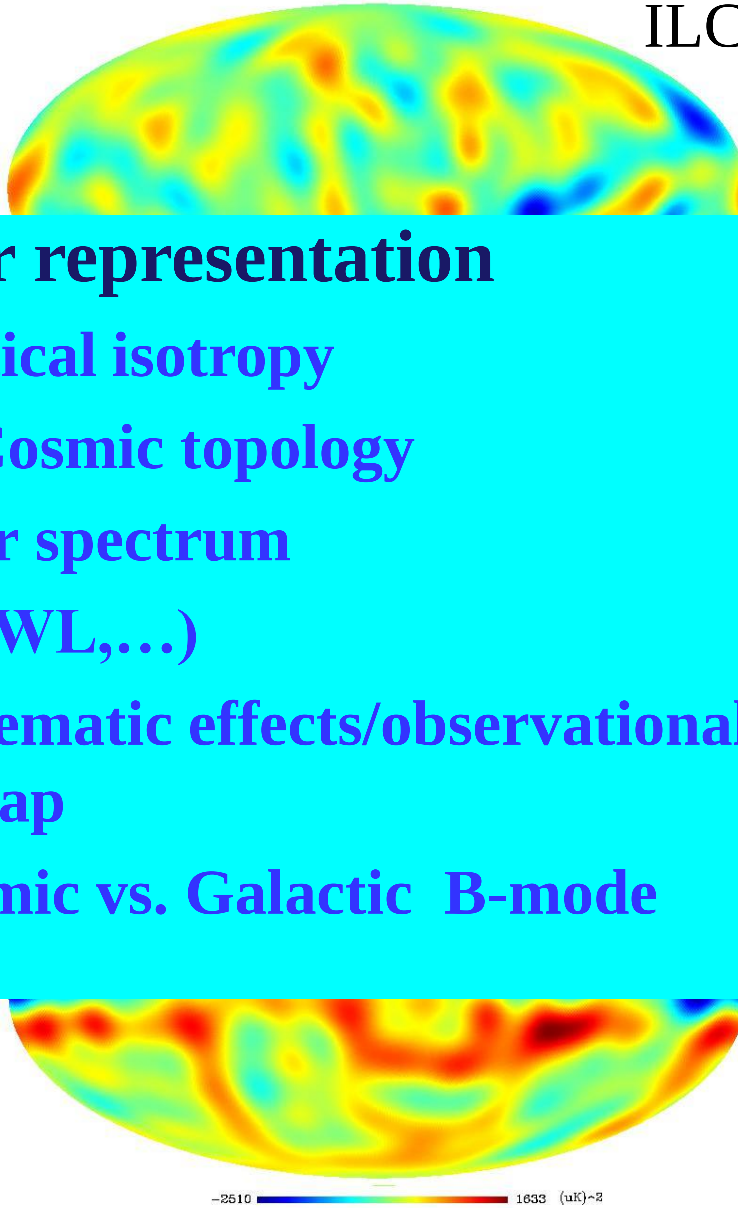
Statistical Isotropy
i.e., NO Patterns

$$\Rightarrow \kappa^{\ell} = \kappa^{0}\delta_{\ell 0}$$

BIPOLAR maps of WMAP

Hajian & Souradeep (PRD 2007)

ILC-3



Reduced BipoSH

$$A_{\ell M} = \sum_{ll'} A$$

Bipolar m

$$\theta(\hat{n}) = \sum_{\ell M}$$

Bipolar representation

- Measure of statistical isotropy
- Spectroscopy of Cosmic topology
- Anisotropic power spectrum
- Deflection fields (WL,…)
- Diagnostic of systematic effects/observational artifacts in the map
- Differentiate Cosmic vs. Galactic B-mode polarization

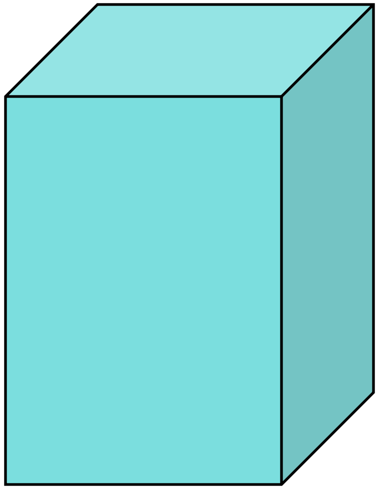
•SI part corresponds to the "monopole" of the map.

