ON THE COUNTING OF RADIO SOURCES IN THE STEADY-STATE COSMOLOGY

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Summary

The problem of the number count of radio sources as a function of the incident flux is shown to depend on two crucial features: (i) the size and behaviour of condensations (ii) the dependence on age of the probability of a galaxy being a radio source. Provided the probability rises by a factor $\sim 10^2$ for galaxies with ages from H^{-1} to about $2{\cdot}5H^{-1}$, and provided the primary condensations of the steady-state theory possess initial dimensions of order 30 megaparsecs, the radio source count can rise more steeply than is the case for sources uniformly distributed in Euclidean space. Each primary condensation contains of the order of 10^5 galaxies, which in the main expand apart from each other as the universe expands.

It is shown that a luminosity function can be chosen for the radio sources giving results consistent with observation, not only for the source count, but also with the data on angular diameters and with experience in the problem of optical identifications.

1. *The* $\log N - \log P$ *curve.*—Six years ago, Ryle announced in his Halley lecture (**16**) that counts of the number $N(P)$ of radio sources brighter than P did not behave in the manner to be expected of uniformly distributed objects in Euclidean space.

From the point of view of the theoretician, the situation over the last six years has seemed uncertain, since different observers have been in disagreement over the real value of the slope of the $\log N - \log P$ curve. In a comparison over a common portion of the sky, the slope obtained at Cambridge (**1**) was $-2{\cdot}5$ at $81{\cdot}5$ Mc/s and $-2{\cdot}7$ at 159 Mc/s, whereas Mills, Slee and Hill (**2**) obtained $-1{\cdot}8$. Very recently, the Cambridge observers have announced the result of a new survey, yielding about $-1{\cdot}8$ for the slope, and this they regard as superseding their former estimates. Another recent survey, at 960 Mc/s, by Kellermann and Harris (**3**) gives $-2{\cdot}0$.

Although the concordance between the recent Cambridge work and that of MSH suggests that the slope probably is steeper than the Euclidean value of $-1{\cdot}5$ (a view expressed by Bolton in his survey given at the U.R.S.I. General Assembly, London, 1960 September), it must be noted that Mills has remarked on the likely existence of a systematic progressive error in his judgment of the values of P. In a "noise-limited" instrument, such as the Mills Cross, the observer tends to overestimate P more and more as the limit of detectability is reached. Mills has suggested that this effect could reduce the slope to $-1{\cdot}65$, or even to $-1{\cdot}5$ (4)*. On the other hand, the Cambridge instrument is not noise-limited, and no such systematic effect should be present. The concordance between the Sydney and the Cambridge work may therefore be coincidental.

In the present paper we propose to accept the Cambridge claim that the slope is steeper than $-1{\cdot}5$ for values of P less than 3×10^{-25} watt m^{-2} (c/s)$^{-1}$. We do not propose, however, to attach importance to the precise value of $-1{\cdot}8$. In fact, our best result for the slope will turn out close to $-1{\cdot}6$. We do not regard the difference between this value and the recent Cambridge result as a significant discrepancy.

At $P > 3 \times 10^{-25}$ w.m^{-2} (c/s)$^{-1}$ the slope may well be less steep. Indeed, Mills (4) gives a slope no greater than $-1{\cdot}3$ at high flux values. Certainly it does not seem necessary for the theory to provide a slope steeper than the Euclidean value at large P. (The values of P mentioned above refer to a frequency of 160 Mc/s. This frequency will be taken as standard throughout the paper.)

2. *The general problem.*—Before the steady-state cosmology was proposed, ambiguities made the predictions of theory uncertain: three choices were available for the world curvature, and a double infinity of parametric values was also available for adjustment (the parameters being the cosmical constant and a constant of integration). Steady-state cosmology does not suffer from these ambiguities. There is no uncertainty concerning the line element, except for the numerical value of Hubble's constant. Once this has been fixed from observation, purely geometrical questions can be settled explicitly.

A further feature adds a still greater degree of definiteness. Because the time axis is open, into the past as well as the future, and because the line element is stationary (but not of course static), all observable properties must reproduce themselves, otherwise the continuing expansion would weaken and ultimately destroy the property in question. For example, galaxies must continue to condense, otherwise the spatial density of galaxies would tend to zero. Dynamo action must preserve any intergalactic magnetic field that may exist; and so on.

These advantages seem to have encouraged an erroneous optimism that the theory is capable of handling explicitly a physical property whose nature is not specified—for example, that the theory will predict the slope of the $\log N - \log P$ curve for radio sources without it being necessary to specify in any precise way the properties of radio sources. In a recent criticism of the steady-state theory Ryle and his colleagues (15) obtained a $\log N - \log P$ curve without apparently needing to use any of the physical properties of radio sources. Manifestly, there is a paradox in this, for one cannot count objects without some close specification of what are the objects to be counted. The paradox is actually resolved by the circumstance that the mathematical analysis necessary to arrive at the $\log N - \log P$ curve used by the Cambridge radio-astronomers itself imposes physical properties on the radio sources. If it should turn out that the properties thus imposed are indeed satisfied by the radio sources, then the claim that observation contradicts the steady-state theory may well be correct.

In the following sections we shall construct a counter-example, showing that properties can be imposed on the radio sources that lead to a $\log N - \log P$ in

* We are indebted to Drs Allen, Palmer and Rowson for the information that the effect reported by Mills has been investigated also at Jodrell Bank. The effect is found to be present, and of quantitatively the order stated by Mills.

satisfactory agreement with the observations. It will be necessary to use a definite picture of the mode of formation of galaxies. Before beginning the cosmological discussion it will be useful to review briefly the astrophysical features of this picture. It is perhaps relevant to add that the essential features of the following section are not new (cf. (**5**, **6**)). We shall therefore be using an astrophysical picture that arose quite independently of the present radio source problem, not a picture invented specifically for explaining the radio observations.

3. *Condensations in the steady-state theory.*—It has been realized for some years that the formation of galaxies, and of clusters of galaxies, implies the existence of a distance scale apparently quite independent of the cosmological distance scale. In the steady-state cosmology, for example, the constants appearing in the cosmological equations are c, H, and G; and from these cH^{-1} is a length. But this is the so-called "radius of the observable Universe", about 3000 mpc ($H = 100\ \mathrm{km\,s^{-1}\,(mpc)^{-1}}$), very much larger than ~ 3 mpc given by the galaxies and their clusters.

To obtain a distance scale appropriate to the galaxies and their clusters, it was suggested (**5**) that condensation may occur from an intergalactic gas at high temperature. Write V for the velocity of sound in the gas, then VH^{-1} is a length. To obtain ~ 3 mpc, with $H = 100\ \mathrm{km\,s^{-1}\,(mpc)^{-1}}$, V must be $\sim 300\ \mathrm{km\,sec^{-1}}$. It appeared immediately significant that this latter value is just of the order of the internal motions found in the galaxies, and also of the order of their peculiar motions.

It was further suggested (**5**) that condensation is promoted by cooling in the hot gas—locally-cooled regions being compressed by surrounding hotter gas. The idea that pressure gradients are necessary for producing condensation has been supported by an investigation of Harwit (**7**), who has shown that in the steady-state model condensation cannot take place through the operation of purely gravitational forces. This is clearly connected with the remarks of the previous paragraph—if condensation through gravitational forces could take place, a distance scale for the condensation process involving only cosmological constants would be expected to turn up in the theory, and this we know to be impossible. A similar argument applies to other systems of cosmology; since the appropriate scale for the condensations cannot be built from the cosmological constants c, G, λ, it is improbable that condensation can take place under purely gravitational forces. This argument is strongly supported by an investigation of Lifshitz (**8**). In other systems of cosmology it is difficult, however, to ascribe a high kinetic temperature to the intergalactic medium *at the onset of condensation of the galaxies* (a temperature of 10^7 °K is required at the expansion stage where the density falls below $10^{-25}\ \mathrm{g\,cm^{-3}}$).

Investigation of radiation by free-free electron-proton collisions showed that cooling can take place in a time scale of the required order, viz. the reproduction time of the steady-state theory, $\frac{1}{3}H^{-1}$, provided the gas density is not lower than $\sim 10^{-27}\ \mathrm{g\,cm^{-3}}$. Initially, the cooling is slow, the energy of contraction being mostly dissipated by radiation. A large measure of dissipation implies that a bound, stable structure will very likely be formed. However, after contraction to about $\frac{1}{3}$ of the initial dimension—i.e. to about 1 mpc, the density rising meanwhile to $\sim 10^{-26}\ \mathrm{g\,cm^{-3}}$—the cooling accelerates rapidly, and thereafter only a comparatively small fraction of the energy of contraction is dissipated as radiation. The energy of contraction then appears as dynamical energy, implying a fragmentation into sub-units, the sub-units being individual galaxies.

Cooling within an intergalactic medium of temperature $\sim 10^7$ °K and density 10^{-27} g cm^{-3} therefore fitted the broad requirements of a theory of the formation of galaxies and of clusters. It gave the dimensions, time-scales, and fragmentation properties correctly. Moreover, rapid cooling by hydrogen ceases at about 10^4 °K. The requirement that the hydrostatic pressure within and without a highly cooled zone be essentially the same, thus yielded a density $\sim 10^{-24}$ g cm^{-3} inside a highly cooled zone, and this is precisely the order of the densities that must have occurred in the galaxies *before* flattening to a disk structure took place in many cases. If, further, we ask: what mass of gas, at density $\sim 10^{-24}$ g cm^{-3}, can control dynamical motions of order 200–300 km sec^{-1}, the answer is about $10^{11} M_\odot$. This was an additional cogent result—the theory immediately determined the general order of the masses of the main class of galaxy (smaller galaxies were thought to be splinters from the dynamical disruption of lesser masses). Finally, the hydrostatic pressure in question, $\sim 10^{-12}$ dyne cm^{-2}, is a value that still affects the structure of our own galaxy in a remarkable degree.

So far, the former theory appeared to be on a sound basis—the results obtained were sufficiently numerous to be reckoned a satisfactory return for the single important hypothesis of a high kinetic temperature (the assumption of a gas density $\sim 10^{-27}$ g cm^{-3} cannot be regarded as a serious assumption, since this is just the general order of the mean densities actually found observationally to be present in many clusters of galaxies). To bring these considerations into closer relation with the steady-state cosmology it was found necessary to introduce further ideas, however, and these were more removed from the astronomical and astrophysical evidence. Accordingly, it is of great interest to find these further ideas turning out to be of particular relevance to the subject of the present paper. It does not seem likely that without these ideas the radio source counts could be explained within the framework of the steady-state theory.

The mean intergalactic density appearing in the steady-state theory is of order 10^{-29} to 3×10^{-29} g cm^{-3}, the exact value depending on H and on the particular mathematical formulation of the theory. Hence 10^{-27} g cm^{-3} cannot be used everywhere in space but only in regions that are already partially condensed. The question thus arose as to how such regions became partially condensed.

