

Evidence for Majorana Neutrinos: Dawn of a new era in spacetime structure

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Abstract. We show that Majorana particles belong to the Wigner class of fermions in which the charge conjugation and the parity operators commute, rather than anticommute. Rigorously speaking, Majorana spinors do not satisfy the Dirac equation [a result originally due to M. Kirchbach, which we re-render here]. Instead, they satisfy a different wave equation, which we derive. This allows us to reconcile Stückelberg-Feynman interpretation with the Majorana construct. We present several new properties of neutral particle spinors and argue that discovery of Majorana particles constitutes dawn of a new era in spacetime structure.

1. Introduction

The long sought-after signal from experiments on neutrinoless double beta decay has finally been reported by the Heidelberg-Moscow (HM) collaboration [1, 2]. In its most natural explanation the HM events suggest neutrinos to be fundamentally neutral particles in the sense of Majorana [3].

It is our intention to argue that the discovery a Majorana particle, taken to its logical implications, opens a new era in the structure of spacetime. It constitutes a discovery in which spacetime is not merely a classical object, a mere $SU(2)_R \otimes SU(2)_L$ realization of the Lorentz algebra (in the sense of Ryder [4]), but the underlying representation spaces exploit additional relative phases. These phases between the two $SU(2)$ building blocks encode in them important C, P, and T properties. Furthermore, Majorana particles belong to a new and unusual Wigner class – a class necessary for implementing supersymmetry. Even though neutrino itself may not be a supersymmetric particle, its Majorana nature tells us that spacetime does realize a construct that is central to construction of supersymmetric theories.

The Lorentz algebra, associated with the generators of rotation, \mathbf{J} , and boosts, \mathbf{K} , fails to incorporate fermionic fields. As is well known, see, e.g., Ref. [4], this circumstance is remedied by the introduction of two $SU(2)$ generators:

$$SU(2)_R : \quad \mathbf{A} = \frac{1}{2} (\mathbf{J} + i\mathbf{K}) , \quad (1)$$

$$SU(2)_L : \quad \mathbf{B} = \frac{1}{2} (\mathbf{J} - i\mathbf{K}) . \quad (2)$$

The resulting algebra is no longer that of the Lorentz group. In the notation of Ref. [4], the Weyl spinors belong to $(1/2, 0)$ and $(0, 1/2)$ representation spaces; and the parity covariant spin-1/2 constructs belong to the $(1/2, 0) \oplus (0, 1/2)$ representation space. The vector indexed objects, such as x^μ , A^μ , transform as $(1/2, 1/2)$, and so on. Therefore, while the Lorentz algebra has served us well, the underlying representation spaces for the quantum field

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theoretic description of nature belong, at least in the non-Planckian realm, to the two $SU(2)$'s introduced above. In reference [5] we have taken an *ab initio* look at the representation spaces which are appropriate to the description of spin $1/2$ charged particles, vector particles, and Rarita-Schwinger particles. Here, we present an *ab initio* formalism that embodies the original spirit of Majorana but extends/completes it in a non-trivial way. In doing so we build upon, but do not confine, to already existing original literature [3, 6, 7] and exploit our experience in the spacetime structure of massive particles to benefit us [5, 8, 9, 10, 11].

Based upon our studies in Refs. [5, 11] we take as given that wave equation for spinors underlying the description of spin one half charged particles carries a symmetry under the operation of $(1/2, 0) \oplus (0, 1/2)$ -representation-space charge conjugation operator:

$$C = \begin{pmatrix} 0_2 & i\Theta \\ -i\Theta & 0_2 \end{pmatrix} K. \quad (3)$$

Here, operator K complex conjugates any object that appears on its right, and Θ is the Wigner's spin- $1/2$ time reversal operator[§]

$$\Theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (4)$$

We further assume that the C operator, and related P , and T operators, are intrinsic to the $(1/2, 0) \oplus (0, 1/2)$ representation space up to certain phases which may be fixed by additional physical requirements. The mentioned phases may affect commutative and anticommutative properties of the C , P , and T as these properties depend not only on the form of the operators but also on these phases and the spinors on which the operators act upon.

The operator C appeared on the physics scene not in expectation but as a surprise that lay hidden in spacetime symmetries and it revealed itself in the now famous Dirac construct for spin one half. The well-known symmetry associated with this operator brought into existence prediction of an entirely new type matter, called antimatter. Yet, the particles associated with matter, and those associated with antimatter, are not eigenstates of this very operator. Instead, this operator takes particle spinors into antiparticle spinors and vice versa. The notion readily extends to a fully quantum field theoretic framework.

Beyond Dirac, Stückelberg in 1942 and Feynman in 1948 proposed to interpret antiparticles as particles scattered backward in time [13, 14, 15]. This latter proposal, as noted by Hatfield [16], carries the advantage that it applies equally well to fermions as to bosons. However, already in 1937 Majorana identified particle creation operators with antiparticle creation operators. In that quantum-field-theoretic proposal, Majorana did not consistently alter the relevant representation space since he still used Dirac's $u_h(\mathbf{p})$ and $v_h(\mathbf{p})$ spinors. Moreover, given the time sequence of ideas, he could not foresee the impact of Stückelberg-Feynman interpretation for his proposal. The former deficiency was remedied, though only partly, by 1957 papers of McLennan and Case [6, 7], and by the 1996 work of Ref. [9]. Here we hope to attend to all these question, and in the process bring to attention additional structure in the theory of neutral particles.

At this stage of the *paper* it is, therefore, to be concluded that as far as neutral particles are concerned the existing state of theory is unsatisfactory. It calls for an *ab initio* construction based on the eigenspinors of the C operator. After the reader has examined our presentation, it is our contention that she/he will find it absurd, though "workable," to describe charged particles in terms of the neutral-particle framework we present. We carry

[§] For an arbitrary spin it is defined by the property $\Theta \mathbf{J} \Theta^{-1} = -\mathbf{J}^*$. We refrain from identifying Θ with " $-i\sigma_2$," as is done implicitly in all considerations on the subject – see, e.g., Ref. [12] – because such an identification does not exist for higher-spin $(j, 0) \oplus (0, j)$ representation spaces. The existence of Wigner time reversal operator for all j , allows, for fermionic j 's, the introduction of $(j, 0) \oplus (0, j)$ neutral particles.

$(1/2, 0) \oplus (0, 1/2)$ $\{C, P\} = 0$ FERMIONIC MATTER FIELDS	$(1/2, 0) \oplus (0, 1/2)$ $[C, P] = 0$ FERMIONIC GAUGE FIELDS
$(1, 0) \oplus (0, 1)$ $\{C, P\} = 0$ BOSONIC MATTER FIELDS	$(1/2, 1/2)$ $[C, P] = 0$ BOSONIC GAUGE FIELDS

Table 1. The diagonal Wigner blocks are the ordinary fermionic matter and bosonic gauge fields, while the off-diagonal blocks refer to new structure in spacetime (refer to text for details). This primitive block can be extended to incorporate higher-spin particles of supergravity.

the same sentiments for the existing description of neutral particles (for it carries a strong dependence on charged-particle framework). Charged and neutral particles demand their own independent frameworks. Once that is done one may, if one wishes, seek differences and similarities between the two. But, not before. This *paper* undertakes this task.

The stated assertion that discovery of a spin-1/2 neutral particle shall be a step into a new realm of spacetime structure, arises, in part, from our ability to construct Table 2. In that table the diagonal Wigner blocks are the ordinary fermionic-matter and bosonic-gauge fields. The top off-diagonal block is one of the key results of this *paper*. Supersymmetric fermionic gauge bosons live in that block. The bottom off-diagonal block is populated by bosonic matter fields and awaits experimental confirmation. It was constructed in the 1993 paper cited as Ref. [11]. The gauge aspect of the top off-diagonal block tentatively refers either to the gauginos, or for neutrinos (should they be confirmed to belong there) it should be interpreted as an internal fermionic line as it appears in neutrinoless double beta decay. The C and P operators belong to the indicated representation space for each of the Wigner blocks. Possibility, but without explicit construction (for which we take credit), of such blocks is due to Wigner, and his colleagues [17].

We now present a detailed and systematic development of the theory of neutral particles.