Preferred Directions in the universe

Correlation function on a Torus (periodic box)

$$C(q_1, q_2) = \sum_n P(k_n) \exp[-i \tfrac{2\pi R}{L}\vec{n} \cdot (\hat{q}_1 - \hat{q}_2)]$$

SI violation at low wave-number, *kR<1*



$$C(q_1, q_2) \approx C_0 [1 - \sum_i \varepsilon_i^{\,2} (\Delta q_i)^2]$$

$$\kappa_\ell = \kappa_0(\varepsilon_i)\ \delta_{\ell 0} + \kappa_2(\varepsilon_i)\ \delta_{\ell 2}$$

(Hajian & Souradeep 2003)

More generally,

$$P(\vec{k}) = P_0(k)[1 + g_{LM}(k) Y_{LM}(\hat{k})]$$

(Pullen & Kamionkowski 2008)

$$A_{ll'}^{LM} = C^{L0}_{\ell 0 \ell' 0} \int \frac{dk}{k} P_0(k)\, g_{LM}(k)\, \Delta_\ell(k, \tau_o) \Delta$$

Talk: Moumita Aich

BipoSH for Torus

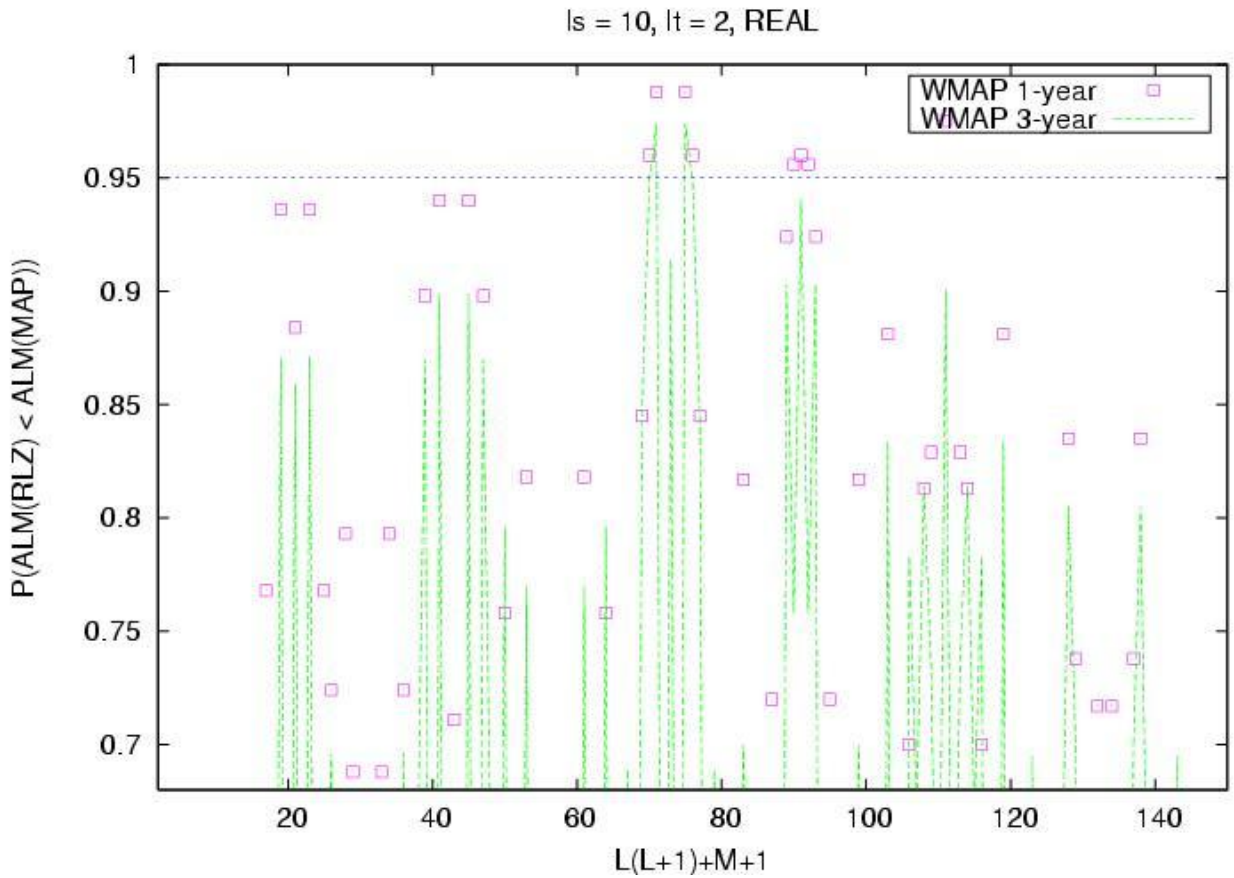
Testing Statistical Isotropy of WMAP-3yr