It was suggested that condensation occurs in two stages, a primary stage from 10^{-29} g cm^{-3} to 10^{-27} g cm^{-3}, and a secondary stage from $\sim 10^{-27}$ g cm^{-3} to $\sim 10^{-24}$ g cm^{-3}, the secondary stage being controlled by radiative cooling in the manner described above. Pressure gradients also play an important role in promoting primary condensation, the basic concept being one of equal hydrostatic pressure, the separate stages following the lines of the following table:

	Density (g cm^{-3})	Temperature (°K)	Distance scale (mpc)	Mass ($M_\odot$)
Before primary condensation	10^{-29}	10^9	30	10^{16}
After primary condensation	10^{-27}	10^7	3	10^{15}
After secondary condensation	10^{-24}	10^4	0·1	10^{11}

The differences of mass in the last column take account of the effects of fragmentation during condensation. Since the distance scales are not accurate within a factor ~ 2, the masses should not be taken as accurate within ~ 10, except in the last line of the table, where the mass is determined from the criterion of stability given above, not simply from multiplying volume by density.

A primary temperature of $\sim 10^9$ °K implies an energy of $\sim 2 \times 10^{17}$ erg per gram of hydrogen. It was noticed that this is just the amount of energy to be expected if the newly created matter of the steady-state theory consists of neutrons, allowance being made for the energy lost by neutrinos in the neutron decay. The existence of a high kinetic temperature could thus be explained in terms of the form of the newly created material.

The main question that now arose was: how does primary condensation take place? Not by radiative cooling; it was shown (5) that even under rather favourable conditions not more than about 5 per cent of the thermal energy can be dissipated by ordinary radiative processes. A possibility considered by Gold and Hoyle (9) was that, since the intergalactic material on the present picture possesses widely different temperatures in different places, "heat engines" may operate without the necessary dissipation needing to be even as high as 5 per cent. Such an engine could convert thermal energy almost entirely into the dynamical energy of mass motions of the gas. The precise nature of the heat engine was not specified, however, by Gold and Hoyle. It was thought that where the thermodynamic possibility exists for the operation of heat engines of great efficiency such engines are almost certain to arise. The situation has a logical analogy to the case of convection, where one considers the phenomenon to exist even though the detailed properties of the convective cells are not precisely specified.

If the fraction of energy dissipated in primary condensation is small, thermal energy being converted largely into the dynamical motions of a number of condensed units (which subsequently undergo secondary condensation) then the units are most unlikely to stay bound together as an aggregate. Rather do we expect primary condensation to result in a number of clusters of galaxies in motion with respect to each other at speeds of the order of the speed of sound in a gas of kinetic temperature 10^9 °K, viz. ~ 3000 km sec^{-1}. That is to say, we expect primary condensation to result in a number of comparatively dense blobs of gas ($\sim 10^{-27}$ g cm^{-3}) moving with speeds ~ 3000 km sec^{-1} with respect to each other. The largest of these blobs possess radii ~ 3 mpc, while the whole group of blobs possesses a radius ~ 30 mpc. For each blob of radius ~ 3 mpc there may well be many smaller blobs.

Although the blobs possess kinetic energy large enough to expand the group, there is no reason why the initial motions should all be directed radially outward. Inward motions can occur, and may well be more common than outward ones, particularly if the system has experienced compression from the surrounding hot gas. But inward motions can serve only to delay, and not to prevent, the eventual expansion of the group—inward motions eventually become outward motions. The situation is similar to an element of hot gas without bounding walls. Even without collisions between the constituent particles, the element soon begins to expand, and eventually all the particles move outward. In a similar way, the blobs of our group expand apart from each other, assisted in the cosmological case by the expansion of the Universe.

We may sum up the features of the picture as follows:

(i) A primary condensation consists of an appreciable number of blobs of gas with density $\sim 10^{-27}\,\mathrm{g\,cm^{-3}}$.

(ii) *All galaxies forming within the blobs of a particular primary condensation are age-correlated.*

(iii) The blobs move initially with criss-crossing motions, their velocities with respect to each other being $\sim 3000\,\mathrm{km\,sec^{-1}}$.

(iv) After the initial criss-crossing, the blobs expand apart from each other.

(v) The whole age-correlated region has an initial radius $\cong 30$ mpc, and may contain from $\sim 10^5$ galaxies.

(vi) The largest of the blobs may contain upwards of 10^3 galaxies, but small blobs are probably much more common than large ones.

The need for criss-crossing motions, followed by expansion, was noted in (**5**), where it was pointed out that an intergalactic magnetic field cannot be maintained unless some such phenomenon exists.

We have now reached the stage where it is necessary to consider the relation of primary condensations one to another. This will be conveniently treated in the following section. It remains in this section only to point out that in special cases blobs with a primary condensation may possess small relative velocities, and may come close enough to enable them to resist the general expansion of the Universe. It is of interest in this connection that both Shane and Wirtanen (**10**) and Abell (**11**) find cases where clusters of galaxies form groups of dimensions ~ 20 mpc, a value satisfactorily close to the radii of our primary groups.

(Although Abell gives a value ~ 20 mpc, this refers to $H \cong 180\,\mathrm{km\,s^{-1}\,(mpc)^{-1}}$. For the distance scale of the present paper, $H = 100\,\mathrm{km\,s^{-1}\,(mpc)^{-1}}$, Abell's value should be increased to ~ 40 mpc.)

4. *The discrete model.*—The existence of a condensation distance scale introduces a discrete element into the steady-state theory. In accordance with the discussion of the previous section, this distance scale will be taken as that associated with the primary condensations.

To preserve the steady-state characteristic it is necessary that every feature of the Universe shall repeat itself in a time interval $\sim \frac{1}{3}H^{-1}$. The distribution of primary condensations existing at any moment is maintained, in spite of their expansion apart from each other, by the formation of new condensations. The interval $\frac{1}{3}H^{-1}$ may be spoken of as the "length of a generation".

The discreteness referred to above is inherent in the theory. For the sake of simplicity, we shall introduce a second discreteness representing a convenient approximation, recognizing however that this second discreteness would not enter into a strict theory. We shall regard the inherently discrete primary condensations as arising at particular moments of time, the moments being separated by interval $\frac{1}{3}H^{-1}$. Suppose that time t is one of these moments. Then at time t a system of primary condensations comes discretely into being. Between t and $t+\frac{1}{3}H^{-1}$ no further primary condensations arise, the system of condensations born at time t simply expand apart from each other in accordance with the de Sitter expansion factor $R(t) = \exp(Ht)$. But at time $t+\frac{1}{3}H^{-1}$ a new system of primary condensations again comes into being, the individual condensations and their mean spacing apart being exactly the same as was the case for the system born at time t. And so on for subsequent generations.

And we shall introduce the further artificial discreteness that the lattice formed by the primary condensations of each generation always has the same structure. For simplicity, we take the lattice to be cubic, and we also take the primary condensations to be spheres having centres at the lattice points of the cubic structure. After an initial delay we expect each primary condensation to join in the general expansion—once the delay caused by the criss-crossing motions discussed in the previous section is over, expansion of the lattice units keeps step with the expansion of the lattice itself. That is to say, the radius of a lattice unit maintains a constant ratio to the lattice distance. We take the ratio to be $\frac{1}{3}$. This requires the lattice units of each generation to be well separated, since the choice of this ratio requires the distance separating two neighbouring condensations A and B to be made up in the following way: $\frac{1}{3}$ in A, $\frac{1}{3}$ in neither A nor B, $\frac{1}{3}$ in B.

Our choice of this particular ratio has been guided by the consideration that if the lattice units were initially almost touching each other, and if a time $\sim \frac{1}{3}H^{-1}$ be allowed for the delay in the expansion of the lattice units, then the ratio must be $\frac{1}{2}e^{-1/3}$, which is close to $\frac{1}{3}$.

Write a_1 for the radius of each lattice unit and l_1 for the lattice spacing at the beginning of uniform expansion. Then, from what has been said, $a_1 \sim 30$ mpc, $l_1 \sim 90$ mpc. The precise numerical values of a_1, l_1 will not affect the following work in any important respect—their values could be changed by a factor of at least 2 without appreciable consequences. On the other hand, the ratio a_1/l_1 does turn out to be important. A decrease in this ratio would make the number count problem simpler, although a marked decrease would probably raise a different difficulty in that non-isotropic effects would then arise in marked degree. The latter question will be taken up later in the paper.

It is convenient to describe the lattice characterized by (a_1, l_1) as the "first generation" lattice. This will be the "youngest" lattice that an observer will expect to find. The observer will also detect older lattices. Those older by $(n-1)/3H$, $n=2, 3, \ldots$ will possess lattice units and lattice spacing characterized by (a_n, l_n) given by

$$(a_n, l_n) = e^{n-1/3}(a_1, l_1). \tag{1}$$

We shall refer to (a_n, l_n) as the "nth generation lattice". Allowing for the initial delay period of $\frac{1}{3}H^{-1}$, the age of the nth generation is just $\frac{1}{3}nH^{-1}$.

The galaxies to be found in the lattice units of a particular generation will not possess ages as long as that of the lattice in question, since an interval of time must elapse before the galaxies and the stars within them are formed. We tentatively set this interval as $\sim \frac{2}{3}H^{-1}$. Then galaxies containing stars appear in the second generation. Galaxies possessing stars of age $\sim H^{-1}$ appear, however, only in the 5th generation. The 5th generation lattice has $a_5 \sim 100$ mpc, $l_5 \sim 300$ mpc, although once again, we note that these numerical values could easily be in error by a factor ~ 2.

Since the majority of definitive identifications of radio sources are with giant E galaxies, and since the stars of E galaxies give a late colour index, and are almost certainly at least of age H^{-1} (**12**), it appears that radio sources occur mainly in old lattices, in generations later than the 5th. We therefore see that the discreteness inherent in steady-state cosmology must enter the radio source problem on a scale at least as great as $l_5 \sim 300$ mpc. For the most part in what follows we shall be able to employ simple averages derived from the usual continuous form of the steady-state theory. The present discrete picture is not then needed. At certain stages of the argument, however, the discreteness of the condensation picture is required, indeed at one crucial point. It is important to notice that this crucial aspect is concerned with the inherent discreteness of the condensation picture, not with the artificial discreteness of the sequence of regular lattices.

Several further points will be used at later stages. The probability that an observer at random lies inside a lattice unit belonging to the nth generation is given by

$$f = \frac{4\pi}{3}\left(\frac{a_n}{l_n}\right)^3 = \frac{4\pi}{3}\left(\frac{a_1}{l_1}\right)^3 \cong \frac{1}{6\cdot 4} \tag{2}$$

the ratio $\frac{1}{3}$ being used for a_1/l_1. The probability is therefore the same for all generations.

Although f is appreciably less than unity, the observer must lie in lattice units belonging to an infinite sub-set of $n = 1, 2, \ldots$. But since the space density of galaxies weakens with increasing n as $\exp(-n)$ the main contribution to the density of galaxies at the observer's position comes from the smallest value of n in the sub-set.