2. Neutral particle spinors as Eigenspinors of charge conjugation operator: $\lambda(\mathbf{p})$ and $\rho(\mathbf{p})$

The boost generator for the $(1/2, 0)$ representation space is $-i\sigma/2$, and that for $(0, 1/2)$ is $+i\sigma/2$. Consequently, the respective boosts are

$$\kappa^{(\frac{1}{2}, 0)} = \exp\left(+\frac{\boldsymbol{\sigma}}{2} \cdot \boldsymbol{\varphi}\right) = \sqrt{\frac{E+m}{2m}} \left(1_2 + \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m}\right), \quad (5)$$

$$\kappa^{(0, \frac{1}{2})} = \exp\left(-\frac{\boldsymbol{\sigma}}{2} \cdot \boldsymbol{\varphi}\right) = \sqrt{\frac{E+m}{2m}} \left(1_2 - \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{E+m}\right), \quad (6)$$

where the linear boost parameter is defined as:

$$\cosh(\varphi) = \frac{E}{m}, \quad \sinh(\varphi) = \frac{|\mathbf{p}|}{m}, \quad \hat{\boldsymbol{\varphi}} = \frac{\mathbf{p}}{|\mathbf{p}|}. \quad (7)$$

The boosts take a particle at rest to a particle moving with momentum \mathbf{p} in the “boosted frame.” We use the notation in which 1_n and 0_n represent $n \times n$ identity and null matrices, respectively.

Thus each of the $SU(2)$ ’s is separately endowed with the dispersion relation $E^2 = \mathbf{p}^2 + m^2$ as encoded in Eqs. (7) via the identity, $\cosh^2(\varphi) - \sinh^2(\varphi) = 1$.

The explicit expressions for $\kappa^{(\frac{1}{2},0)}$ and $\kappa^{(0,\frac{1}{2})}$ allow for the observation:

$$\left(\kappa^{(0,\frac{1}{2})}\right)^{-1} = \left(\kappa^{(\frac{1}{2},0)}\right)^\dagger, \quad \left(\kappa^{(\frac{1}{2},0)}\right)^{-1} = \left(\kappa^{(0,\frac{1}{2})}\right)^\dagger. \quad (8)$$

Further, Θ , the Wigner's spin-1/2 time reversal operator, has the property

$$\Theta [\boldsymbol{\sigma}/2] \Theta^{-1} = - [\boldsymbol{\sigma}/2]^*, \quad (9)$$

When combined, these observations imply that [12]:

- (i) If $\phi_L(\mathbf{p})$ transforms as a left handed spinor, then $(\zeta_\lambda \Theta) \phi_L^*(\mathbf{p})$ transforms as a right handed spinor – where, ζ_λ is an unspecified phase.
- (ii) If $\phi_R(\mathbf{p})$ transforms as a right handed spinor, then $(\zeta_\rho \Theta)^* \phi_R^*(\mathbf{p})$ transforms as a left handed spinor – where, ζ_ρ is an unspecified phase.

As a consequence, the following spinors belong to the $(1/2, 0) \oplus (0, 1/2)$ representation space :

$$\lambda(\mathbf{p}) = \begin{pmatrix} (\zeta_\lambda \Theta) \phi_L^*(\mathbf{p}) \\ \phi_L(\mathbf{p}) \end{pmatrix}, \quad \rho(\mathbf{p}) = \begin{pmatrix} \phi_R(\mathbf{p}) \\ (\zeta_\rho \Theta)^* \phi_R^*(\mathbf{p}) \end{pmatrix}. \quad (10)$$

Demanding $\lambda(\mathbf{p})$ and $\rho(\mathbf{p})$ to be self/anti-self conjugate under C ,

$$C\lambda(\mathbf{p}) = \pm\lambda(\mathbf{p}), \quad C\rho(\mathbf{p}) = \pm\rho(\mathbf{p}), \quad (11)$$

restricts the phases, ζ_λ and ζ_ρ , to two values:

$$\zeta_\lambda = \pm i, \quad \zeta_\rho = \pm i. \quad (12)$$

The plus sign in the above equation yields self conjugate, $\lambda^S(\mathbf{p})$ and $\rho^S(\mathbf{p})$ spinors; while the minus sign results in the anti-self conjugate spinors, $\lambda^A(\mathbf{p})$ and $\rho^A(\mathbf{p})$.

Several remarks appear appropriate:

- (i) The $\lambda^{S,A}(\mathbf{p})$ and $\rho^{S,A}(\mathbf{p})$ are eigenspinors of the charge conjugation operator, C , in the $(1/2, 0) \oplus (0, 1/2)$ representation space. They are counterpart of the Dirac's $u(\mathbf{p})$ and $v(\mathbf{p})$ spinors, which are eigenspinors of the charge operator in the same representation space.
- (ii) The self-conjugate spinors are the standard textbook material.|| However, they must be supplemented by anti-self conjugate spinors to span the entire $(1/2, 0) \oplus (0, 1/2)$ representation space. Their neglect results in internal inconsistencies and the wrong conclusion on the true number of degrees of freedom for neutral particles. Any attempt to discard the anti-self conjugate spinors would parallel a call to discard the $v(\mathbf{p})$ spinors. The latter would amount to throwing away the antiparticles from one's theory. This would not only go against the observed reality but would make the theory internally inconsistent. Similar conclusions shall be seen to hold for our theory, and we shall duly examine the entire set of eigenspinors associated with C . It shall, however, suffice to confine to the set, $\lambda^{S,A}(\mathbf{p})$, or to the physically equivalent set, $\rho^{S,A}(\mathbf{p})$.

|| See, e.g., Eq. (1.4.52) of Ref. [12]. However, we take issue with the colloquial assertion that ‘‘Majorana spinors,’’ i.e., $\lambda^S(\mathbf{p})$, are Weyl spinors in four-component form. Such a misunderstanding has perhaps arisen due to implicit, or inadvertent, neglect of the $\lambda^A(\mathbf{p})$ spinors. A Weyl spinor transform as a $(1/2, 0)$, or as a $(0, 1/2)$, spinor; while the neutral particle spinors transform as $(1/2, 0) \oplus (0, 1/2)$ four-component spinors. The Weyl space is a two dimensional representation space. Hence, it cannot be spanned by four independent neutral particle spinors, i.e., $\lambda^S(\mathbf{p})$ and $\lambda^A(\mathbf{p})$. That honor belongs to the the four-dimensional $(1/2, 0) \oplus (0, 1/2)$ representation space.

- (iii) The necessary presence of the Wigner time reversal operator in the neutral particle spinors, as we shall sometime call the eigenspinors of the C operator, endows them with their own unique time evolution. We shall examine this aspect below.
- (iv) Both the the $\lambda^{S,A}(\mathbf{p})$, as well as $u(\mathbf{p})$ and $v(\mathbf{p})$ spinors, can be expressed in any realization (i.e., in Weyl, in Dirac, or in Majorana realizations; or whatever realization serves a particular task at hand). This brings in no new physics beyond convenience.

3. The explicit form of $\lambda(\mathbf{p})$ spinors

To obtain explicit expressions for $\lambda(\mathbf{p})$, we first write down the rest spinors. These are:

$$\lambda^S(\mathbf{0}) = \begin{pmatrix} +i\Theta\phi_L^*(\mathbf{0}) \\ \phi_L(\mathbf{0}) \end{pmatrix}, \quad \lambda^A(\mathbf{0}) = \begin{pmatrix} -i\Theta\phi_L^*(\mathbf{0}) \\ \phi_L(\mathbf{0}) \end{pmatrix}. \quad (13)$$

Next, we choose the $\phi_L(\mathbf{0})$ to be helicity eigenstates,

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{p}} \phi_L^\pm(\mathbf{0}) = \pm \phi_L^\pm(\mathbf{0}), \quad (14)$$

and concurrently note that

$$\boldsymbol{\sigma} \cdot \hat{\mathbf{p}} \Theta [\phi_L^\pm(\mathbf{0})]^* = \mp \Theta [\phi_L^\pm(\mathbf{0})]^*. \quad (15)$$

Derivation of Eq. (15): Complex conjugating Eq. (14) gives,

$$\boldsymbol{\sigma}^* \cdot \hat{\mathbf{p}} [\phi_L^\pm(\mathbf{0})]^* = \pm [\phi_L^\pm(\mathbf{0})]^*.$$

Substituting for $\boldsymbol{\sigma}^*$ from Eq. (9) then results in,

$$\Theta \boldsymbol{\sigma} \Theta^{-1} \cdot \hat{\mathbf{p}} [\phi_L^\pm(\mathbf{0})]^* = \mp [\phi_L^\pm(\mathbf{0})]^*.$$

But $\Theta^{-1} = -\Theta$. So,

$$-\Theta \boldsymbol{\sigma} \Theta \cdot \hat{\mathbf{p}} [\phi_L^\pm(\mathbf{0})]^* = \mp [\phi_L^\pm(\mathbf{0})]^*.$$

Or, equivalently,

$$\Theta^{-1} \boldsymbol{\sigma} \Theta \cdot \hat{\mathbf{p}} [\phi_L^\pm(\mathbf{0})]^* = \mp [\phi_L^\pm(\mathbf{0})]^*.$$

Finally, left multiplying both sides of the preceding equation by Θ , and moving Θ through $\hat{\mathbf{p}}$, yields Eq. (15).