Hajian & Souradeep, (PRD 2007)

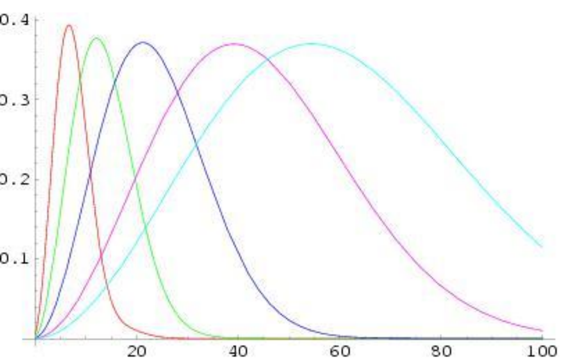
Retain orientation information: Reduced BipoSH

Outliers (> 95%)

$$A_{\ell M} = \sum_{ll'} A_{ll'}^{\ell M}$$



$$\ell(\ell+1)+M+1$$



Filters

BIPOLAR measurements by WMAP-7 team

(Bennet et al. 2010)

$$\frac{A^{(+)LM}_{l_1 l_2}}{C^{L0}_{l0l'0}}$$

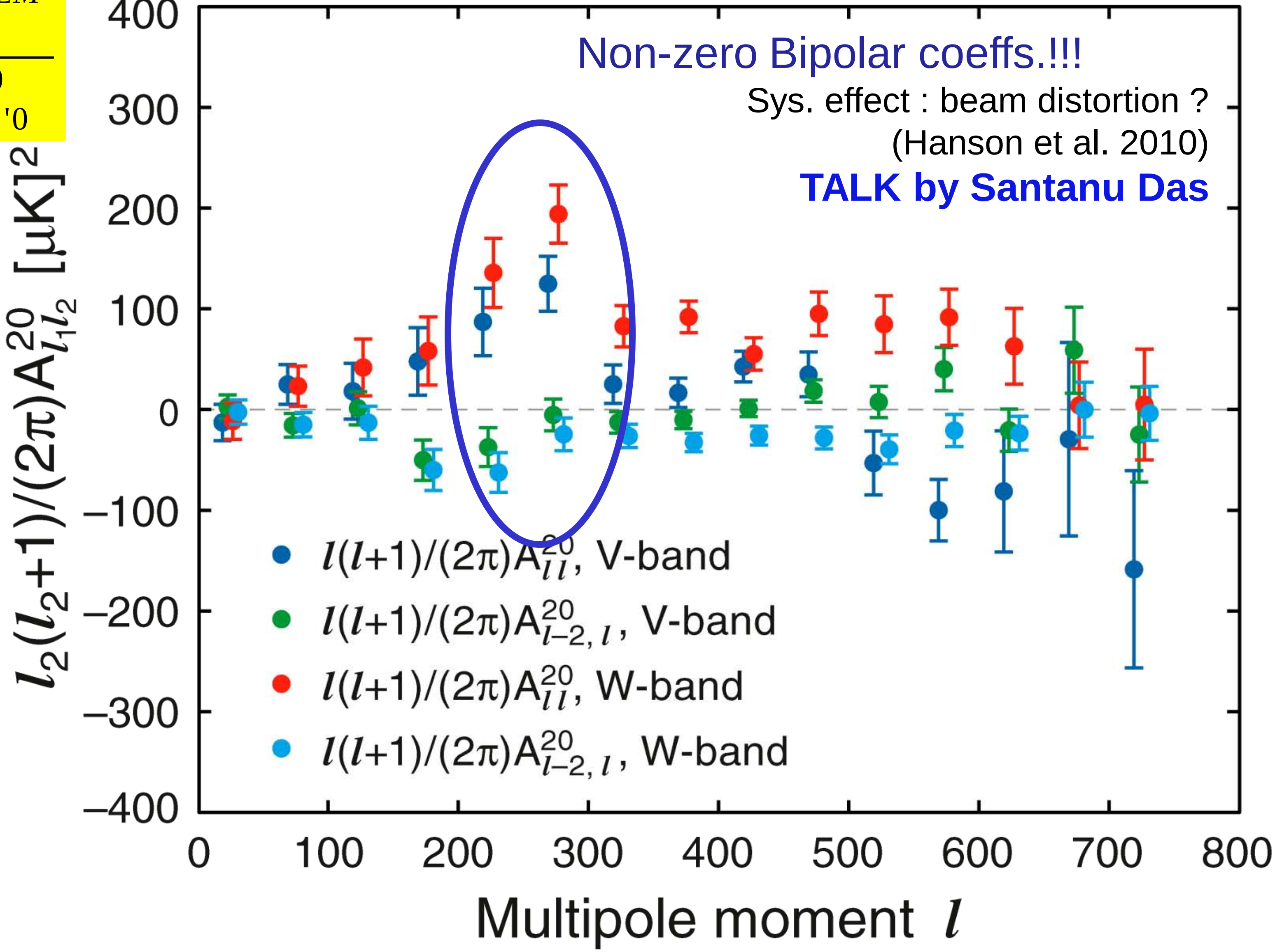
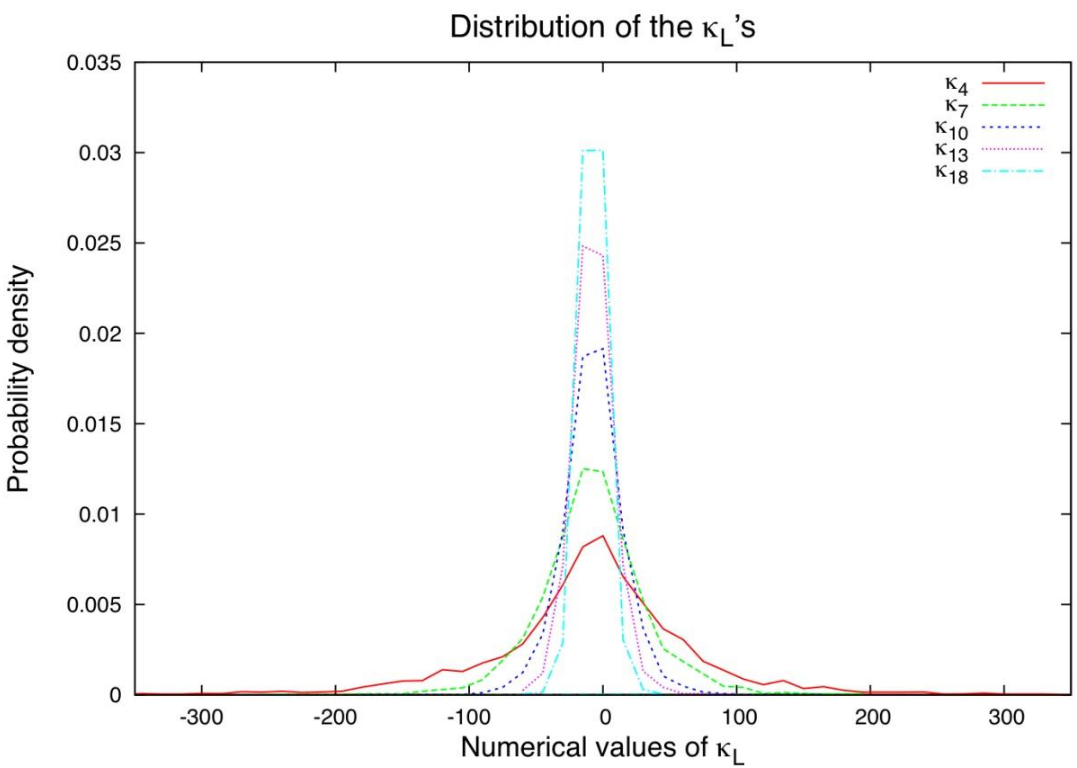