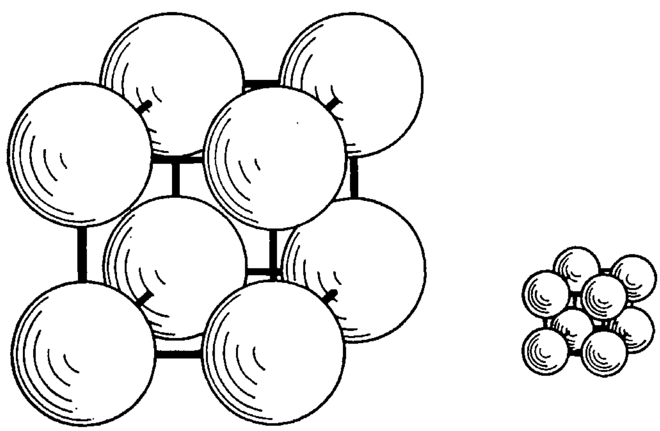

FIG. 1

The oldest stars in our Galaxy appear to have ages of order $1\cdot 5 \times 10^{10}$ years $\cong 4H^{-1}/3$. This implies that we ourselves are inside a sixth generation lattice unit. We may also lie inside a younger lattice unit, and we must certainly lie inside older systems. The latter make no more than a small contribution to the total density of galaxies, while galaxies belonging to any younger lattice may still be largely undeveloped. In this connection we note that our presence here on the Earth guarantees some such situation, since our presence demands a galaxy, stars, planetary formation, etc.

It has sometimes been objected to the steady-state theory that the mean age of all galaxies should be $\frac{1}{3}H^{-1}$, whereas in our locality observation suggests ages that are generally of order H^{-1}. We now see that our presence inside a fifth or sixth generation lattice unit would give systematic ages $\sim 1\cdot 5 \times 10^{10}$ years for the developed galaxies around us.

In Fig. 1 we show the forms of lattices for the first and the fourth generations, the scale in the latter case being greater by e. The positions of the centres of the lattice units of any particular generation are probably *anticorrelated* with those of preceding and succeeding generations. This is because the formation of a primary condensation at a particular place probably inhibits the formation of a condensation at the same place in the succeeding generation. Since f is only $\sim \frac{1}{8}$ there is opportunity for the lattice units of any generation to form in the interstices of the lattice of the preceding generation.

5. *The general mathematical formulation.*—The line element of the steady-state theory is

$$ds^2 = c^2\,dt^2 - \exp(2Ht)[dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)], \tag{3}$$

where $r = 0$ is the observer's position. Our discussion of the lattice distances, given in the preceding section, refers to a particular value of t, distances being given by $\exp(Ht)\int du$. The line integral with respect to u is here to be taken along a straight line in the Euclidean sub-space

$$du^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2). \tag{4}$$

Indeed, if we choose the arbitrary zero of t to be the present, then the present lattice distances are simply coordinate distances.

The lattice points of any generation possess coordinates r, θ, ϕ independent of t. And once the lattice units reach the stage of uniform expansion, the clusters of galaxies that make up each lattice unit may also be taken as having constant coordinates, if we neglect the finite size of a cluster.

Taking $t = 0$ as the present, we ask: what is the r-coordinate of a galaxy that we observe (at the present) to give a red-shift $\Delta\lambda/\lambda$? The answer in the steady-state theory is

$$r = cH^{-1}z, \tag{5}$$

where

$$z = \frac{\Delta\lambda}{\lambda}. \tag{6}$$

A galaxy of intrinsic bolometric luminosity L and red-shift z will be observed as having an apparent bolometric intensity P given by

$$P = \frac{LH^2}{4\pi c^2 z^2 (1+z)^2} \tag{7}$$

where P is the energy flux normal to the direction θ, ϕ of the galaxy in question.

Suppose now that we wish to count the number of galaxies brighter than some specified P, on the basis that all galaxies possess the same intrinsic L. First we define a value of z, z_m say, by solving (7) for z, using the specified P. Our problem is to count the number of galaxies with $z < z_m$. We attempt this, first using the simple continuous steady-state theory. In this the number of galaxies per unit proper volume is a constant, independent of the epoch. We consider observations made at $t = 0$. The proper volume in the coordinate range r to $r + dr$ is $4\pi r^2\,dr$ at $t = 0$. The number of galaxies in this coordinate range is therefore

$$(\text{constant}) \times r^2\,dr.$$

By (5), the coordinate range r to $r + dr$ corresponds to a red-shift range z to $z + dz$, where z is given by (5), and $dz = c^{-1}H\,dr$. Thus the number of galaxies giving red-shifts in the range z to $z+dz$ is of the form

$$(\text{constant}) \times z^2\, dz.$$

Hence, at first sight, the answer to our problem is given by integrating the latter expression from $z=0$ to $z=z_m$, viz. (constant) $\times z_m^3$. Since the constant is independent of the epoch, and also of P, a similar calculation for varying P gives the number count as a function of P. The count is not absolute, of course, since we have not specified any absolute proper density of galaxies.

This simple procedure overlooks, however, the point that some of the galaxies we have counted, and which now exist at $t=0$, did not exist at the time the radiation would need to have been emitted in order to reach us at the present day. For red-shift z the radiation would need to have been emitted a time $H^{-1}\ln(1+z)$ ago.

To cope with this complication we must introduce the age distribution of the galaxies. In the continuous steady-state theory the number of galaxies with present-day ages between τ and $\tau+d\tau$ is

$$(\text{constant}) \times \exp(-3H\tau)\, d\tau \text{ per unit proper volume.}$$

Thus if we are considering galaxies giving red-shifts between z and $z+dz$ we must restrict ourselves to those in the age distribution with $\tau \geqslant H^{-1}\ln(1+z)$. This number is

$$(\text{constant}) \times z^2\, dz \int_{H^{-1}\ln(1+z)} \exp(-3H\tau)\, d\tau, \tag{8}$$

where the upper limit of the integration with respect to τ remains to be stated.

The three-dimensional sphere at $t=0$ defined by a red-shift z, or by the corresponding $r=cH^{-1}z$, must now be compared with the lattice spacings of our model discussed in the previous section. If for generation n the lattice distance l_n exceeds $2r$ then certainly not more than one lattice unit of the nth generation can fall completely inside the sphere. The phenomenon is then essentially discrete. Continuous averages are no longer applicable.

The point of transition from the continuous picture to the discrete model is of necessity rather uncertain. We could, for instance, require that at least *two* adjacent lattice units of generation n should fall completely inside the sphere of radius r in order that the continuous picture be applicable up to the age implied by generation n. This requires $l_n < 6r/5$.

To deal with the slight measure of uncertainty involved in the transition to discreteness we apply the continuous picture up to a generation n defined by

$$l_n = kcH^{-1}z, \tag{9}$$

where k is a constant between 1 and 2. In a later section we shall show that the precise value of k is not important.

It remains to relate l_n to the lattice distance of the first generation. We have

$$l_n = l_1 \exp\frac{n-1}{3} = cH^{-1}z_1 \exp\frac{n-1}{3}.$$

With l_1 and z_1 specified for the first generation, the limit of applicability of the continuous theory occurs at generation n, determined by solving

$$kz \simeq z_1 \exp\frac{n-1}{3}. \tag{10}$$

The age of the nth generation is $\frac{1}{3}nH^{-1}$. By (10) this is

$$H^{-1}\left(\tfrac{1}{3}+\ln\frac{kz}{z_1}\right).$$

However, this does not give the correct upper limit of the integral in (8), since the age of generation n has been reckoned from the time of formation of the primary condensations, whereas the zero of τ in (8) refers to *galaxies* of zero age. There is an interval of time, Δ say, between the formation of a primary condensation and the formation of galaxies within it. Above we took Δ to be $\sim\frac{2}{3}H^{-1}$. The correct upper limit is thus

$$H^{-1}\left(\tfrac{1}{3}+\ln\frac{kz}{z_1}\right)-\Delta,$$

which can be written as $H^{-1}\ln kz/z_1'$, where

$$z_1'=z_1\exp\left(H\Delta-\tfrac{1}{3}\right). \tag{11}$$

Thus the count for the range z to $z+dz$, given by the continuous picture, is

$$(\text{constant})\, z^2\,dz\int_{H^{-1}\ln(1+z)}^{H^{-1}\ln\frac{kz}{z_1'}}\exp(-3H\tau)\,d\tau. \tag{12}$$

The transformation $\chi=H\tau-\ln(1+z)$ reduces (12) to

$$(\text{constant})\ .\ \frac{z^2\,dz}{(1+z)^3}\int_0^{\ln\frac{kz}{(1+z)z_1'}}\exp(-3\chi)\,d\chi. \tag{13}$$

To obtain the total count down to flux level P, i.e. for z up to z_m, we integrate (13) as follows

$$(\text{constant})\ .\ \int_{\frac{z_1'}{k-z_1'}}^{z_m}\frac{z^2\,dz}{(1+z)^3}\int_0^{\ln\frac{kz}{(1+z)z_1'}}\exp(-3\chi)\,d\chi. \tag{14}$$

The integral is taken over an area in the (χ, z) plane enclosed by the z-axis, the line $z=z_m$, and the curve

$$\chi=\ln\frac{kz}{(1+z)z_1'}. \tag{15}$$

This gives the number of galaxies brighter than P, *in so far as this can be determined from the continuous theory.*

Over and above (14) there is the possibility that a further contribution arises from discrete effects. These do not have much importance, except in the case where a sphere of radius r, $r=cH^{-1}z$, $(z<z_m)$, is invaded by a lattice unit of a generation even older than the value of n corresponding to $z=z_m$ in (10). Such a case represents a genuine fluctuation that must be given special consideration. We shall return to this question at a later stage.

It is already a matter of interest that (15) vanishes for

$$z_m=\frac{z_1'}{k-z_1'}\simeq\frac{z_1'}{k}\qquad (k\sim 1\cdot 5\quad\text{and}\quad z_1'\ll 1).$$

This implies that as z_m increases above $\sim z_1'/k$ the product $NP^{3/2}$ increases to a maximum value and thereafter declines. Hence we have a $\log N-\log P$ curve that has a slope initially greater than $-1 \cdot 5$. This is not our answer to the criticism of Ryle, since the increased slope only applies for z_m somewhat greater than z_1'/k, which will turn out to be about $1/15$, whereas the observed slope maintains its steepness up to $z_m \cong 0 \cdot 3$ or $0 \cdot 4$. Ryle has argued that the slope is maintained up to $z_m = 1$ or more, but it seems to us that the evidence concerning angular diameters of radio sources scarcely supports this extreme view.

The simplest hypothesis capable of steepening the $\log N - \log P$ curve at a larger value of z_m is the following: only galaxies of age greater than T, say, become strong radio sources. We must then count the sub-class of galaxies with age greater than T. The number of these galaxies varies with z_m, and hence with P according to

$$(\text{constant}) \int^{z_m} \frac{z^2\,dz}{(1+z)^3} \int_{HT}^{\ln \frac{kz}{(1+z)z_1'}} \exp(-3\chi)\,d\chi. \tag{16}$$

In order that the integral with respect to χ shall yield a positive result it is necessary that

$$\frac{kz}{(1+z)z_1'} \geqslant \exp(HT). \tag{17}$$

For this to be certainly the case the lower limit of the integral with respect to z must be taken at the value of z given by considering the equality sign in (17). The count can only be applied for values of z_m greater than this particular value of z. Indeed, the count falls to zero at z_m equal to this z, and the $\log N - \log P$ curve is steeper than $-1 \cdot 5$ for z_m somewhat greater. That is to say, the slope exceeds $-1 \cdot 5$ for z_m somewhat greater than the value of z given by

$$\frac{kz}{(1+z)z_1'} = \exp(HT). \tag{18}$$

For T between $2H^{-1}$ and $3H^{-1}$ this yields a value of z of $\sim 0 \cdot 3$.

These considerations are a special case only, however, of the general situation that arises when the probability of a galaxy being a radio source is a function of the age of the galaxy. We consider the problem in the following terms.