That is, $\Theta [\phi_L^\pm(\mathbf{0})]^*$ has opposite helicity of $\phi_L^\pm(\mathbf{0})$. Since $\boldsymbol{\sigma} \cdot \hat{\mathbf{p}}$ commutes with the boost operator $\kappa^{(1/2,0)}$ the above result applies for all momenta. In conjunction with the definition of the neutral spinors we are thus lead to the result that neutral spinors are *not* single helicity objects. Instead, they invite an interpretation of dual helicity spinors. This shall allow for processes like neutrinoless double beta decay.

In the process we are led to four rest spinors. Two of which are self-conjugate,

$$\lambda_{\{-,+\}}^S(\mathbf{0}) = \begin{pmatrix} +i\Theta [\phi_L^+(\mathbf{0})]^* \\ \phi_L^+(\mathbf{0}) \end{pmatrix}, \quad \lambda_{\{+,-\}}^S(\mathbf{0}) = \begin{pmatrix} +i\Theta [\phi_L^-(\mathbf{0})]^* \\ \phi_L^-(\mathbf{0}) \end{pmatrix}, \quad (16)$$

and the other two, which are anti-self conjugate,

$$\lambda_{\{-,+\}}^A(\mathbf{0}) = \begin{pmatrix} -i\Theta [\phi_L^+(\mathbf{0})]^* \\ \phi_L^+(\mathbf{0}) \end{pmatrix}, \quad \lambda_{\{+,-\}}^A(\mathbf{0}) = \begin{pmatrix} -i\Theta [\phi_L^-(\mathbf{0})]^* \\ \phi_L^-(\mathbf{0}) \end{pmatrix}. \quad (17)$$

The first helicity entry refers to the $(1/2, 0)$ transforming component of the $\lambda(\mathbf{p})$, while the second entry encodes the helicity of the $(0, 1/2)$ component.

The boosted spinors are now obtained via the operation:

$$\lambda_{\{h,-h\}}(\mathbf{p}) = \begin{pmatrix} \kappa^{(\frac{1}{2},0)} & 0_2 \\ 0_2 & \kappa^{(0,\frac{1}{2})} \end{pmatrix} \lambda_{\{h,-h\}}(\mathbf{0}). \quad (18)$$

In the boosts, we replace $\boldsymbol{\sigma} \cdot \mathbf{p}$ by $\boldsymbol{\sigma} \cdot \hat{\mathbf{p}} |\mathbf{p}|$, and then exploit Eq. (15). After simplification, Eq. (18) yields:

$$\lambda_{\{-,+\}}^S(\mathbf{p}) = \sqrt{\frac{E+m}{2m}} \left(1 - \frac{|\mathbf{p}|}{E+m}\right) \lambda_{\{-,+\}}^S(\mathbf{0}), \quad (19)$$

which, in the massless limit, *identically vanishes*, while

$$\lambda_{\{+,-\}}^S(\mathbf{p}) = \sqrt{\frac{E+m}{2m}} \left(1 + \frac{|\mathbf{p}|}{E+m}\right) \lambda_{\{+,-\}}^S(\mathbf{0}), \quad (20)$$

does not. We hasten to warn the reader that one should not be tempted to read the two different prefactors to $\lambda^S(\mathbf{0})$ in the above expressions as the boost operator that appears in Eq. (18). For one thing, there is only one (not two) boost operator(s) in the $(1/2, 0) \oplus (0, 1/2)$ representation space. The simplification that appears here is due to a fine interplay between Eq. (15), the boost operator, and the structure of the $\lambda^S(\mathbf{0})$. Similarly, the anti-self conjugate set of the boosted spinors reads:

$$\lambda_{\{-,+\}}^A(\mathbf{p}) = \sqrt{\frac{E+m}{2m}} \left(1 - \frac{|\mathbf{p}|}{E+m}\right) \lambda_{\{-,+\}}^A(\mathbf{0}), \quad (21)$$

$$\lambda_{\{+,-\}}^A(\mathbf{p}) = \sqrt{\frac{E+m}{2m}} \left(1 + \frac{|\mathbf{p}|}{E+m}\right) \lambda_{\{+,-\}}^A(\mathbf{0}). \quad (22)$$

In the massless limit, the first of these spinors *identically vanishes*, while the second does not. Representing the unit vector along \mathbf{p} , as,

$$\hat{\mathbf{p}} = \left(\sin(\theta) \cos(\phi), \sin(\theta) \sin(\phi), \cos(\theta) \right), \quad (23)$$

the $\phi_L^\pm(\mathbf{0})$ take the explicit form:

$$\phi_L^+(\mathbf{0}) = \sqrt{m} e^{i\vartheta_1} \begin{pmatrix} \cos(\theta/2) e^{-i\phi/2} \\ \sin(\theta/2) e^{i\phi/2} \end{pmatrix}, \quad (24)$$

$$\phi_L^-(\mathbf{0}) = \sqrt{m} e^{i\vartheta_2} \begin{pmatrix} \sin(\theta/2) e^{-i\phi/2} \\ -\cos(\theta/2) e^{i\phi/2} \end{pmatrix}. \quad (25)$$

On setting ϑ_1 and ϑ_2 to be zero — a fact that we explicitly note [9] — we find the following *bi-orthonormality* relations for the self-conjugate neutral spinors,

$$\bar{\lambda}_{\{-,+\}}^S(\mathbf{p}) \lambda_{\{-,+\}}^S(\mathbf{p}) = 0, \quad \bar{\lambda}_{\{-,+\}}^S(\mathbf{p}) \lambda_{\{+,-\}}^S(\mathbf{p}) = +2im, \quad (26)$$

$$\bar{\lambda}_{\{+,-\}}^S(\mathbf{p}) \lambda_{\{-,+\}}^S(\mathbf{p}) = -2im, \quad \bar{\lambda}_{\{+,-\}}^S(\mathbf{p}) \lambda_{\{+,-\}}^S(\mathbf{p}) = 0. \quad (27)$$

Their counterpart for antiself-conjugate neutral spinors reads,

$$\bar{\lambda}_{\{-,+\}}^A(\mathbf{p}) \lambda_{\{-,+\}}^A(\mathbf{p}) = 0, \quad \bar{\lambda}_{\{-,+\}}^A(\mathbf{p}) \lambda_{\{+,-\}}^A(\mathbf{p}) = -2im, \quad (28)$$

$$\bar{\lambda}_{\{+,-\}}^A(\mathbf{p}) \lambda_{\{-,+\}}^A(\mathbf{p}) = +2im, \quad \bar{\lambda}_{\{+,-\}}^A(\mathbf{p}) \lambda_{\{+,-\}}^A(\mathbf{p}) = 0, \quad (29)$$

while all combinations of the type $\bar{\lambda}^A(\mathbf{p})\lambda^S(\mathbf{p})$ and $\bar{\lambda}^S(\mathbf{p})\lambda^A(\mathbf{p})$ identically vanish.