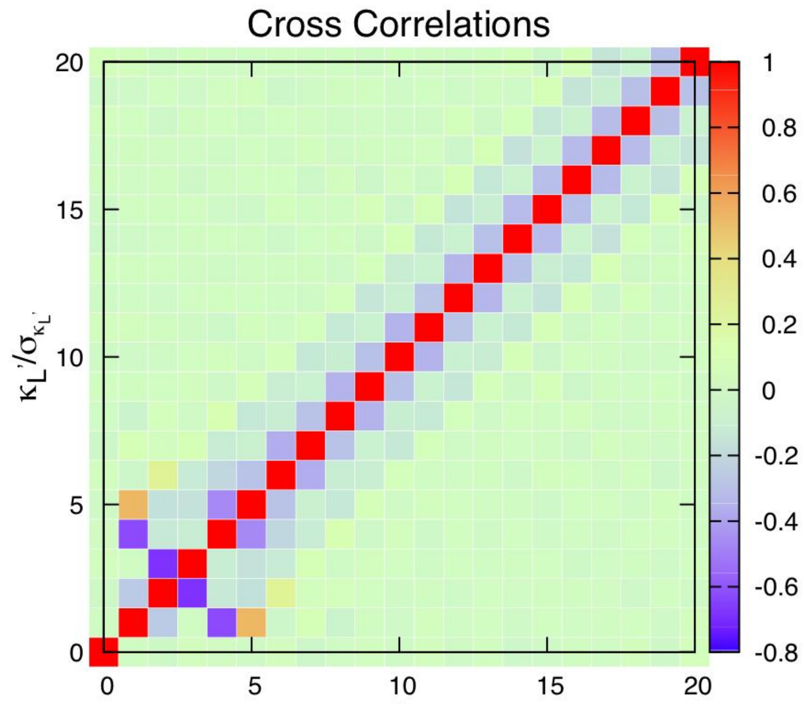
Image Credit: NASA / WMAP Science Team

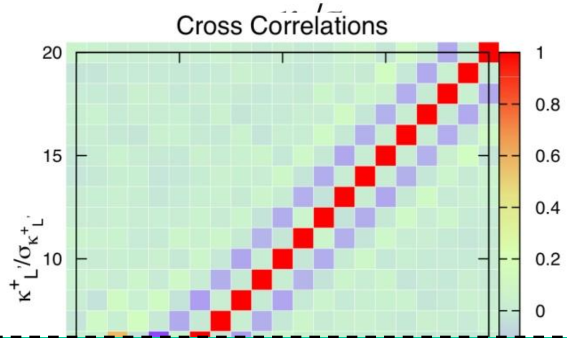
Revisting Statistics of BipoSH

(*Nidhi Joshi*, Aditya Rotti, *TS, 2011*)

- ~Analytic PDF for BipoSH
- Confirmed with MC simulations
- Faster BipoSH code (20x)
- Compute up to high multipoles







Talk: Nidhi Joshi

Day 2

Even & odd Bipolar coefficients

$$C(\hat{n}_1,\hat{n}_2) = \sum_{l_1 l_2 LM} A^{LM}_{l_1 l_2}\{Y_{l_1}(\hat{n}_1)\otimes Y_{l_2}(\hat{n}_2)\}_{LM}$$

$$\Rightarrow C(\hat{n}_1,\hat{n}_2) = \sum_{ll'LM} A^{(+)LM}_{ll'}\frac{[1+(-1)^{L+l+l'}]}{2}\{Y_{l_1}(\hat{n}_1)\otimes Y_{l_2}(\hat{n}_2)\}_{LM}$$

$$+\sum_{ll'LM} A^{(-)LM}_{ll'}\frac{[1-(-1)^{L+l+l'+1}]}{2}\{Y_{l_1}(\hat{n}_1)\otimes Y_{l_2}(\hat{n}_2)\}_{LM}$$

If only even $L+l'+l$ contribute, it is becoming popular to use

$$A^{(+)LM}_{l_1 l_2} \rightarrow \frac{A^{(+)LM}_{l_1 l_2}}{C^{L0}_{l0l'0}}\sqrt{\frac{2L+1}{(2l_1+1)(2l_2+1)}}$$