(1) It may be that only galaxies with some special property become radio sources of the intense type that contributes appreciably to the $\log N - \log P$ curve. Examples of such properties might perhaps be those galaxies that are members of physically connected pairs, or perhaps E-type galaxies. So long as the property is not age-correlated, however, we simply choose out the sub-class of galaxies that possess the property, or properties, in question. This changes the counting scale but not the relative counts for variable P.

(2) The intrinsic radio luminosity will certainly not be the same for all radio sources. So long as the intrinsic luminosity is not age-correlated, however, it is possible to represent the luminosity function by a histogram, considering each section of the histogram in the manner described above, each section leading to an expression of the form (14), but each with a different z_m (for the same P). Varying P gives a $\log N - \log P$ curve for each section of the histogram. If none of these separate curves possesses slope greater than $-1 \cdot 5$, the combined curve cannot possess slope greater than $-1 \cdot 5$. The luminosity function can be represented to any required degree of accuracy by taking a sufficient number of sections in the histogram.

(3) The situation is changed only if either the probability of a galaxy being a radio source or the luminosity function is age-dependent (or of course both of these). The dependence of the probability on age we denote by $K(\tau)$. To deal with an age dependence of the luminosity function, we represent the luminosity function by a histogram at each value of τ, the histograms being chosen in the following related way. The luminosity function is first normalized to unit area for each τ, and the histograms are then made up of the same number of sections of the same areas. At different τ the sections of the histograms are related serially, the first to the first, the second to the second, and so on. In the following considerations we consider only one set of related sections of the histograms, and we denote the dependence of the intrinsic radio emission for this section by $L(\tau)$. The results for other sets are similar to those obtained below.

Consider galaxies with present ages between τ and $\tau+d\tau$ giving red-shifts between z and $z+dz$. The number of these galaxies that are radio sources is given by an expression of the form

$$(\text{constant})\, z^2\, dz K(\tau - H^{-1}\ln \overline{1+z}) \exp(-3H\tau)\, d\tau \tag{19}$$

provided the values of z, τ can be related within the terms of the continuous theory. The relevant condition is that already discussed above, that

$$H\tau \leqslant \ln \frac{kz}{z_1'}. \tag{20}$$

Introducing $\chi = H\tau - \ln(1+z)$ (19) and (20) can be written as

$$(\text{constant}) \quad \frac{z^2}{(1+z)^3} K(\chi) \exp(-3\chi)\, d\chi\, dz, \tag{19'}$$

$$\chi \leqslant \ln \frac{kz}{(1+z)z_1'}. \tag{20'}$$

A second condition is imposed by the requirement that the apparent bolometric flux be greater than P, viz.

$$P \leqslant \frac{H^2 L(\chi)}{4\pi c^2 z^2 (1+z)^2}. \tag{21}$$

The radio astronomer does not of course measure the bolometric flux, but the flux per unit frequency range at some specified frequency. However, it can readily be shown that for sources with frequency spectra $d\nu/\nu$ the measurement of flux per unit frequency range yields the same relative counts of sources as does the total bolometric flux. Most radio sources possess spectra of about the form $d\nu/\nu^{0\cdot 8}$. This we regard as being sufficiently close to $d\nu/\nu$ for the difference to be neglected. The effect of $d\nu/\nu^{0\cdot 8}$, and of regarding P as the flux per unit frequency range, is to replace the factor $(1+z)^2$ in (21) by $(1+z)^{1\cdot 8}$. We have not troubled to take account of this slight modification.

To obtain the number of sources with flux greater than P, (19′) must be integrated over the portion of the χ, z plane defined by the conditions (20′) and (21). The (χ, z) plane is shown in Fig. 2. First, draw the curve

$$\chi = \ln kz/(1+z)z_1'.$$

Next draw the curve along which

$$z^2(1+z)^2 = \frac{H^2}{4\pi c^2 P} L(\chi). \tag{22}$$

The relevant area is bounded by these two curves and by the z-axis, in the manner shown in Fig. 2.

It will be noticed that the age dependence of L does not affect the weighting function (19′), although it does affect the area of integration. The two functions $K(\chi)$, $L(\chi)$ do not therefore affect the counts in the same way. Indeed, it turns out that while $K(\chi)$ can have an extremely important effect on the $\log N - \log P$

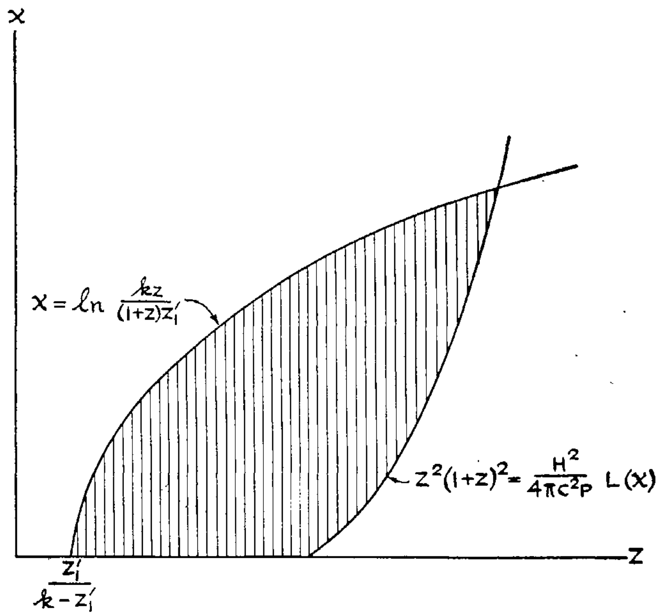

FIG. 2

curve, the dependence of the luminosity L on age is much less effective. This is demonstrated by an explicit example given in the Appendix. Since the form of $L(\chi)$ is comparatively unimportant, it will be sufficient in later work to take L independent of χ, keeping only the probability function $K(\chi)$. In this case the general integral reduces to a form very similar to (14), viz.

$$(\text{constant}) \int_{\frac{z_1'}{k-z_1'}}^{z_m} \frac{z^2\,dz}{(1+z)^3} \int_0^{\ln\frac{kz}{(1+z)z_1'}} K(\chi)\exp(-3\chi)\,d\chi. \tag{23}$$

Writing $q = \exp\chi$ gives the somewhat more convenient expression

$$(\text{constant}) \int_{\frac{z_1'}{k-z_1'}}^{z_m} \frac{z^2\,dz}{(1+z)^3} \int_1^{\frac{kz}{(1+z)z_1'}} K(q)\frac{dq}{q^4}. \tag{24}$$

6. *The basic $\log N - \log P$ distribution.*—In order that (24) give a result of an essentially new character, it is necessary that an appreciable contribution to the integral with respect to q shall come from the upper part of the range of q. This will be the case if $K(q)$ varies with q according to some power $\geqslant 3$. The simplest case is where $K \propto q^4$, when the integral is equally weighted over the whole range of q. We shall adopt this distribution throughout the following work, noting, however, that essentially similar results are obtained for any power law between

about 3 and 5. In this connection it may be noted that it is always possible, over a limited range of q, to represent any general function $K(q)$ by a power law. Thus $K\propto q^4$ simply means that over the range of q in question we regard K as being represented by this particular power law. It will turn out that the relevant range of q covers the generations of our lattice model from $n=5$ to $n=10$, and that values of z from about 0·1 to about 0·3 are mainly involved. The corresponding range of q is from 1 to about 3. Our dependence $K\propto q^4$ therefore implies that the probability of a galaxy (perhaps of a certain restricted type —e.g. members of pairs) being a radio source increases over the relevant generations by a factor $\sim 3^4$. A fuller discussion of these questions will be given at a later stage when numerical results have been derived, and when the various other physical factors of relevance have been discussed.

With $K\propto q^4$, (24) gives

$$\text{(constant)} \int_{\frac{z_1'}{k-z_1'}}^{z_m} \frac{z^2\,dz}{(1+z)^3}\left[\frac{kz}{(1+z)z_1'}-1\right]. \tag{25}$$

An important point now emerges. We shall be concerned mainly with values of z_m in the range 0·15 to 0·3. Since the integrand of (25) is heavily weighted towards its upper limit, we can consider the factor

$$\frac{kz}{(1+z)z_1'}-1$$

for values of z near z_m. It is seen then that $kz/(1+z)z_1'$ appreciably exceeds unity (k we expect to lie between 1 and 2, and k/z_1' will be found to be ~ 15). This means that, without severe approximation, we could neglect the -1 in this factor, in which case the uncertain factor k simply goes outside the integral and is absorbed into the multiplying constant. Hence we see that the choice of k does not affect the $\log N-\log P$ curve in any really important degree.

To obtain numerical values, k/z_1' must be specified. The value of z_1' is related to the first generation lattice by

$$z_1'=z_1\exp\left(H\Delta-\tfrac{1}{3}\right),$$

where $z_1=cH^{-1}l_1\simeq 0{\cdot}03$ for $l_1\simeq 90$ mpc ($H=100$ km s^{-1} (mpc)$^{-1}$) and Δ is the time delay between the formation of a primary condensation and the development of the first radio sources within it (more precisely of the first radio sources that make an appreciable contribution to the count). The optical identifications available show a marked association of radio sources with galaxies of late colour index, the stars of which can scarcely have ages less than H^{-1}. Thus, allowing $\frac{2}{3}H^{-1}$ for galaxy formation, together with a further H^{-1} for the ageing of the stars we have $z_1'\simeq z_1\exp\frac{4}{3}=0{\cdot}11$ for $z_1=0{\cdot}03$. The value given to k defines the cut-off from the continuous theory. We have seen that a value between 1 and 2 would seem appropriate. With $k\simeq 1{\cdot}6$ $z_1'\simeq 0{\cdot}11$, $k/z_1'\simeq 15$. This will be the value used in succeeding work. As already emphasized, the results are not sensitive to this particular choice.

The integral (25) can be evaluated explicitly to give

$$\left[\left(1-\frac{z_1}{k}\right)\log(1+z)+\frac{3z^2+2z}{2(1+z)^2}\cdot\frac{z_1'}{k}-\frac{11z^3+15z^2+6z}{6(1+z)^3}\right]_{\frac{z_1'}{k-z_1'}}^{z_m}. \tag{26}$$

Taking $k/z_1' = 15$ counts on an arbitrary scale can be worked out numerically as a function of z_m. Since $P \propto z_m^{-2}(1+z_m)^{-2}$ the corresponding energy fluxes can also be given on an arbitrary scale. (It will be recalled that so far we are only considering sources of a single fixed intrinsic luminosity.) Results are given in Table I. The associated value of $NP^{3/2}$, again on an arbitrary scale, are also given in Table I.