We take note that the bi-orthogonal norms of the Majorana spinors are intrinsically *imaginary*. The associated completeness relation is:

$$\boxed{-\frac{1}{2im} \left(\left[\lambda_{\{-,+\}}^S(\mathbf{p}) \bar{\lambda}_{\{+,-\}}^S(\mathbf{p}) - \lambda_{\{+,-\}}^S(\mathbf{p}) \bar{\lambda}_{\{-,+\}}^S(\mathbf{p}) \right] - \left[\lambda_{\{-,+\}}^A(\mathbf{p}) \bar{\lambda}_{\{+,-\}}^A(\mathbf{p}) - \lambda_{\{+,-\}}^A(\mathbf{p}) \bar{\lambda}_{\{-,+\}}^A(\mathbf{p}) \right] \right) = 1_4.} \quad (30)$$

4. Neutral particle spinors in Majorana realization

The $\lambda^{S,A}(\mathbf{p})$ obtained above are in Weyl realization – locally, in this section, we denote them by, $\lambda_W^{S,A}(\mathbf{p})$. In Majorana realization (subscripted by, M) these spinors are given by:

$$\lambda_M^{S,A}(\mathbf{p}) = \mathcal{S} \lambda_W^{S,A}(\mathbf{p}), \quad (31)$$

where

$$\mathcal{S} = \frac{1}{2} \begin{pmatrix} 1_2 + i\Theta & 1_2 - i\Theta \\ -(1_2 - i\Theta) & 1_2 + i\Theta \end{pmatrix}. \quad (32)$$

The $\lambda_M^S(\mathbf{p})$ are real, while $\lambda_M^A(\mathbf{p})$ are pure imaginary.

5. The $\rho(\mathbf{p})$ spinors are not independent

Now, $(1/2, 0) \oplus (0, 1/2)$ is a four dimensional representation space. Therefore, there cannot be more than four independent spinors. Consistent with this observation, we find that the $\rho(\mathbf{p})$ spinors are related to the $\lambda(\mathbf{p})$ spinors via the following identities:

$$\rho_{\{-,+\}}^S(\mathbf{p}) = -i\lambda_{\{+,-\}}^A(\mathbf{p}), \quad \rho_{\{+,-\}}^S(\mathbf{p}) = +i\lambda_{\{-,+\}}^A(\mathbf{p}), \quad (33)$$

$$\rho_{\{-,+\}}^A(\mathbf{p}) = +i\lambda_{\{+,-\}}^S(\mathbf{p}), \quad \rho_{\{+,-\}}^A(\mathbf{p}) = -i\lambda_{\{-,+\}}^S(\mathbf{p}). \quad (34)$$

Using these identities, one may immediately obtain the bi-orthonormality and completeness relations for the $\rho(\mathbf{p})$ spinors. In the massless limit, $\rho_1^S(\mathbf{p})$ and $\rho_1^A(\mathbf{p})$ *identically vanish*. A particularly simple orthonormality, as opposed to bi-orthonormality, relation exists between the $\lambda(\mathbf{p})$ and $\rho(\mathbf{p})$ spinors:

$$\bar{\lambda}_{\{-,+\}}^S(\mathbf{p})\rho_{\{-,+\}}^A(\mathbf{p}) = -2m = \bar{\lambda}_{\{-,+\}}^A(\mathbf{p})\rho_{\{-,+\}}^S(\mathbf{p}) \quad (35)$$

$$\bar{\lambda}_{\{+,-\}}^S(\mathbf{p})\rho_{\{+,-\}}^A(\mathbf{p}) = -2m = \bar{\lambda}_{\{+,-\}}^A(\mathbf{p})\rho_{\{+,-\}}^S(\mathbf{p}). \quad (36)$$

An associated completeness relation also exists, and it reads:

$$\begin{aligned} & -\frac{1}{2m} \left(\left[\lambda_{\{-,+\}}^S(\mathbf{p}) \bar{\rho}_{\{-,+\}}^A(\mathbf{p}) + \lambda_{\{+,-\}}^S(\mathbf{p}) \bar{\rho}_{\{+,-\}}^A(\mathbf{p}) \right] \right. \\ & \quad \left. + \left[\lambda_{\{-,+\}}^A(\mathbf{p}) \bar{\rho}_{\{-,+\}}^S(\mathbf{p}) + \lambda_{\{+,-\}}^A(\mathbf{p}) \bar{\rho}_{\{+,-\}}^S(\mathbf{p}) \right] \right) = 1_4. \end{aligned} \quad (37)$$

The results of this section are in spirit of Refs. [6, 7, 10, 9].

The completeness relation (30) confirms that a physically complete theory of neutral particle spinors must incorporate the self as well as antiself conjugate spinors. However, one

has a choice. One may either work with the set $\{\lambda^S(\mathbf{p}), \lambda^A(\mathbf{p})\}$, or with the physically and mathematically equivalent set, $\{\rho^S(\mathbf{p}), \rho^A(\mathbf{p})\}$. One is also free to choose some appropriate combinations of neutral particle spinors from these two sets.

6. Comparison with the Dirac framework: The $\lambda(\mathbf{p})$ do not satisfy Dirac equation

The main result of this section is a re-rendering of a proof given by M. Kirchbach[18]. Any mistake, if any, that the reader may notice is entirely due to my failure.

The bi-orthonormality relations (26-29) and the completeness relation (30) are counterpart of the following relations for the charged, i.e. Dirac, particle spinors:

$$\bar{u}_h(\mathbf{p}) u_{h'}(\mathbf{p}) = +2m\delta_{hh'}, \quad \bar{v}_h(\mathbf{p}) v_{h'}(\mathbf{p}) = -2m\delta_{hh'}, \quad (38)$$

$$\frac{1}{2m} \left[\sum_{h=\pm 1/2} u_h(\mathbf{p}) \bar{u}_h(\mathbf{p}) - \sum_{h=\pm 1/2} v_h(\mathbf{p}) \bar{v}_h(\mathbf{p}) \right] = 1_4. \quad (39)$$

Furthermore, if one wishes (with certain element of hazard to become apparent below), one can write the the *momentum-space* neutral spinor set $\{\lambda^S(\mathbf{p}), \lambda^A(\mathbf{p})\}$, in terms of charged particle spinor *momentum-space* set $\{u(\mathbf{p}), v(\mathbf{p})\}$. This task is best accomplished by introducing the following – to be used only locally – notation:

$$d_1 \equiv u_+(\mathbf{p}), \quad d_2 \equiv u_-(\mathbf{p}), \quad d_3 \equiv v_+(\mathbf{p}), \quad d_4 \equiv v_-(\mathbf{p}), \quad (40)$$

$$m_1 \equiv \lambda_{\{-,+\}}^S(\mathbf{p}), \quad m_2 \equiv \lambda_{\{+,-\}}^S(\mathbf{p}), \quad m_3 \equiv \lambda_{\{-,+\}}^A(\mathbf{p}), \quad m_4 \equiv \lambda_{\{+,-\}}^A(\mathbf{p}). \quad (41)$$

Then, the neutral particle spinors can be written as,

$$m_i = \sum_{j=1}^4 \Omega_{ij} d_j, \quad (42)$$

where

$$\Omega_{ij} = \begin{cases} + (1/2m) \bar{d}_j m_i 1_4, & \text{for } j = 1, 2 \\ - (1/2m) \bar{d}_j m_i 1_4, & \text{for } j = 3, 4 \end{cases}. \quad (43)$$

In matrix form, the Ω reads:

$$\Omega = \frac{1}{2} \begin{pmatrix} 1_4 & -i_4 & -1_4 & -i_4 \\ i_4 & 1_4 & i_4 & -1_4 \\ 1_4 & i_4 & -1_4 & i_4 \\ -i_4 & 1_4 & -i_4 & -1_4 \end{pmatrix}, \quad (44)$$

where, $i_4 \equiv i1_4$. Equations (42) and (44) immediately tell us that a neutral particle momentum-space spinor is a linear combination of the charged particle momentum-space *particle and antiparticle* spinors. In momentum space, the charged-particle spinors are annihilated by $(\gamma^\mu p_\mu \pm m1_4)$,

$$\begin{cases} \text{For particles:} & (\gamma^\mu p_\mu - m1_4) u(\mathbf{p}) = 0, \\ \text{For antiparticles:} & (\gamma^\mu p_\mu + m1_4) v(\mathbf{p}) = 0. \end{cases} \quad (45)$$