- Anisotropic P(k)
- Linear order templates

Even & odd parity BipoSH

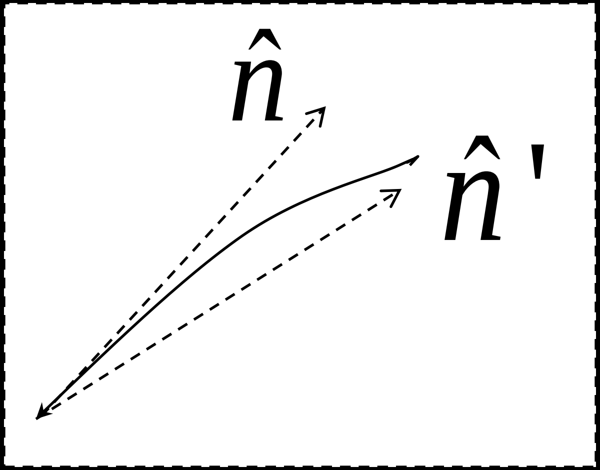
$$A_{l_2 l_1}^{(+)LM} = A_{l_1 l_2}^{(+)LM}$$ symmetric

$$A_{l_2 l_1}^{(-)LM} = -A_{l_1 l_2}^{(-)LM}$$ antisymm.

$$[A_{l_1 l_2}^{(+)LM}]^* = (-1)^M A_{l_1 l_2}^{(+)L,-M}$$ Even parity

$$[A_{l_1 l_2}^{(-)LM}]^* = (-1)^{M+1} A_{l_1 l_2}^{(-)L,-M}$$ Odd parity

SI violation : Deflection field



$$T(\hat{n}') = T(\hat{n} + \vec{\Theta}) = T(\hat{n}) + \vec{\Theta} \bullet \vec{\nabla} T(\hat{n})$$

$$\vec{\Theta} = \vec{\nabla} \phi(\hat{n}) + \vec{\nabla} \times \Omega(\hat{n})$$

$$= \nabla_i \phi(\hat{n}) + \varepsilon_{ij} \nabla_j \Omega(\hat{n})$$

Gradient	Curl
WL:scalar	WL: tensor/GW

SI violation : Deflection field

Brooks, Kamionkowski & Souradeep

$$T(\hat{n}) \rightarrow a_{lm} = a^S_{lm} + \delta a_{lm}, \quad \phi(\hat{n}), \Omega(\hat{n}) \rightarrow \phi_{LM}, \Omega_{LM}$$

$$\delta a_{lm} = \frac{1}{2} \sum_{LM} \sum_{l'm'} a^S_{l'm'} \left[\phi_{LM} E^L_{ll'} - i\Omega_{LM} O^L_{ll'} \right] G^L_{ll'} C^{LM}_{lml'm'}$$

$$G^L_{ll'} = [\ldots]\, C^{L1}_{l0l'-1} \qquad (= [\ldots.] C^{L0}_{l0l'0} \text{ for } l + l' + L : \textit{even})$$

$$\text{even: } E^L_{ll'} = \frac{1}{2}\left[1 + (-1)^{l+l'+L}\right] \text{ \& odd: } O^L_{ll'} = \frac{1}{2}\left[1 - (-1)^{l+l'+L}\right]$$

$$\text{SI violation: } \langle a_{lm} a_{l'm'} \rangle \neq C_l \delta_{ll'} \delta_{mm'}$$

Deflection field: Even & Odd parity BipoSH

Brooks, Kamionkowski & Souradeep 2011

$$A_{ll'}^{(+)LM} = \phi_{LM}\left[\frac{C_l G_{l'l}^L}{\sqrt{l'(l'+1)}} + \frac{C_{l'} G_{ll'}^L}{\sqrt{l(l+1)}}\right]$$

WL: scalar

$$A_{l_2 l_1}^{(-)LM} = i\Omega_{LM}\left[\frac{C_l G_{l'l}^L}{\sqrt{l'(l'+1)}} - \frac{C_{l'} G_{ll'}^L}{\sqrt{l(l+1)}}\right]$$

WL: tensor

BipoSH Measures of deflection field

Estimators

$$\tilde{\phi}_{LM} = \frac{\sum_{ll'} Q^{+}_{ll'} A^{(+)LM}_{ll'} / \sigma^{2\,LM}_{ll'}}{\sum_{ll'} (Q^{+}_{ll'})^2 / \sigma^{2\,LM}_{ll'}}$$

$$\tilde{\Omega}_{LM} = \frac{\sum_{ll'} Q^{-}_{ll'} A^{(-)LM}_{ll'} / \sigma^{2\,LM}_{ll'}}{\sum_{ll'} (Q^{-}_{ll'})^2 / \sigma^{2\,LM}_{ll'}}$$