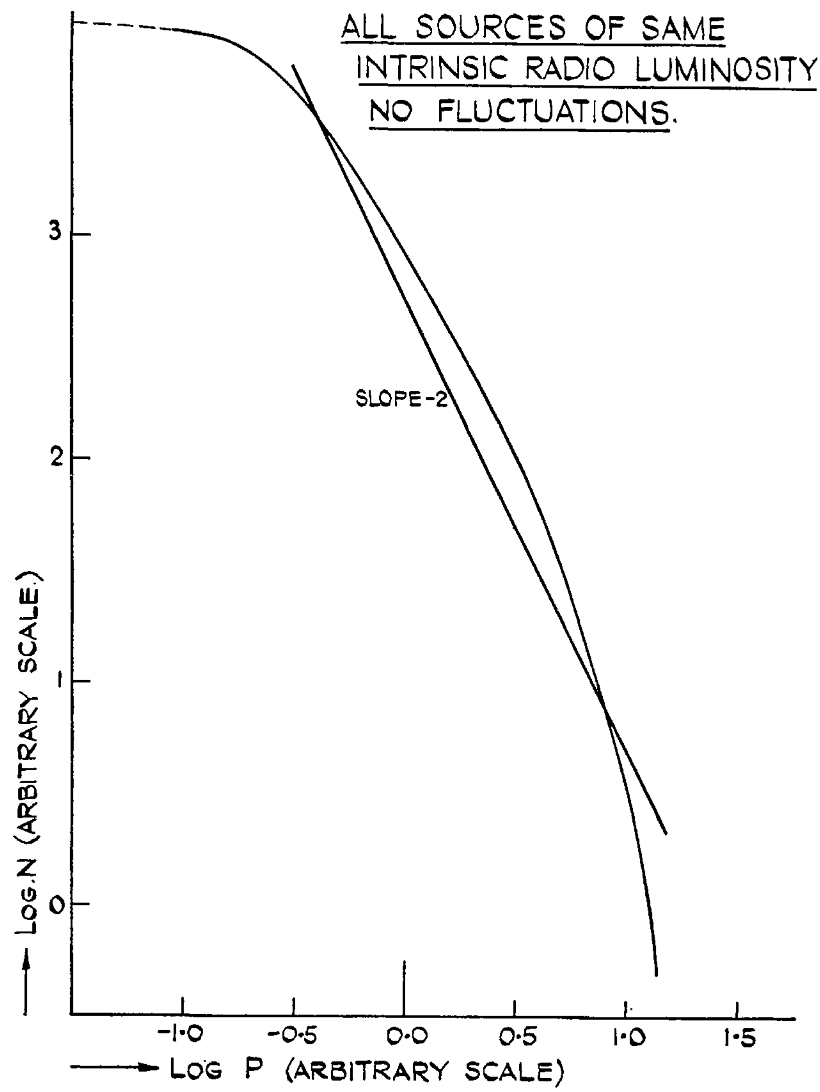

FIG. 3

The $\log N - \log P$ curve is plotted in Fig. 3. It is evidently steeper than $-1{\cdot}5$ over the range from $z = 0{\cdot}1$ to $z = 0{\cdot}3$. This range of z corresponds to a change by a factor $12{\cdot}5$ in P, as compared to about a factor 15 for the results upon which Ryle based his criticism of the steady-state theory. The slope of the $\log N - \log P$ curve is close to $-1{\cdot}8$ for z between $0{\cdot}2$ and $0{\cdot}3$, and the slope even exceeds -2 for z between $0{\cdot}1$ and $0{\cdot}2$.

To relate these results to observation it is necessary that the hitherto arbitrary scales of N and P be fixed so that N and P are in reasonable relationship to the values obtained in the observational surveys. To give a suitable association we take the unit of P to be 10^{-26} w.m^{-2} (c/s)$^{-1}$ and we regard N as the number of

TABLE I

z	·08	·09	·1	·11	·12	·13	·14	·15	·175	·2	·25	·3	·4	·5	·6
log P	1·61	1·50	1·40	1·31	1·23	1·15	1·08	1·01	·86	·72	·49	·30	$\bar{1}$·99	$\bar{1}$·73	$\bar{1}$·52
N	·15	·81	2·20	4·56	8·06	13·0	19·4	27·9	58·8	107	269	542	1356	2385	2498

TABLE II

I: 10^{26}w (c/s)$^{-1}$ at 160 Mc/s^{-1}. II: 3×10^{26}w (c/s)$^{-1}$. III: 10^{27}w (c/s)$^{-1}$. IV: 3×10^{27}w (c/s)$^{-1}$. V: 10^{28}w (c/s)$^{-1}$. VI: 3×10^{28}w (c/s)$^{-1}$
(P in 10^{-26}w.m^{-2}(c/s)$^{-1}$)

log P (10^{26}w.m^{-2}(c/s)$^{-1}$)	−·75	−·5	−·25	0	·25	·5	·75	1	1·25	1·5	1·75	2
log $NP^{3/2}$ (Arbitrary scale for N)	·56	·76	·92	·99	1·04	1·03	1·01	·98	·93	·88	·83	·76
I	85	74[21]	59	48[19·75]	35	26[18·5]	17	14[17·25]	13	F[16]	F	F[14·75]
II	15	26[22·25]	7	43[21]	45	44[19·75]	38	30[18·5]	19	18[17·25]	16	F[16]
III			4	9[22·25]	8	25[21]	31	31[19·75]	28	24[18·5]	15	15[17·25]
IV					2	5[22·25]	11	16[21]	19	22[19·75]	19	20[18·5]
V							3	9[22·25]	20	32[21]	40	47[19·75]
VI									1	4[22·25]	10	18[21]

The values entered for the six luminosity classes give the percentage of each class at the various flux levels. F represents a fluctuation contribution. Estimated optical photographic magnitudes in brackets [ ], uncorrected for red-shift.

sources per steradian. This transfer to absolute scales yields information concerning absolute values of the probability function K and of the intrinsic emission L. We shall defer a discussion of the probability function until a later section.

The value of L can now be worked out from (7), by inserting $z = 0{\cdot}3$, $P = 2 \times 10^{-26}\,\text{w.m}^{-2}\,(\text{c/s})^{-1}\,\text{s}^{-1}$. The result is $L = 3{\cdot}3 \times 10^{26}\,\text{w}\,(\text{c/s})^{-1}$, the values of P and L being taken at ~ 160 Mc/s. For a source with spectrum $d\nu/\nu^{0{\cdot}7}$ extending up to $\nu = 3000$ Mc/s, the total energy emission would be $\sim 4 \times 10^{42}\,\text{log}\,\text{s}^{-1}$, a value that falls within a factor of about 2 of the total emission of the source Hydra A. It is of course the case that a value must be ascribed to H in (7). In the present calculation $H = 100\,\text{km}\,\text{s}^{-1}\,(\text{mpc})^{-1}$ was used.

7. *Fluctuations in the number count.*—Two quite different types of fluctuation enter the present theory: fluctuations in the number count due to the presence of the observer in or near one or more lattice units of the discrete model, and fluctuations from isotropy in the distribution of sources on the sky. The two types of fluctuation are affected by the discrete properties of the model in markedly different ways. The effect on the count is more extreme when the observer lies inside a lattice unit. This lifts the $\log N - \log P$ curve at large values of P, thereby reducing the overall slope of the curve to a greater degree than if the observer were to lie immediately outside a lattice unit—in the latter case the curve is unaffected at both large and small values of P, but receives a wiggle at intermediate values. Departures from isotropy only become an issue, on the other hand, when the discrete lattice units are projected on to small areas of the sky. Hence departures from isotropy must be considered for the case where the observer is distant from a lattice unit, not when he is inside it. The question of non-isotropy will be considered later in the paper. For the present, we are concerned with effects on the number count, particularly with placing the observer inside a lattice unit, or units.

First we note that an observer cannot lie in more than one lattice unit of a particular generation. If he lies in two lattice units of different generations, one of the units will have appreciably greater weight than the other. Thus if the two units belong to generations with ages great enough for them both to belong to the range of q in which $K(q)$ is rapidly increasing, but not so great that $K(q)$ has reached an ultimate saturation level, then the older of the two units contributes the greater weight. It follows that we need only consider the presence of the observer inside one lattice unit, the one giving the greatest weight.

The lattice units are never so large that the $1 + z \simeq 1$ cannot be used as a tolerable approximation for the sources within the particular lattice unit in which the observer happens to be situated. These particular sources, taken by themselves, therefore follow a $\log N - \log P$ curve with slope close to $-1{\cdot}5$. The general effect is shown by Fig. 4, where $\log NP^{3/2}$ is plotted against $\log P$. Curve I is simply a plot of the values of Table I. At large P, the curve falls steeply away. Curve II of Fig. 4 shows the effect of a fluctuation. At large P the fluctuation dominates and $NP^{3/2}$ is effectively constant. Then as P decreases the curve first falls to a minimum, and thereafter rises steeply. The minimum is caused by reaching the limit of the local lattice unit at $\log P \simeq 2{\cdot}0$, the contribution of the lattice unit ceasing at this value of P.

The reason why the fluctuation has been set at the level shown in Fig. 4 will now be explained. We have seen that the age of our own Galaxy suggests that we are situated inside a lattice unit belonging to about the 6th generation. The 6th generation lattice has spacing $l_1 \exp 5/3$. The distance of an observer at the centre of a unit cube of this lattice from each of the nearest eight vertices is $(\sqrt{3}/2)l_1 \exp 5/3$. This distance corresponds to a red-shift $(\sqrt{3}/2)c^{-1}Hl_1 \exp 5/3$. With $l_1 \cong 90$ mpc, $H = 100\,\text{km sec}^{-1}\,(\text{mpc})^{-1}$ (we have used these values throughout), the resulting value is 0·137. According to Table I the value of N at $z = 0{\cdot}137$ is ~ 17. If the whole of this count were contributed by the 6th generation alone then each of the eight lattice units of the unit cube would contribute ~ 2 to the count. This would ignore, however, the contribution of the 5th generation,

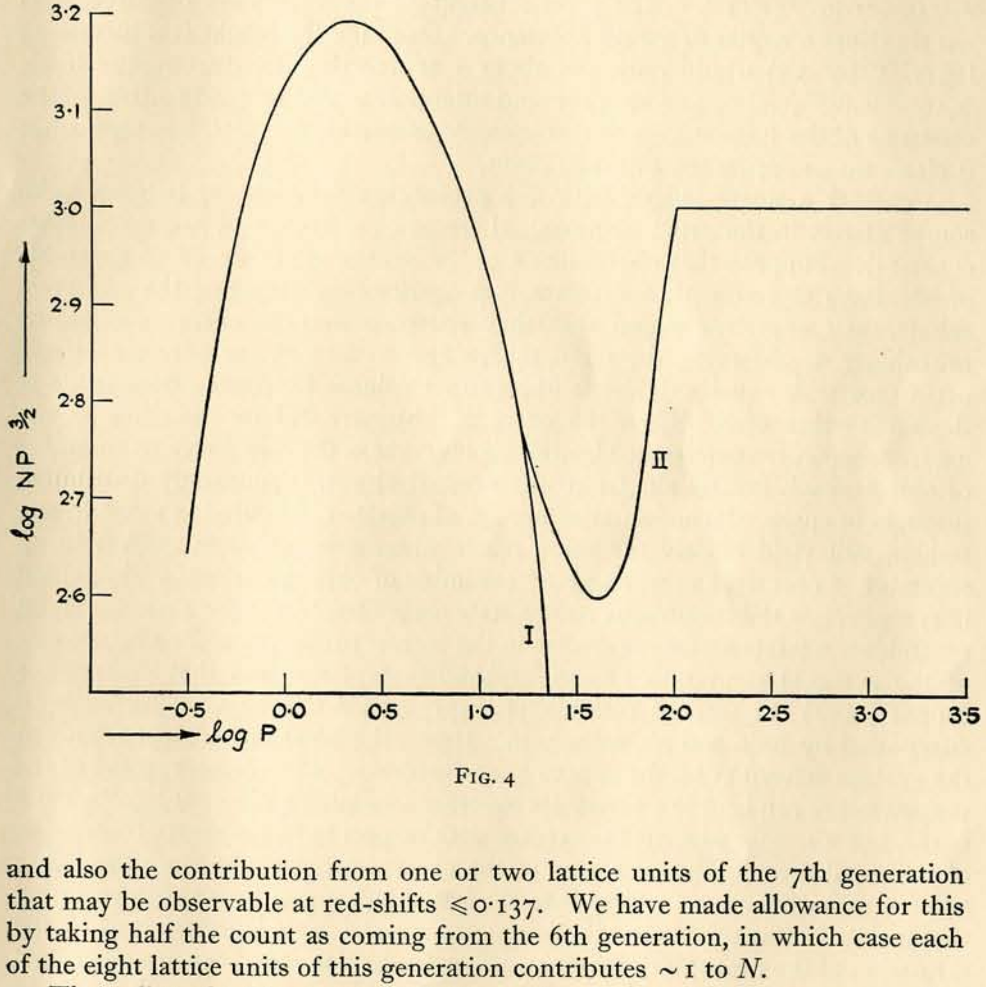

FIG. 4

and also the contribution from one or two lattice units of the 7th generation that may be observable at red-shifts $\leqslant 0{\cdot}137$. We have made allowance for this by taking half the count as coming from the 6th generation, in which case each of the eight lattice units of this generation contributes ~ 1 to N.