Since the mass terms carry opposite signs, hence are different for the particle and antiparticle, the neutral particle spinors cannot be annihilated by $(\gamma^\mu p_\mu - m1_4)$, or, by $(\gamma^\mu p_\mu + m1_4)$. Moreover, in the configuration space, since the time evolution of the of $u(\mathbf{p})$ occurs via $\exp(-ip_\mu x^\mu)$ while that for $v(\mathbf{p})$ spinors occurs via $\exp(+ip_\mu x^\mu)$ one cannot naively go from momentum-space expression (42) to its configuration space counterpart. In fact several conceptual and technically subtle hazards are confronted if one begins to mix the two set of

spinors. One ought to, as we intend to and shall, develop the theory of neutral particle spinors entirely in its own right. We thus end this digression by making part of the above argument more explicitly. For that purpose we introduce:

$$M \equiv \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}, \quad D \equiv \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}, \quad \Lambda \equiv \begin{pmatrix} \gamma_\mu p^\mu & 0_4 & 0_4 & 0_4 \\ 0_4 & \gamma_\mu p^\mu & 0_4 & 0_4 \\ 0_4 & 0_4 & \gamma_\mu p^\mu & 0_4 \\ 0_4 & 0_4 & 0_4 & \gamma_\mu p^\mu \end{pmatrix}. \quad (46)$$

In this language, equation (42) becomes

$$M = \Omega D. \quad (47)$$

Now, applying from left the operator Λ and using, $[\Lambda, \Omega] = 0$, we get

$$\Lambda M = \Omega \Lambda D. \quad (48)$$

But, Eqs. (45) imply

$$\Lambda D = \begin{pmatrix} m1_4 & 0_4 & 0_4 & 0_4 \\ 0_4 & m1_4 & 0_4 & 0_4 \\ 0_4 & 0_4 & -m1_4 & 0_4 \\ 0_4 & 0_4 & 0_4 & -m1_4 \end{pmatrix}. \quad (49)$$

Therefore, on using $D = \Omega^{-1} M$ we obtain,

$$\Lambda M = \Omega (\text{r.h.s. of Eq. 49}) \Omega^{-1} M. \quad (50)$$

An explicit evaluation of, $\mu \equiv \Omega (\text{r.h.s. of Eq. 49}) \Omega^{-1}$, reveals it to be,

$$\mu = \begin{pmatrix} 0_4 & -im1_4 & 0_4 & 0_4 \\ im1_4 & 0_4 & 0_4 & 0_4 \\ 0_4 & 0_4 & 0_4 & im1_4 \\ 0_4 & 0_4 & -im1_4 & 0_4 \end{pmatrix}. \quad (51)$$

Thus, finally giving us the result,

$$\begin{pmatrix} \gamma_\mu p^\mu & 0_4 & 0_4 & 0_4 \\ 0_4 & \gamma_\mu p^\mu & 0_4 & 0_4 \\ 0_4 & 0_4 & \gamma_\mu p^\mu & 0_4 \\ 0_4 & 0_4 & 0_4 & \gamma_\mu p^\mu \end{pmatrix} \begin{pmatrix} \lambda_{\{-,+\}}^S(\mathbf{p}) \\ \lambda_{\{+,-\}}^S(\mathbf{p}) \\ \lambda_{\{-,+\}}^A(\mathbf{p}) \\ \lambda_{\{+,-\}}^A(\mathbf{p}) \end{pmatrix} - im \begin{pmatrix} -\lambda_{\{+,-\}}^S(\mathbf{p}) \\ \lambda_{\{-,+\}}^S(\mathbf{p}) \\ \lambda_{\{+,-\}}^A(\mathbf{p}) \\ -\lambda_{\{-,+\}}^A(\mathbf{p}) \end{pmatrix} = 0, \quad (52)$$

which explicitly establishes the result that $(\gamma^\mu p_\mu \pm m1_4)$ do not annihilate the neutral particle spinors. \mathbf{P} The text-book assertions that Majorana mass term is ‘off-diagonal’ is a rough translation of this equation.

7. Commutativity of C and P for neutral particle spinors

The parity operation is slightly subtle for neutral particle spinors.

With a reminder to remarks made immediately after Eq. (4), the parity operator in the $(1/2, 0) \oplus (0, 1/2)$ representation space is,

$$P = e^{i\phi_P} \gamma^0 \mathcal{R}, \quad (53)$$

\mathbf{P} The result contained in the above equation confirms earlier result of Ref. [19].

where

$$\gamma^0 = \begin{pmatrix} 0_2 & 1_2 \\ 1_2 & 0_2 \end{pmatrix}. \quad (54)$$

The \mathcal{R} is defined as,

$$\mathcal{R} \equiv \{\theta \rightarrow \pi - \theta, \phi \rightarrow \phi + \pi, p \rightarrow p\}. \quad (55)$$

This has the consequence that eigenvalues, h , of the helicity operator

$$\mathbf{h} = \frac{\boldsymbol{\sigma}}{2} \cdot \hat{\mathbf{p}} \quad (56)$$

change sign under the operation of \mathcal{R} ,

$$\mathcal{R} : h \rightarrow h' = -h. \quad (57)$$

Furthermore,

$$P u_h(\mathbf{p}) = e^{i\phi_P} \gamma^0 \mathcal{R} u_h(\mathbf{p}) = e^{i\phi_P} \gamma^0 u_{-h}(-\mathbf{p}) = -i e^{i\phi_P} u_h(\mathbf{p}) \quad (58)$$

Similarly,

$$P v_h(\mathbf{p}) = i e^{i\phi_P} v_h(\mathbf{p}). \quad (59)$$

We now require the eigenvalues of the P to be real. This fixes the phase factor,

$$e^{i\phi_P} = \pm i. \quad (60)$$

The remaining ambiguity, as contained in the sign, still remains. It is fixed by recourse to text-book convention by taking the sign on the right-hand side of Eq. (60) of to be positive. This very last choice shall not affect our conclusions (as it should not). The parity operator is thus fixed to be,

$$P = i \gamma^0 \mathcal{R}. \quad (61)$$

Thus,

$$P u_h(\mathbf{p}) = + u_h(\mathbf{p}), \quad (62)$$

$$P v_h(\mathbf{p}) = - v_h(\mathbf{p}). \quad (63)$$

The consistency of Eqs. (62) and (63) requires,

$$\text{CHARGED PARTICLE SPINORS : } P^2 = 1_4, \quad [\text{cf. Eq.(71)}]. \quad (64)$$

To calculate the anticommutator, $\{C, P\}$, when acting on the $u_h(\mathbf{p})$ and $v_h(\mathbf{p})$ we now need, in addition, the action of C on these spinors. This action can be summarized as follows:

$$C : \begin{cases} u_{+1/2}(\mathbf{p}) \rightarrow -v_{-1/2}(\mathbf{p}), u_{-1/2}(\mathbf{p}) \rightarrow v_{+1/2}(\mathbf{p}), \\ v_{+1/2}(\mathbf{p}) \rightarrow u_{-1/2}(\mathbf{p}), v_{-1/2}(\mathbf{p}) \rightarrow -u_{+1/2}(\mathbf{p}). \end{cases} \quad (65)$$

Using Eqs. (62), (63), and (65) one can readily obtain the action of anticommutator, $\{C, P\}$, on the four $u(\mathbf{p})$ and $v(\mathbf{p})$ spinors. For each case it is found to vanish: $\{C, P\} = 0$.

The P acting on the neutral particle spinors yields the result,

$$\boxed{\begin{aligned} P \lambda_{\{-,+\}}^S(\mathbf{p}) &= +i \lambda_{\{+,-\}}^A(\mathbf{p}), P \lambda_{\{+,-\}}^S(\mathbf{p}) = -i \lambda_{\{-,+\}}^A(\mathbf{p}), \quad (66) \\ P \lambda_{\{-,+\}}^A(\mathbf{p}) &= -i \lambda_{\{+,-\}}^S(\mathbf{p}), P \lambda_{\{+,-\}}^A(\mathbf{p}) = +i \lambda_{\{-,+\}}^S(\mathbf{p}). \quad (67) \end{aligned}}$$

Following the same procedure as before, we now use (66), (67), and (65) to evaluate the action of the commutator $[C, P]$ on each of the four neutral particle spinors. We find it vanishes for each of them: $[C, P] = 0$. It confirms the claim we made in Table 2.

The commutativity and anticommutativity of the C and P operators is a deeply profound result and it establishes that the theory of neutral and charged particles must be developed in their own rights. This is the task we have undertaken and are developing here in this *Paper*.