Variance

$$\mathrm{var}\left(\tilde{\phi}_{LM}\right) = \left[\sum_{ll'} (Q^{+}_{ll'})^2 / \sigma^{2\,LM}_{ll'}\right]^{-1}$$

$$\mathrm{var}\left(\tilde{\Omega}_{LM}\right) = \left[\sum_{ll'} (Q^{-}_{ll'})^2 / \sigma^{2\,LM}_{ll'}\right]^{-1}$$

$$A^{(\pm)LM}_{l_1 l_2} \rightarrow \frac{A^{(\pm)LM}_{l_1 l_2}}{C^{L1}_{l0l'1}}[\ldots] = \frac{A^{(\pm)LM}_{l_1 l_2}}{G^{L}_{ll'}}$$

CMB BipoSHs & Bispectra

(Kamionkowski & Souradeep, PRD 2011)

For deflection field $$a_{lm} = a^{S}_{lm} + \delta a_{lm}$$

$$A^{LM}_{ll'} \square \ \phi_{LM} \sum_{mm'} \left\langle a^{s}_{lm} a^{s}_{l'm'} \right\rangle C^{LM}_{lml'm'}$$

$$\phi_{LM} \rightarrow a_{LM} \qquad \Rightarrow A^{LM}_{ll'} \square \sum_{mm'} \left\langle a_{LM} a_{lm} a_{l'm'} \right\rangle C^{LM}_{lmlm'}$$

BipoSH related to Bispectrum

$$B_{Lll'} \square \sum_{Mmm'} \left\langle a_{LM} a_{lm} a_{l'm'} \right\rangle C^{LM}_{lml'm'} \quad (\ldots)$$

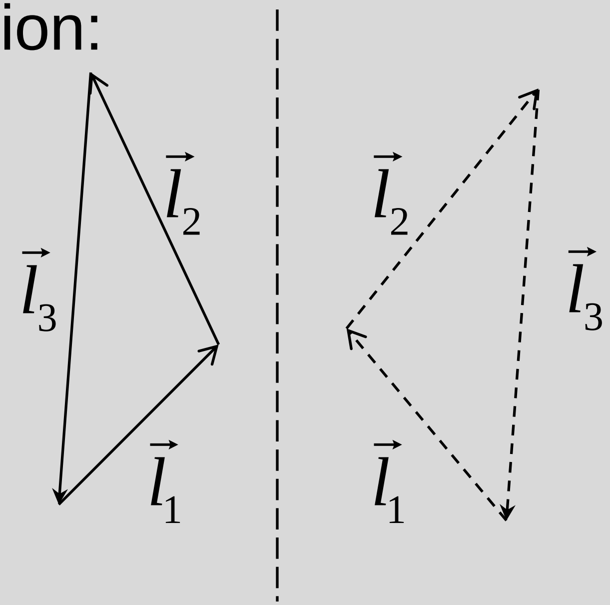
$$\square \quad \sum_{M} A^{(+)LM}_{ll'}$$

Consider only: $l + l' + L =$ even

Odd parity Bispectra ?

$$B^{(-)}_{Lll'} \propto \sum_{M} A^{(-)LM}_{ll'}$$

Flat sky intuition:



$$l_1 < l_2 < l_3$$

$$B^{(-)} \propto \frac{\vec{l}_1 \times \vec{l}_2}{l_1 l_2}$$

has opposite sign in the two mirror configurations.

Odd parity Bispectra

For local NG model

Flat sky approx

$$B(\vec{l}_1,\vec{l}_2) = 2\left[f_{\rm nl} + f_{\rm nl}^{\rm odd} \frac{\vec{l}_1 \times \vec{l}_2}{l_1 l_2} \right] (C_{l_1} C_{l_2} + \text{ perms.})$$