The radius of a 6th generation lattice unit $\sim \frac{1}{6}l_1 \exp 5/3$. A source at such a distance has red-shift $\sim \frac{1}{3}z_1 \exp 5/3$ $(z_1 = 0{\cdot}03)$. Thus an observer situated at the centre of a 6th generation lattice unit will find that all the local sources coming from the unit in question give red-shifts $z \leqslant 0{\cdot}01 \exp 5/3 = 0{\cdot}053$. The value of $\log P$ at $z = 0{\cdot}053$, on the scale of Table I, is about 2·0. Thus with $N \cong 1$ at $\log P = 2{\cdot}0$, the constant level of $\log NP^{3/2}$ to be used for curve II of Fig. 4 is $\sim 3{\cdot}0$.

The precise depth of the minimum of curve II depends on the steepness in the region $\log P \cong 1{\cdot}5$ of the fluctuation-free curve. This is affected by our choices for z_1, k, Δ. Since our choices are somewhat uncertain, the depth of the minimum, and its position with respect to P, is not well determined. All we can properly assert is that a minimum is to be expected in the region from $\log P \cong 1{\cdot}5$ to $\log P \cong 2{\cdot}0$.

If 10^{-26} w.m^{-2} (c/s)$^{-1}$ is the unit of P, then the values of N in Table I at $P \cong 10^{-25}$ w.m^{-2} (c/s)$^{-1}$ are approximately correct for the number of radio sources counted per steradian. On this basis $N \cong 1$ implies only about one source per steradian. It is clear, therefore, that although the fluctuation appears impressive when plotted in the form of Fig. 4, the actual number of sources introduced by the fluctuation is very small compared with the count at $P < 10^{-25}$ w.m^{-2} (c/s)$^{-1}$. The precise height of the fluctuation of curve II is not therefore a matter of great importance. Doubling the height (i.e. increasing $\log NP^{3/2}$ by 0·3) would only add about 1 source per steradian at flux levels $\geqslant 10^{-24}$ w.m^{-2} (c/s)$^{-1}$, and no profound attention would be paid to this. The existence of the minimum is of considerable theoretical interest, however, since it clears up an apparently puzzling point.

Consider a single object, with any probability function for being a radio source at rest in the usual cosmological frame of reference (r, θ, ϕ coordinates constant). Suppose that observations of the source are made by an ensemble of observers at rest with constant r, θ, ϕ coordinates. Imagine the observers collate their experiences and that they draw an average curve relating the probability of observing the radio source against flux P, the averaging being performed with equal weight per unit proper volume for observations made at the same value of t. Then the curve so obtained will be the same as the $\log N - \log P$ curve determined by a single observer in the case where an ensemble of objects, each exactly similar to the original object, is uniformly distributed throughout space. Hence an arbitrary set of objects distributed in space in any fashion will yield exactly the same result when averaged with respect to an ensemble of observers as will a grand ensemble of objects uniformly distributed in space. Now the continuous steady-state theory treats just the case of a grand ensemble—it treats a mixture of ages in the correct proportions. Taking account of the eventual saturation of any probability functions (so that divergences cannot occur) the smooth continuous theory yields the form of $\log N - \log P$ curve used by Ryle and his colleagues. Hence the latter curve must represent the average arrived at by the ensemble of observers. The discrete model of the present paper can lead to a particular observer obtaining a different $\log N - \log P$ curve, but it cannot prevent the average with respect to all observers from being of the usual form. The averaging of the observers must take place through a volume with dimensions greater than the lattice spacing of the oldest generation that makes an important contribution to the count. This is determined by the saturation of the probability function. The results derived above are unaffected by a saturation at the 10th generation. The relevant lattice spacing could therefore be $l_1 \exp 3 \cong 20 l_1 \cong 1800$ mpc. Averaging must therefore take place over the whole "observable" region of the Universe (this is defined as $cH^{-1} \cong 3000$ mpc, for $H = 100$ km s^{-1} (mpc)$^{-1}$). While it would be possible to raise logical objections to observers actually collating their experiences over regions as large as this, one can still carry through a mathematical average. The question arises as to how the results derived above could lead, when averaged for all observers, to the $\log N - \log P$ curve given by the usual continuous theory.

The answer lies in the minimum shown in curve II of Fig. 4. Some observers will be situated inside late generation lattice units, e.g. the 8th and 9th. The fluctuation level of $NP^{3/2}$ for an observer in such lattice units is not only lifted but the fluctuation extends to smaller values of P. The minimum of curve II of Fig. 4 is shifted left by $\sim 1\cdot0$ in $\log P$. The observer would find a $\log N - \log P$ curve with slope $-1\cdot5$ at high P, the slope would become somewhat less than $-1\cdot5$ as P decreased (due to the usual red-shift effect), but then for P around 10^{-25} w.m^{-2} (c/s)$^{-1}$ the slope would fall precipitately away to the minimum—the slope would be *markedly less* than that given by the completely continuous theory. Eventually at low values of P the slope might increase somewhat, but $NP^{3/2}$ would not rise back to its value at high P.

We expect observers of the sort described in the previous paragraph to have a large weight in determining the average for the whole ensemble of observers. This is because such observers add comparatively large numbers of sources at high values of P. The slope of a $\log N - \log P$ curve can be appreciably changed by adding only a moderate number of sources provided these are added at high and at medium P values. No great differences in the counts at low P values are needed. Indeed we expect all observers to obtain much the same counts at low flux values.

Can any explanation be given of why we do not ourselves lie inside a late generation lattice unit? On the basis of sheer probability the chance against lying inside a lattice unit of any particular generation is about 5/6 (cf. Section 4). The chance against lying inside lattice units belonging to the 7th, 8th and 9th generation would thus be $\sim (5/6)^3$, and this is not small. Moreover there could be biological and physical reasons of relevance, relating to our presence on the Earth. For life to be possible (in our particular form) rather stringent chemical conditions, as well as astronomical conditions, must very likely be satisfied. It is entirely possible that these conditions are only met with in galaxies of about the age $\frac{4}{3}H^{-1}$, in which case our presence inside a 6th generation lattice unit would not be an accident. And this could inhibit our chance of lying inside a 7th, or even an 8th, generation unit, since the lattice positions of succeeding generations are very likely anticorrelated. Nor is it an entirely unreasonable speculation that the degree of electromagnetic activity associated with 8th and 9th generation lattice units might be so great that biological development would be significantly affected by a very high cosmic ray background inside such units.

Quite apart from these points, however, it is important to notice that although a discrete model does not change the average of an ensemble of observers, *it does make each observation yield a $\log N - \log P$ curve that is different from the average curve*. The situation is not similar to the case of a Maxwellian distribution of particles, where most particles have energies close to the mean energy. A typical observer does not obtain a $\log N - \log P$ curve close to the average of the whole ensemble. Observers obtain widely differing curves depending on their individual relationships to the discrete structure of the condensations. Only after a very large-scale averaging process has been carried out does the mean $\log N - \log P$ curve appear.

Our answer to Ryle's criticism lies at exactly this point. Every observer taken individually obtains a $\log N - \log P$ curve differing from the mean curve. Hence every observer could interpret his result as disproving the steady-state theory, even though in the mean their results actually verified the theory. For example, an observer situated inside a late generation lattice unit (8 or 9) could argue that as P decreased below 10^{-25} w.m^{-2} (c/s)$^{-1}$ the slope differed markedly from the continuous steady-state curve, but in just the opposite sense to that of the terrestrial observations. We are dealing with a property in which the fluctuations make an important contribution, particularly fluctuations arising from old lattice units. The property is due to the weighting effect of the probability function $K(\tau)$ which causes the mean situation to have no close resemblance to the situations in particular cases. Such an effect does not arise when K is independent of τ, as it is in the simple counting of galaxies.

8. *The effect of the luminosity function.*—In the previous sections we have shown how a $\log N - \log P$ curve appreciably steeper than -1.5 can be obtained. In the present section we shall be concerned less with the precise value of the steepness of the $\log N - \log P$ curve than with the discussion of a realistic luminosity function, and with the relation of the luminosity function to an absolute scale for P. To obtain the best fit of the theory to known data concerning the luminosity function some sacrifice of steepness has to be made—the value obtained below for the slope resulting from the combination of a number of luminosity classes will turn out close to -1.6.

We have seen in Section 6 that the unit of the $\log P$ scale in Table I and in Figs. 2, 3, 4 can be taken as 10^{-26} w.m^{-2} (c/s)$^{-1}$ if the intrinsic luminosity of the sources in question is $\sim 3 \times 10^{26}$ w. (c/s)$^{-1}$ at frequency 160 Mc s^{-1}. Such sources we refer to as of class II. We consider five other classes: I, $L = 10^{26}$ w. (c/s)$^{-1}$; III, $L = 10^{27}$ w. (c/s)$^{-1}$; IV, $L = 3 \times 10^{27}$ w. (c/s)$^{-1}$; V, $L = 10^{28}$ w. (c/s)$^{-1}$; VI, $L = 3 \times 10^{28}$ w. (c/s)$^{-1}$. Each class of source is taken as possessing a $\log (NP^{3/2}) - \log P$ curve of the form of curve II of Fig. 4, but with an appropriate displacement of the flux scale. Thus class I is displaced by ~ 0.5 to the left (with respect to Fig. 4 for class II), class III by ~ 0.5 to the right, class IV by 1.0 to the right, class V by ~ 1.5 to the right, and class VI by 2.0 to the right. The maxima for I and II were taken as of equal height in the $\log NP^{3/2} - \log P$ plane, the maximum of III was reduced by 0.2 in the logarithm, that of IV by 0.44, that of V by 0.24, that of VI by 0.64. These choices were made in order to give what we feel to be reasonable agreement with all the available data, such as it is known to us. The result of combining the six classes is shown in Table II. The percentages of each class at the various five levels are given in the second part of the table.