8. Parity asymmetry for neutral particle spinors

Unlike the charged particle spinors, Eqs. (66) and (67) reveal that neutral particle spinors are not eigenstates of P . Furthermore, a rather apparently paradoxical asymmetry is contained in these equations. For instance, the second equation in (66) reads:

$$P\lambda_{\{+,-\}}^S(\mathbf{p}) = -i\lambda_{\{-,+\}}^A(\mathbf{p}). \quad (68)$$

Now, in a normalization-independent manner

$$\lambda_{\{+,-\}}^S(\mathbf{p}) \propto \left(1 + \frac{|\mathbf{p}|}{E+m}\right) \lambda_{\{+,-\}}^S(\mathbf{0}), \quad (69)$$

while

$$\lambda_{\{-,+\}}^A(\mathbf{p}) \propto \left(1 - \frac{|\mathbf{p}|}{E+m}\right) \lambda_{\{-,+\}}^A(\mathbf{0}). \quad (70)$$

Consequently, in the massless/high-energy limit the P -reflection of $\lambda_{\{+,-\}}^S(\mathbf{p})$ identically vanishes. The same happens to the $\lambda_{\{-,+\}}^A(\mathbf{p})$ spinors under P -reflection. This situation is in sharp contrast to the charged particle spinors. The consistency of Eqs. (66) and (67) requires $P^2 = -1_4$ and in the process shows that the remaining two, i.e. first and third equation in that set, do not contain additional physical content:

NEUTRAL PARTICLE SPINORS : $P^2 = -1_4$. [cf. Eq.(64)].
(71)

That is, for neutral particle spinors:

NEUTRAL PARTICLE SPINORS : $P^4 = 1_4$, .
(72)

The origin of the asymmetry under P -reflection resides in the fact that the $(1/2, 0) \oplus (1/2, 0)$ neutral particles spinors, in being dual helicity objects, combine Weyl spinors of *opposite* helicities. However, in the massless limit, the structures of $\kappa^{(\frac{1}{2}, 0)}$ and $\kappa^{(0, \frac{1}{2})}$ force only positive helicity $(1/2, 0)$ -Weyl and negative helicity $(0, 1/2)$ -Weyl spinors to be non-vanishing. For this reason, in the massless limit the neutral particle spinors, $\lambda_{\{-,+\}}^S(\mathbf{p})$ and $\lambda_{\{-,+\}}^A(\mathbf{p})$, carrying negative helicity $(1/2, 0)$ -Weyl and positive helicity $(0, 1/2)$ -Weyl spinors identically vanish.

So we have the following situation: The $(1/2, 0) \oplus (0, 1/2)$ is a P covariant representation space. Yet, in the neutral particle formalism, it carries P -reflection asymmetry. This circumstance has a precedence in the Velo-Zwanziger observation, who noted [20], “the main lesson to be drawn from our analysis is that special relativity is not automatically satisfied by writing equations which transform covariantly.” We conjecture that this asymmetry may underlie the phenomenologically known parity violation. Even though the latter is incorporated, by hand, in the standard model of the electroweak interactions its true physical origin has remained unknown.

9. A Master wave equation for spinors

To study time evolution of neutral particle spinors we need appropriate wave equation. This we do in the following manner. First, we obtain the momentum-space wave equation satisfied by the $\lambda(\mathbf{p})$ spinors. Next, we ascertain the time evolution via a “ $p_\mu \rightarrow i\partial_\mu$ ” prescription.

Since the method we use, and the results we obtain, appear somewhat unusual we exercise extra care in presenting our results. We, therefore, present a unified method which

applies not only to neutral particle spinors but applies equally well to other cases (such as the Dirac formalism). The method is a generalization of the textbook procedure [4] with corrections noted in Refs. [11, 21, 22, 8].

Thus, define a general $(1/2, 0) \oplus (0, 1/2)$ spinor

$$\chi(\mathbf{p}) = \begin{pmatrix} \chi^{(\frac{1}{2}, 0)}(\mathbf{p}) \\ \chi^{(0, \frac{1}{2})}(\mathbf{p}) \end{pmatrix}, \quad (73)$$

such that in particle's rest frame, where, $\mathbf{p} = \mathbf{0}$, by definition,

$$\chi^{(\frac{1}{2}, 0)}(\mathbf{0}) = \mathcal{A} \chi^{(0, \frac{1}{2})}(\mathbf{0}). \quad (74)$$

Here, the 2×2 matrix \mathcal{A} encodes C , P , and T properties of the spinor and is left unspecified at the moment except that we require it to be invertible.

Once $\chi^{(\frac{1}{2}, 0)}(\mathbf{0})$ and $\chi^{(0, \frac{1}{2})}(\mathbf{0})$ are specified the $\chi^{(\frac{1}{2}, 0)}(\mathbf{p})$ and $\chi^{(0, \frac{1}{2})}(\mathbf{p})$ follow from,

$$\chi^{(\frac{1}{2}, 0)}(\mathbf{p}) = \kappa^{(\frac{1}{2}, 0)} \chi^{(\frac{1}{2}, 0)}(\mathbf{0}), \quad (75)$$

$$\chi^{(0, \frac{1}{2})}(\mathbf{p}) = \kappa^{(0, \frac{1}{2})} \chi^{(0, \frac{1}{2})}(\mathbf{0}). \quad (76)$$

Below, we shall need their inverted forms also. These we write as follows:

$$\chi^{(\frac{1}{2}, 0)}(\mathbf{0}) = \left(\kappa^{(\frac{1}{2}, 0)} \right)^{-1} \chi^{(\frac{1}{2}, 0)}(\mathbf{p}), \quad (77)$$

$$\chi^{(0, \frac{1}{2})}(\mathbf{0}) = \left(\kappa^{(0, \frac{1}{2})} \right)^{-1} \chi^{(0, \frac{1}{2})}(\mathbf{p}). \quad (78)$$

Equation (74) implies,

$$\chi^{(0, \frac{1}{2})}(\mathbf{0}) = \mathcal{A}^{-1} \chi^{(\frac{1}{2}, 0)}(\mathbf{0}) \quad (79)$$

which on immediate use of (77) yields,

$$\chi^{(0, \frac{1}{2})}(\mathbf{0}) = \mathcal{A}^{-1} \left(\kappa^{(\frac{1}{2}, 0)} \right)^{-1} \chi^{(\frac{1}{2}, 0)}(\mathbf{p}). \quad (80)$$

However, since

$$\left(\kappa^{(\frac{1}{2}, 0)} \right)^{-1} = \kappa^{(0, \frac{1}{2})} \quad (81)$$

we have:

$$\chi^{(0, \frac{1}{2})}(\mathbf{0}) = \mathcal{A}^{-1} \kappa^{(0, \frac{1}{2})} \chi^{(\frac{1}{2}, 0)}(\mathbf{p}). \quad (82)$$

Similarly,

$$\chi^{(\frac{1}{2}, 0)}(\mathbf{0}) = \mathcal{A} \kappa^{(\frac{1}{2}, 0)} \chi^{(0, \frac{1}{2})}(\mathbf{p}). \quad (83)$$

Substituting for $\chi^{(\frac{1}{2}, 0)}(\mathbf{0})$ from Eq. (83) in Eq. (75) and re-arranging gives:

$$- \chi^{(\frac{1}{2}, 0)}(\mathbf{p}) + \kappa^{(\frac{1}{2}, 0)} \mathcal{A} \kappa^{(\frac{1}{2}, 0)} \chi^{(0, \frac{1}{2})}(\mathbf{p}) = 0; \quad (84)$$

while similar use of Eq. (82) in Eq. (76) results in:

$$\kappa^{(0, \frac{1}{2})} \mathcal{A}^{-1} \kappa^{(0, \frac{1}{2})} \chi^{(\frac{1}{2}, 0)}(\mathbf{p}) - \chi^{(0, \frac{1}{2})}(\mathbf{p}) = 0. \quad (85)$$

The last two equations when combined into a matrix form result in the *momentum-space master equation for $\chi(\mathbf{p})$* ,

$$\boxed{\begin{pmatrix} & -1_2 & \kappa^{(\frac{1}{2},0)} \mathcal{A} \kappa^{(\frac{1}{2},0)} \\ \kappa^{(0,\frac{1}{2})} \mathcal{A}^{-1} \kappa^{(0,\frac{1}{2})} & & -1_2 \end{pmatrix} \chi(\mathbf{p}) = 0. \quad (86)}$$

Thus, the momentum-space equation for $\chi(\mathbf{p})$ is entirely determined by the boosts $\kappa^{(\frac{1}{2},0)}$ and $\kappa^{(0,\frac{1}{2})}$ and the CPT-property encoding matrix \mathcal{A} .