In general

$$\tilde{f}_{\rm nl} = \sigma_{f_{\rm nl}}^2 \sum_{l_1<l_2<l_3} 6\, G_{l_1 l_2}^{l_3} \frac{(C_{l_1} C_{l_2} + \text{ perms.})}{C_{l_1}C_{l_2} + C_{l_3}C_{l_2} + C_{l_1}C_{l_3}} E_{l_1 l_2}^{l_3}$$

$$\tilde{f}_{\rm nl}^{\rm odd} = \sigma_{f_{\rm nl}}^2 \sum_{l_1<l_2<l_3} 6\, G_{l_1 l_2}^{l_3} \frac{(C_{l_1} C_{l_2} + \text{ perms.})}{C_{l_1}C_{l_2} + C_{l_3}C_{l_2} + C_{l_1}C_{l_3}} O_{l_1 l_2}^{l_3}$$

$$\sigma_{f_{\rm nl}}^{-2} = \sum_{l_1<l_2<l_3} \frac{\left[6\, G_{l_1 l_2}^{l_3} (C_{l_1} C_{l_2} + \text{ perms.})\right]^2}{C_{l_1}C_{l_2} + C_{l_3}C_{l_2} + C_{l_1}C_{l_3}}$$

Summary

Thank you !!!

- Current observations now allow a meaningful search for deviations from the `standard' flat, ΛCDM cosmology.
- Anomalies in WMAP suggest possible breakdown down of statistical isotropy.

- **Bipolar harmonics provide a mathematically complete, well defined, representation of SI violation.**
  - Possible to include SI violation in CMB arising **both** from ***direction dependent Primordial Power Spectrum*** , as well as, ***SI violation in the CMB photon distribution function.***
  - ***BipoSH*** provide a well structured representation of the systematic breakdown of rotational symmetry.
  - Bipolar observables have been measured in the WMAP data.
  - Systematic effects are important and must be quantified similar to signal

- **BipoSH coefficients can be separated into even and odd parity parts.**
  - For a general deflection field, gradient & curl parts are represented by even & odd parity BipoSH, respectively. Eg., Weak lensing by scalar & tensor (or 2^{nd} order scalar) perturbations.
  - Estimators for grad/curl deflections field harmonics in terms of even/odd BipoSH
- **BipoSH** for correlated deflection field relate to **Bispectra**
  - ***Pointed to, hitherto unexplored, odd-parity bispectrum***.
  - Minor modification to existing estimation methods for even-parity bispectra
  - Odd parity bispectrum may arise in exotic parity violations, but, also an interesting null test for usual bispectrum analysis.

Statistical Isotropy: CMB Photon distribution

(Moumita Aich & TS, PRD 2010)

Statistical isotropy

$$\Delta_\ell(k,\tau_{rec}) \xrightarrow{\text{Free stream}} \Delta_\ell(k,\tau_0)$$

$$= \sum_{\ell' l} [\ldots]\, j_l(k\Delta\tau) \left[C^{l0}_{\ell 0 \ell' 0}\right]^2 \Delta_{\ell'}(k,\tau_{rec})$$

General: Non-Statistical isotropy

$$\Delta^{LM}_{\ell_3\ell_4}(k,\tau_{rec}) \xrightarrow{\text{Free stream}} \Delta^{LM}_{\ell_1\ell_2}(k,\tau_0)$$

$$= \sum_{\ell\ell_3\ell_4} [\ldots]\, j_l(k\Delta\tau)\, C^{L0}_{\ell 0 \ell_1 0} C^{L0}_{\ell 0 \ell_1 0} \begin{Bmatrix} \ell_4 & L & \ell_3 \\ \ell_1 & l & \ell_2 \end{Bmatrix}$$

$$\times\, \Delta^{LM}_{\ell_3\ell_4}(k,\tau_{rec})$$

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Non-Statistical isotropic terms also free-stream to large multipole values

$$\begin{Bmatrix} \ell_4 & L & \ell_3 \\ \ell_1 + s & l + s & \ell_2 + s \end{Bmatrix} \xrightarrow{\text{Large } s} \approx [\ldots] C^{\ell_3 \ (\ell_1 - l)}_{\ell_4 \ (\ell_2 - l) \ L \ (\ell_1 - \ell_2)}$$

$$\Delta^{LM}_{\ell_1 \ell_2}(k, \tau_0) \xrightarrow{l >> \ell_1 - l, \ell_2 - l, L} \sum_l [\ldots] \, j_l(k \Delta\tau) \, \Delta^{LM}_{\ell_1 - l, \ell_2 - l}(k, \tau_{rec})$$