The fluctuation introduced at high values of P brings in sources only at the places entered as F in Table II. The precise contribution of the fluctuation is uncertain within a factor 2. For curve II of Fig. 4, sources of class I would increase the total count at $\log P = 1.5$, and at $\log P = 1.75$ by about 30 per cent, while the count at $\log P = 2.0$ would be approximately doubled.

Optical photographic magnitudes, uncorrected for spectrum red-shift effects, and making no allowance for galactic obscuration, are attached in brackets to the entries in the second part of Table II for the columns $\log P = -0.5$, 0, 0.5, 1, 1.5, 2. Since optical and radio magnitudes vary with distance in the same way when spectrum red-shift effects are omitted (these are in any case small for the radio magnitudes) the optical magnitudes change by 1.25 when the $\log P$ changes by 0.5. And assuming that all radio sources are associated with galaxies of the same intrinsic optical luminosity, the optical magnitudes vary as $2.5 \log L$ for the various radio classes. Since $\log L$ changes by 0.5 from one radio class to the next, the optical magnitude therefore also changes by 1·25 as we pass from one radio class to the next at fixed P.

Remembering that Table I refers to class II sources (when the scale of P is 10^{-26} w.m^{-2} (c/s)$^{-1}$), we note that the red-shift for $\log P=1$ is close to 0·15 for a class II source. At this red-shift, Humason, Mayall and Sandage give about +18·5 for the photographic magnitude of the brightest galaxy of the cluster 0025+2223. This value has been taken as the standard in Table II, other values being determined by differencing in the manner described in the previous paragraph.

The magnitudes in Table II are therefore chosen as bright as seems reasonably possible. They assume that all radio sources are associated with galaxies of abnormally great optical emissivity. While it is true that known optical identifications support this assumption, it is obvious that a selection effect exists in favour of making identifications with galaxies of abnormal luminosity. Also we may note that cluster 0025+2223 is at comparatively high galactic latitude $b=-40°$, where galactic absorption is only about 0·1 mag. In attempting identification with radio sources it will usually be necessary to work at lower galactic latitudes where absorption is greater. Hence it does not seem likely that we have set the optical magnitudes too faint in Table II.

The situation concerning the possibility of making optical identifications stands out clearly in the table. A satisfactory identification demands a highly accurate radio position, say $\pm 0{\cdot}5'$ in both coordinates, and so far it has not been found possible to obtain such positions except for particularly bright sources. And the remarkable point emerges from Table II that the bright radio sources are just the ones that contain the highest proportion of classes IV, V and VI, all of which are distant objects associated with faint optical magnitudes.

A thoroughgoing attempt to secure optical identifications has recently been made by Bolton and Minkowski, who have considered about 100 bright sources with flux levels above about $\log P \simeq 1{\cdot}5$ (160 Mc/s). The attempted identifications with galaxies brighter than +19·5 were successful in about 45 per cent of cases, which is about the percentage indicated by Table II, if the contribution F of class I is omitted. Table II suggests that some 35 per cent will be likely to fall between +17 and +19·5, as compared with the 30 per cent actually identified. The contribution F might have been expected also to yield some 30 per cent, whereas about 15 per cent were found at optical magnitudes bright enough to be associated with the nearby class I objects.

The values of Table II do not hold out much hope that many further optical identifications will be forthcoming, unless either it proves possible to work to magnitude +20 (or even +21 if spectrum red-shift effects become important) or accurate radio positions become available down to $\log P \simeq 1$.

It is our impression that the following argument has surrounded the whole question of optical identifications. Taking the bright radio sources, roughly half of them cannot be identified with galaxies brighter than +19. Accordingly it seems fairly certain that these radio sources are distant objects. Then how much more distant must the faint radio sources be? Our table shows that the last step of this argument may well be quite erroneous. The effect of the red-shift (not in the spectrum now, but on the apparent bolometric magnitude) can produce a situation in which classes IV, V, VI dominate the count at high, and even medium, values of P, but in which the intrinsically faint classes I and II entirely dominate the count at low flux values. This effect has already been noted by Mills (4).

It is of interest to consider the relation of our luminosity function to the recent work of Allen, Palmer and Rowson, on the surface temperatures of radio sources*. For the purpose of relating our results to the observations, we note that the L value for class VI is close to that of Cygnus A. Write T_{Cyg} for the measured surface temperature of Cygnus A Then if all radio sources possess the same physical size we expect the temperatures of our six classes to be given by

$$(\text{I, II, III, IV, V, VI}) \cong (3 \cdot 10^{-3}, 10^{-2}, 3 \cdot 10^{-2}, 10^{-1}, 3 \cdot 10^{-1}, 1) T_{\text{Cyg}}. \quad (27)$$

Some uncertainty is attached to the appropriate value of T_{Cyg}. In the general surface temperature experiments, in which Cygnus A was treated in the same way as other sources, a value $T_{\text{Cyg}} \cong 8 \times 10^8\,^{\circ}\text{K}$ was found. Special information from the fuller analysis of Jennison *et al.* suggests a lower value of $T_{\text{Cyg}} \cong 3 \times 10^8\,^{\circ}\text{K}$, however. A satisfactory compromise appears to be to accept the lower value of T_{Cyg}, but to continue to ignore the effect of the red-shift on the surface temperature, as we have already done in (27). The red-shift lowers the surface temperature by $(1+z)^{-4}$. If we restrict ourselves to sources that make the important contributions to Table II, the values of z are never greater than 0·3 to 0·4, so that the $(1+z)^{-4}$ factor is not larger than the measure of uncertainty in T_{Cyg}. For the Jodrell Bank experiments, $\log P$ is on the average near 1·2, and at this value of P the red-shift correction is small for sources of classes I, II and III, although it amounts to a factor $\sim 3^{-1}$ for class V.

We set the surface temperatures in relation to Cygnus A as follows

$$T_{\text{I}} \cong 10^6,\ T_{\text{II}} \cong 3 \times 10^6,\ T_{\text{III}} \cong 10^7,\ T_{\text{IV}} \cong 3 \times 10^7,\ T_{\text{V}} \cong 10^8,\ T_{\text{VI}} \cong 3 \times 10^8. \quad (28)$$

If the sources are not of the same physical size, if the intrinsically weaker sources are smaller in their dimensions, then the values of T for the lower classes have been set too low. Turning to the percentages given in Table II for $\log P = 1{\cdot}25$, we see that the most frequent source is of class III with surface temperature $\sim 10^7\,^{\circ}\text{K}$ and that classes II and IV with temperatures $\sim 3 \times 10^6$ and 3×10^7 are symmetrically distributed about $\sim 10^7\,^{\circ}\text{K}$. This result is in close accord with the work of Allen, Palmer and Rowson, as also is the 20 per cent of sources at temperatures of order $10^8\,^{\circ}\text{K}$.

These considerations, both of the optical identifications and of the surface temperatures of radio sources, show that we have been realistic in the absolute values of L that have been associated with the various classes, and hence that our taking $10^{-26}\,\text{w.m}^{-2}\,(\text{c/s})^{-1}$ as the unit of P is not an improper choice.

As we have already mentioned, the mixing of the various luminosity classes results in a reduced slope for the $\log NP^{3/2} - \log P$ curve. The values given in the first part of Table II are plotted in Fig. 5. The slope of the $\log N - \log P$ curve is now no greater than $-1{\cdot}6$ in the important range of P, although there is a steepening to $-1{\cdot}7$ at $\log P \cong 1{\cdot}5$. The effect of the fluctuation at $\log P > 1{\cdot}5$ has not been shown in Fig. 5. This would lift the curve in the region $1{\cdot}5 \leqslant \log P \leqslant 2$, probably yielding a sub-Euclidean slope in this range, exactly as in the case of curve II of Fig. 4.

* We are indebted to the Jodrell Bank workers for valuable discussions on their observations.

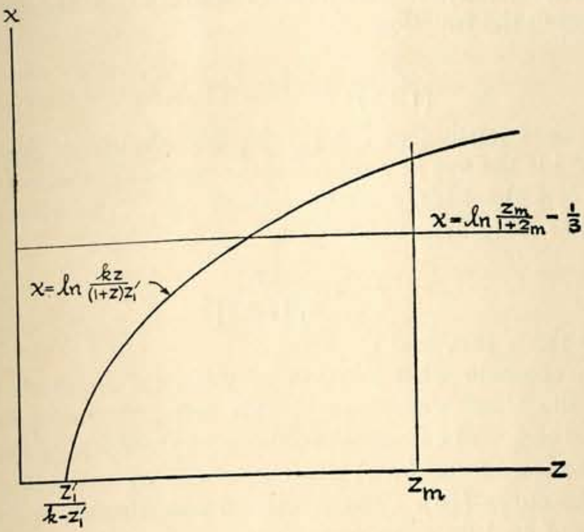

FIG. 5

The present results are of course related to our choice, $K(q) \propto q^4$, for the dependence of the probability function on age over the relevant generations (5th to 9th or 10th). A steeper rise of K would lead to a steeper $\log N - \log P$ curve. For this reason no particular significance we feel can be attached to an amount of the order of 0·1 or 0·2 in the slope. At the outset we noticed that whereas the latest Cambridge results yielded $-1{\cdot}8$, Mills has given reasons for believing that the slope should not exceed $-1{\cdot}5$. In these circumstances our value of $-1{\cdot}6$ does not seem unsatisfactory.

Although we have lost steepness in the slope for $0{\cdot}4 < \log P < 1$ we have gained in the behaviour of the $\log NP^{3/2} - \log P$ curve for $\log P < 0{\cdot}0$. This section is now very close to the ultimate slope suggested by Ryle to be the most probable interpretation of an analysis by Hewish (of deflections rather than source counts) using the method of Scheuer.

9. *The count from individual lattice units.*—We return to a consideration of the luminosity class II alone. The specification of a red-shift z defines a distance $cH^{-1}z$. Within this distance, if z is sufficiently large, there will be a particular generation n for which the volume of a unit cube, l_n^3, is comparable with $4\pi/3(cH^{-1}z)^3$. Observation to red-shift z is expected to include the eight lattice members (or part of them) belonging to such a unit cube. The question arises as to how many sources are to be associated with each of these eight members and how does this number vary with z, and hence with P. An answer to this question can be given to within an uncertainty of a factor of about 2. The uncertainty arises from connecting ideas that relate to the discrete model with values given by the continuous theory. To remove the uncertainty it would be necessary to work entirely in terms of an explicit discrete model. (Such a project is feasible if automatic computing were used. If the present theory turns out to survive any further immediate attacks we shall consider it worth while to abandon altogether the simple methods of the continuous theory and to go over entirely to a discrete situation.)

Referring back to (23), the number of sources counted to red-shift z_m is given by integrating the function

$$\frac{z^2}{(1+z)^3} K(\chi) \exp(-3\chi) \tag{29}$$

over the appropriate portion of the (χ, z) plane shown in Fig. 5. It will be recalled that χH^{-1} is the age of a source at the moment of emission or radiation, viz. $\chi = \tau H - \ln(1+z)$ where τ is the present age. The relevant area of the (χ, z) plane is that lying below the curve

$$\chi = \ln \frac{kz}{z_1'(1+z)}, \tag{30}$$

and enclosed by the z-axis, and by $z = z_m$.