We envisage the most general form of \mathcal{A} to be a unitary matrix with determinant ± 1 :

$$\mathcal{A}_{\pm} = \begin{pmatrix} a e^{i\phi_a} & \sqrt{\pm 1 - a^2} e^{i\phi_b} \\ -\sqrt{\pm 1 - a^2} e^{-i\phi_b} & a e^{-i\phi_a} \end{pmatrix}, \quad (87)$$

with a , ϕ_a , and ϕ_b real. The plus sign yields Determinant of \mathcal{A} to be $+1$, while the minus sign yields it to be -1 . Inserting \mathcal{A} from Eq. (87) into (86), we evaluate the determinant of the operator

$$\mathcal{O} = \begin{pmatrix} & -1_2 & \kappa^{(\frac{1}{2},0)} \mathcal{A} \kappa^{(\frac{1}{2},0)} \\ \kappa^{(0,\frac{1}{2})} \mathcal{A}^{-1} \kappa^{(0,\frac{1}{2})} & & -1_2 \end{pmatrix}, \quad (88)$$

and find it to be:

$$Det[\mathcal{O}] = \frac{(m^2 + p^2 - (2m + E)^2)^2 (m^2 + p^2 - E^2)^2}{(2m(E + m))^4}. \quad (89)$$

The wave operator, \mathcal{O} , supports two type of spinors. Those associated with the usual Einsteinian dispersion relation,

$$E^2 = m^2 + p^2, \quad \text{multiplicity} = 4 \quad (90)$$

and those associated with:

$$E = \begin{cases} -2m - \sqrt{m^2 + p^2}, & \text{multiplicity} = 2 \\ -2m + \sqrt{m^2 + p^2}, & \text{multiplicity} = 2 \end{cases}. \quad (91)$$

The origin of the new dispersion relation must certainly lie, or at least we suspect it to be so, in the new $U(2)$ phases matrix. We shall see below that for the Dirac, as well as Majorana, spinors only the Einsteinian dispersion relation gets invoked.

We hope to take up the other class of spinors, $\chi(\mathbf{p})$, in a subsequent study. Should something of physical interest emerge we shall report it in an appropriate publication.

10. Obtaining Dirac equation from Master equation

To give confidence to our reader in the physical content of the Master equation we now apply it to the charged particle spinors of Dirac formalism. Once we do that we shall return to the task of constructing momentum-space wave equation for the $\lambda(\mathbf{p})$.

The \mathcal{A} can be read off from the Dirac rest spinors. However, we remind the reader, that the writing down of the Dirac rest spinors, as shown by Weinberg and also by our independent studies, follows from the following two requirements:

\mathcal{R}_1 : The conservation of parity [23, 22, 8].

\mathcal{R}_2 : That, in a quantum field theoretic framework, the Dirac field describe fermions [23].

These *physical requirements* determine \mathcal{A} to be:

$$\mathcal{A} = \begin{cases} + 1_2, & \text{for } u(\mathbf{p}) \text{ spinors} \\ - 1_2, & \text{for } v(\mathbf{p}) \text{ spinors,} \end{cases} \quad (92)$$

and correspond to \mathcal{A}_+ with $a = 1, \phi_a = 0$, and $a = 1, \phi_a = \pi$, respectively, with ϕ_b remaining arbitrary. The subscript on \mathcal{A} simply represents that its determinant is plus unity.

Using this information in the Master equation (86), along with the explicit expressions for $\kappa^{(\frac{1}{2}, 0)}$ and $\kappa^{(0, \frac{1}{2})}$, yields:

$$\begin{pmatrix} - 1_2 & \exp(\boldsymbol{\sigma} \cdot \boldsymbol{\varphi}) \\ \exp(-\boldsymbol{\sigma} \cdot \boldsymbol{\varphi}) & - 1_2 \end{pmatrix} u(\mathbf{p}) = 0, \quad (93)$$

$$\begin{pmatrix} 1_2 & \exp(\boldsymbol{\sigma} \cdot \boldsymbol{\varphi}) \\ \exp(-\boldsymbol{\sigma} \cdot \boldsymbol{\varphi}) & 1_2 \end{pmatrix} v(\mathbf{p}) = 0. \quad (94)$$

Exploiting the fact that $\boldsymbol{\sigma}^2 = 1_2$, and using the definition of the boost parameter $\boldsymbol{\varphi}$ given in Eqs. (7), the exponentials that appear in the above equation take the form,

$$\exp(\pm \boldsymbol{\sigma} \cdot \boldsymbol{\varphi}) = \frac{(E 1_2 \pm \boldsymbol{\sigma} \cdot \mathbf{p})}{m}. \quad (95)$$

Using these expansions in Eqs. (93) and (94), multiplying both sides of the resulting equations by m , using $p_\mu = (E, -\mathbf{p})$, and introducing:

$$\gamma^0 = \begin{pmatrix} 0_2 & 1_2 \\ 1_2 & 0_2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0_2 & -\sigma_i \\ \sigma_i & 0_2 \end{pmatrix}, \quad (96)$$

gives Eqs. (93) and (94) the form

$$(p_\mu \gamma^\mu - m 1_4) u(\mathbf{p}) = 0, \quad (97)$$

$$(p_\mu \gamma^\mu + m 1_4) v(\mathbf{p}) = 0. \quad (98)$$

These are the well-known momentum space wave equations for the charged particle spinors (i.e. the Dirac equations). The *linearity* of these equations in p_μ is due to form of \mathcal{A} , and the property of Pauli matrices, $\boldsymbol{\sigma}^2 = 1_2$ – see, Eq. (95).

11. Obtaining wave equation for neutral particle spinors from Master equation

The requirement that the $\lambda(\mathbf{p})$ be eigenstates of the charge conjugation operator completely determines \mathcal{A} for the neutral particle spinors to be:

$$\mathcal{A} = \zeta_\lambda \Theta \alpha, \quad (99)$$

where

$$\alpha = \begin{pmatrix} \exp(i\phi) & 0 \\ 0 & \exp(-i\phi) \end{pmatrix}. \quad (100)$$

Explicitly,

$$\mathcal{A}_-^S = \begin{pmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix}, \quad \mathcal{A}_-^A = \begin{pmatrix} 0 & ie^{-i\phi} \\ -ie^{i\phi} & 0 \end{pmatrix}. \quad (101)$$

The noted \mathcal{A} 's arise from the following choice of the parameters $\{a, \phi_a, \phi_b\}$: $a = 0, \phi_b = -\phi + \pi$ and $a = 0, \phi_b = -\phi$, respectively, with ϕ_a remaining arbitrary. The subscript on \mathcal{A} is to remind that its determinant is minus unity. This difference – summarized in Table 2 – in \mathcal{A} , for Dirac and Majorana spinors, does not allow the $\lambda(\mathbf{p})$ to satisfy the Dirac equation.

SPINOR TYPE	Det[\mathcal{A}]	a	ϕ_a	ϕ_b
Dirac $u(\mathbf{p})$	+1	1	0	arbitrary
Dirac $v(\mathbf{p})$	+1	1	π	arbitrary
Majorana $\lambda^S(\mathbf{p})$	-1	0	arbitrary	$-\phi + \pi$
Majorana $\lambda^A(\mathbf{p})$	-1	0	arbitrary	$-\phi$

Table 2. The parameters $\{a, \phi_a, \phi_b\}$. See text.

Following the same procedure as above, and using

$$\exp\left(\pm \frac{\boldsymbol{\sigma} \cdot \boldsymbol{\varphi}}{2}\right) = \frac{(E + m) 1_2 \pm \boldsymbol{\sigma} \cdot \mathbf{p}}{\sqrt{2m(E + m)}}, \quad (102)$$

we obtain, instead:

$$\left[(p_\mu \gamma^\mu + m \gamma^0) \tilde{\mathcal{A}} (p_\mu \gamma^\mu + m \gamma^0) - 2m(E + m) 1_4 \right] \lambda(\mathbf{p}) = 0; \quad (103)$$

where

$$\tilde{\mathcal{A}} = \begin{pmatrix} 0_2 & \mathcal{A} \\ \mathcal{A}^{-1} & 0_2 \end{pmatrix}. \quad (104)$$

However, $\tilde{\mathcal{A}}$ commutes with $(p_\mu \gamma^\mu + m \gamma^0)$

$$\left[(p_\mu \gamma^\mu + m \gamma^0), \tilde{\mathcal{A}} \right] = 0. \quad (105)$$

Therefore, Eq. (103) after due simplification becomes:

$$\boxed{\left[(p_\mu p^\mu + 2mE + m^2) \tilde{\mathcal{A}} - 2m(E + m) 1_4 \right] \lambda(\mathbf{p}) = 0.} \quad (106)$$

As a check, we calculate the determinant of the operator acting on $\lambda(\mathbf{p})$, and find

$$\begin{aligned} & \text{Det} \left[(p_\mu p^\mu + 2mE + m^2) \tilde{\mathcal{A}} - 2m(E + m) 1_4 \right] \\ &= (m^2 + p^2 - (2m + E)^2)^2 (m^2 + p^2 - E^2)^2. \end{aligned} \quad (107)$$

Furthermore, we make the observation that for all-four $\lambda(\mathbf{p})$, $\tilde{\mathcal{A}} \lambda(\mathbf{p}) = \lambda(\mathbf{p})$. With this observation, Eq. (106) shows that each of the four components of the $\lambda(\mathbf{p})$ satisfies the Klein-Gordon equation: $[(p_\mu p^\mu - m^2) 1_4] \lambda(\mathbf{p}) = 0$. However, the latter equation should not be considered the wave equation for the $\lambda(\mathbf{p})$. The correct equation is (106), and it is this equation when transformed to the configuration space which shall yield the full time evolution.

12. Insensitivity of Majorana spinors to the direction of time

The plane waves for the $\lambda(\mathbf{x}, t)$ and $\rho(\mathbf{x}, t)$ are, $\exp(-i \epsilon p \cdot x) \lambda(\mathbf{p})$ and $\exp(-i \epsilon p \cdot x) \rho(\mathbf{p})$, where $\epsilon = \pm 1$ (depending upon whether the propagation is forward in time, or backward in time). To determine ϵ , it suffices to study the wave equations in the rest frame of the particles. In that frame, the $\lambda(\mathbf{x}, t)$ satisfies the following (simplified) differential equation:⁺

⁺ To obtain the simplified equation below we first multiplied the momentum-space wave equation by $2m(E + m)$. Then, we exploited the fact that in configuration space, $\mathbf{p} = \frac{1}{i} \nabla$. When it acts upon $\lambda(\mathbf{0})$ the resulting eigenvalue vanishes (so we dropped this term), and that $E = i \frac{\partial}{\partial t}$.

$$\begin{pmatrix} \mathcal{O}_a & 0 & 0 & \mathcal{O}_b \\ 0 & \mathcal{O}_a & \mathcal{O}_c & 0 \\ 0 & \mathcal{O}_b & \mathcal{O}_a & 0 \\ \mathcal{O}_c & 0 & 0 & \mathcal{O}_a \end{pmatrix} \lambda(\mathbf{0}) e^{-i\epsilon mt} = 0 \quad (108)$$

where

$$\mathcal{O}_a \equiv -2m \left(i \frac{\partial}{\partial t} + m \right), \quad (109)$$

$$\mathcal{O}_b \equiv -\zeta_\lambda e^{-i\phi} \left(m^2 - \frac{\partial^2}{\partial t^2} + 2im \frac{\partial}{\partial t} \right), \quad (110)$$

$$\mathcal{O}_c \equiv \zeta_\lambda e^{i\phi} \left(m^2 - \frac{\partial^2}{\partial t^2} + 2im \frac{\partial}{\partial t} \right). \quad (111)$$

This equation does not fix the sign of ϵ . It only determines ϵ^2 to be unity. To convince the reader, we give an example result of a simple calculation. Let's consider $\lambda(\mathbf{0})$ to be $\lambda_{\{-,+\}}^S(\mathbf{0})$. Then, set $\zeta_\lambda = \zeta_\lambda^S = i$. For this example, the time evolution Eq. (108) gives:

$$m^2 (\epsilon^2 - 1) \lambda_{\{-,+\}}^S(\mathbf{0}) e^{-i\epsilon mt} = 0. \quad (112)$$

This result does not determine sign of ϵ . That is, Majorana spinors are insensitive to the *forward* and *backward* directions in time so important in the Feynman-Stückelberg interpretation of particles and antiparticles. The *conventional* distinction between particles and antiparticles disappears.

13. Remarks for a quantum field theoretic description for neutral particles

In the Dirac theory of charged particles, the *positive*-definite norms of the $u(\mathbf{p})$ spinors and *negative*-definite norms of the $v(\mathbf{p})$ spinors *and* the anticommutativity of the annihilation and creation operators plays a fundamental role in securing a theory with energy bounded from below. The Dirac-particle dual, $\bar{\psi}(\mathbf{p})$ to a $(1/2, 0) \oplus (0, 1/2)$ spinor, $\psi(\mathbf{p})$, is,

$$\bar{\psi}(\mathbf{p}) = [\psi(\mathbf{p})]^\dagger \gamma^0. \quad (113)$$

It yields real-definite norm for the spinors inhabiting the basis spinors of the Dirac's $(1/2, 0) \oplus (0, 1/2)$ representation space, and it implies the stated spinorial properties.

In order to quantize the theory with neutral particle spinors we find it necessary (as has been verified through a detailed calculation) to define *neutral-particle duals* $\widehat{\lambda}(\mathbf{p})$:

$$\widehat{\lambda}_{\{-,+\}}^S(\mathbf{p}) = + \bar{p}_{\{+,-\}}^A(\mathbf{p}), \quad \widehat{\lambda}_{\{+,-\}}^S(\mathbf{p}) = + \bar{p}_{\{-,+\}}^A(\mathbf{p}), \quad (114)$$

$$\widehat{\lambda}_{\{-,+\}}^A(\mathbf{p}) = - \bar{p}_{\{+,-\}}^S(\mathbf{p}), \quad \widehat{\lambda}_{\{+,-\}}^A(\mathbf{p}) = - \bar{p}_{\{-,+\}}^S(\mathbf{p}). \quad (115)$$

The use of dispersion relation $E = \pm \sqrt{p^2 + m^2}$, yields:

$$\begin{aligned} \widehat{\lambda}_{\{-,+\}}^S(\mathbf{p}) \lambda_{\{-,+\}}^S(\mathbf{p}) &= \widehat{\lambda}_{\{+,-\}}^S(\mathbf{p}) \lambda_{\{+,-\}}^S(\mathbf{p}) = + 2m, \\ \widehat{\lambda}_{\{-,+\}}^A(\mathbf{p}) \lambda_{\{-,+\}}^A(\mathbf{p}) &= \widehat{\lambda}_{\{+,-\}}^A(\mathbf{p}) \lambda_{\{+,-\}}^A(\mathbf{p}) = - 2m. \end{aligned} \quad (116)$$

We construct the projectors:

$$P_S = \frac{1}{2m} \left(\lambda_{\{-,+\}}^S(\mathbf{p}) \widehat{\lambda}_{\{-,+\}}^S(\mathbf{p}) + \lambda_{\{+,-\}}^S(\mathbf{p}) \widehat{\lambda}_{\{+,-\}}^S(\mathbf{p}) \right) \Big|_{E=\pm\sqrt{p^2+m^2}} \quad (117)$$

$$P_A = \frac{-1}{2m} \left(\lambda_{\{-,+\}}^A(\mathbf{p}) \widehat{\lambda}_{\{-,+\}}^A(\mathbf{p}) + \lambda_{\{+,-\}}^A(\mathbf{p}) \widehat{\lambda}_{\{+,-\}}^A(\mathbf{p}) \right) \Big|_{E=\pm\sqrt{p^2+m^2}} \quad (118)$$

and verify that indeed, $P_S^2 = P_S$ and $P_A^2 = P_A$. Furthermore, these degrees of freedom form a complete set:

$$P_S + P_A = I_4. \quad (119)$$

Elsewhere we shall report on the quantum field theory based on these degrees of freedom [25].

14. Conclusion

We showed that Majorana particles belong to the Wigner class of fermions in which the charge conjugation and the parity operators commute, rather than anticommute. We proved, that rigorously speaking, Majorana spinors do not satisfy the Dirac equation. Instead, they satisfy a different wave equation, which we derived. This allowed us to reconcile Stückelberg-Feynman interpretation with the Majorana construct. We presented several new properties of neutral particle spinors and argued that discovery of Majorana particles constitutes dawn of a new era in spacetime structure.

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