It is easy to estimate what fraction of the total count to red-shift z_m is contributed by the "last generation", this being defined as the last range of $\frac{1}{3}$ in the interval of χ. The fraction is obtained by integrating the same function (29) over the portion of the (χ, z) plane enclosed by $\chi = \ln(z_m/1+z_m) - \frac{1}{3}$, by $z = z_m$, and by the curve (30). The fractional contribution is thus the ratio of the two integrals of (29) over the areas shown in Fig. 5. Introducing $q = \exp\chi$ again, we require the ratio of the following integrals

$$\int_{\exp\frac{1}{3}\cdot\frac{z}{1+z}=\frac{z_m}{1+z_m}}^{z_m} \frac{z^2\,dz}{(1+z)^3} \int_{-\frac{1}{3}+\frac{kz}{z_1'(1+z)}}^{\frac{kz}{z_1'(1+z)}} \frac{K(q)}{q^4}\,dq, \tag{31}$$

$$\int_{\frac{z}{1+z}=\frac{z_1'}{k}}^{z_m} \frac{z^2 dz}{(1+z)^3} \int_0^{\frac{kz}{z_1'(1+z)}} \frac{K(q)}{q^4}\,dq. \tag{32}$$

Computations for $0{\cdot}2 < z < 0{\cdot}4$ for $K(q) \propto q^4$ and for $k/z_1' = 15$, give ratios close to 2/9.

We turn now to the actual numbers N given in Table I. We have seen that if these numbers are interpreted per steradian they are in reasonable relation to the actual counts. Thus $\sim 4\pi N(z)$ is the order of the total count on the whole sky to red-shift z. We now see that $\sim 2/9$ of this must be attributed to the "last generation", viz. $\sim 3N(z)$.

The next step brings in the uncertainty of connection to the discrete model. We associate the "last generation", as defined above for the continuous picture, with the situation in the discrete model in which the observer sees the eight lattice units of generation n. Then each lattice unit of the "last generation", contributes a total of $\sim \frac{3}{8}N(z)$ sources. In view of the uncertainty of association we can simply say that the lattice units making the maximum contribution to the count to red-shift z contain of the order of $N(z)$ sources, where the functional dependence of N on z is given in Table I. (The present discussion essentially determines the order of the absolute value of the probability function K.)

Reference to Table II shows that we have made important use of class II sources down to about $\log P = -0{\cdot}5$. Table I shows that such a value of P corresponds to $z \simeq 0{\cdot}6$, and at this red-shift $N(z) \simeq 2500$ sources. It follows that the maximum population of the oldest lattice units (those of the $\sim$ 10th generation) is some 2500 sources of class II. The maximum populations required for the other luminosity classes are easily estimated from Table II. For example, the relative proportions of classes I and II at $\log P = -0{\cdot}5$ is about 3/1. Our considerations down to this flux level therefore require the presence of some 7500 sources of class I. Results for the various classes are as follows:

Maximum number of sources required per lattice unit

Class I	7500
Class II	2500
Class III	250
Class IV	25
Class V	10
Class VI	<1

Since the number of galaxies per lattice unit is $\sim 10^5$ it is clear that the maximum populations in the oldest lattice units only become at all comparable with the number of galaxies for the cases of classes I and II. Only these cases require further consideration.

Class II sources are comparable to the source Hydra A, for which Burbidge (13) has calculated a minimum total energy requirement of $\sim 10^{58}$ ergs. At an emission rate of 4×10^{42} erg sec^{-1} (the value arrived at in Section 6), such an energy reservoir would last for about $2{\cdot}5 \times 10^{15}$ sec. We require the main radio source activity throughout a lattice unit to last for an interval of time of the order of $\frac{1}{3}H^{-1}$. Thus if each of the $\sim 10^5$ galaxies were to become a radio source on one occasion during the relevant generation the number of sources existing at any particular moment would be of order $2{\cdot}5 \times 10^{15}/(\frac{1}{3}H^{-1}) \times 10^5$. H^{-1} must be expressed in seconds, viz. $H^{-1} \cong 3 \times 10^{17}$ sec. The number is thus very close to 2500.

A similar calculation for class I sources yields an equally satisfactory answer. Since the emission rate for class I is lower by a factor ~ 3 the time for which a given energy reservoir can maintain the radiation rate is increased by 3, and the number of sources existing at any moment of time is expressed by the same factor.

It is of course the case that we do not literally require every galaxy to become a radio source. It would be equally sufficient if a sub-class of galaxies became radio sources on an appropriate number of occasions, say 10 per cent of galaxies ten times more frequently. Energy requirements do not forbid such a situation. Thus the gravitational energy of a galaxy of mass M and radius r is of order GM^2/r; and for $M = 3 \times 10^{11} \odot$, $r = 15$ kpc, this is $\sim 5 \times 10^{59}$ erg. Hence an energy reservoir $\sim 10^{58}$ erg could be provided on a number of occasions.

It is entirely possible that the classes III–VI are simply cases where galaxies of abnormally large mass are involved. With M increased by ~ 10, gravitational energies are increased by $\sim 10^2$, which is of the order of difference between II and VI. The lower proportions required for the very strong radio sources could be simply a consequence of the small fraction of galaxies that possess masses of order $3 \times 10^{12} \odot$.

10. *The distribution of sources on the sky.*—The discrete model not only introduces fluctuations in the number counts, of the sort discussed in Section 7, but it also introduces fluctuations from strict isotropy in the distribution of sources on the sky. The discrete lattice units introduce patchiness. We have now to examine whether this patchy characteristic is consistent with the observations or not.

We note that MSH have drawn attention to a possible departure from isotropy, and that more recently Mills, Slee and Hill (14) report that between declinations $-20°$ and $-50°$ they find a marked deficiency of sources in R.A. 00^h to R.A. 02^h. (We have taken Mills' sources and have divided them into 12 groups. Each group covers 2^h in R.A. and the whole range of declination. We find only a negligible probability that the grouping is random. A test applied to the postulate that the source density varies in the ratio 4/3 from one area of sky of order 10^3 square degrees to another yielded a reasonable result.)

We proceed now to the theoretical discussion, and we do this, first, for a single luminosity class. When a limiting flux P is given, the generations of our model can be ordered in the following way. There will be a particular generation, $n(P)$ say, of which the observer will expect to see one complete unit cube with its associated 8 lattice units. For the $n(P)+s$ generation ($s=1, 0, -1, -2, \ldots$) the observer will expect to see $\sim 8\exp(-s)$ lattice units. The series of values of s must be taken only for $s \leqslant 1$, since for $s \geqslant 2$ the formula derived from the continuous theory ceases to have any relevance.

In the simple cubic model the 8 lattice units for $s=0$ present the smallest total solid angle to an observer at equal distances from each of them. With l as the lattice spacing, the observer's distance from the vertices is $(\sqrt{3}/2)l$. Since the radii of the units (in our model) is $\frac{1}{3}l$, the solid angle presented by each unit is 0·485 ster. The 8 units together therefore cover a fraction 0·31 of the whole sky.

For the generation $s=-1$, each lattice unit is smaller by $e^{-1/3}$. Thus a lattice unit of generation $s=-1$ at the same distance l subtends a solid angle smaller by $\sim e^{-2/3}$. There are, however, e times more units from generation $s=-1$ than from generation $s=0$. Generation $s=-1$ therefore subtends a total solid angle that is greater by at least $e^{1/3}$. (Not all the units of $s=-1$ are as far away as l, although the majority of them are at distances of this order.) Hence generation $s=-1$ covers at least a fraction $0{\cdot}31e^{1/3}=0{\cdot}43$ of the whole sky. In a similar way, one can easily see that generation $s=-2$ covers at least a fraction $0{\cdot}31e^{2/3}=0{\cdot}60$ of the whole sky.

The situation is somewhat more complicated for generation $s=1$. A lattice unit of this generation, if completely within range of observation, must subtend so large a solid angle that a simple inverse square law cannot be used. The most distant sources of the 8 lattice units of $s=0$, for an observer at the centre of a unit cube, demand observation not merely to $\sqrt{3}l/2$ but to $l(1/3+\sqrt{3}/2)$. If the sources of an $s=1$ lattice all lie within this distance, then the centre of the lattice unit cannot be further away from the observer than

$$l\left(\frac{1}{3}+\frac{\sqrt{3}}{2}\right)-\frac{1}{3}e^{1/3}$$

the radius of the unit being $\frac{1}{3}l\exp(\frac{1}{3})$. Hence the minimum solid angle subtended by such a unit must be 1·43 ster, *if the unit is to be completely observed*. Three such units would therefore cover at least a fraction 0·342 of the whole sky.

It is of course the case that the formula $8\exp(-s)$, derived from the continuous theory, does not require the value ~ 3, given by $s=1$, to be made up of three complete lattice units of generation $s=1$. The three could be made up by partial contributions from more than three lattice units. Indeed a system of partial contributions is certainly much more probable. Such a system would have a smaller departure from isotropy, since there would be more than three patches on the sky, and each patch would contribute fewer sources. We shall keep, however, to the worst case in which the observer sees three complete lattice units of generation $s=1$. Summarizing we have:

Generation s	Number of lattice units	Solid angle subtended by each lattice unit (ster.)	Minimum fraction of the sky covered
1	3	1·43	0·34
0	8	0·48	0·31
−1	22	0·25	0·43
−2	60	0·13	0·60

It is useful to consider departures from isotropy by dividing these four generations into two pairs $s=-1, -2$; $s=1, 0$. The first of these pairs has so many lattice units that it is essentially free from ambiguities; and the sum of the fractions of the sky covered adds almost exactly to unity, which is a convenient simplification. Since successive generations are expected to be anticorrelated we consider the patches of $s=-1$ to interlace those of $s=-2$.

Write X for the total number of sources contributed by $s=-1, -2, -3, \ldots$. The work of the previous section suggested that $\sim 2/9\ X$ are contributed by $s=-1$, about $\frac{2}{9}\cdot\frac{7}{9}$ by $s=-2$, etc. So many lattice units are contained in $s=-3, -4$, that we regard their contribution as being isotropic. That is to say, $49X/81$ sources are distributed isotropically. It is easily seen that the average source density per unit area for a patch from $s=-1$ to that for a patch from $s=-2$ is

$$\left[0{\cdot}604+\frac{2}{9}(0{\cdot}43)^{-1}\right]\Big/\left[0{\cdot}604+\frac{14}{81}(0{\cdot}57)^{-1}\right]\simeq\frac{5}{4}, \tag{33}$$

where, for simplicity, we have slightly reduced the fraction of the sky covered by the $s=-2$ patches from 0·60 to 0·57.

To test the effect of such a patchiness we took 27 squares arranged in three rows of 9. The squares were of two categories A and B. Equal probabilities were attached to each individual square being of type A or of type B. Decisions were made using random number tables: it turned out that 11 squares were assigned to type A and 16 to type B. The next step was to divide each such square into 100 sub-squares. A probability $\frac{4}{3}p$ was attached to there being a source in each sub-square belonging to type A, and p was attached to there being a source in each sub-square belonging to B. The value of p was chosen so that a total number of sources closely equal to the $3C$ catalogue was obtained, and the probabilities were again controlled by random number tables. As a final detail, when a source happened to fall in a sub-square, its position in that sub-square was also randomized. The result is shown in the second section of Fig. 6, the first part being a plot of the $3C$ sources.

Since the ratio 4/3 used in this test represented a greater measure of anisotropy than was estimated above in (33), it seems unlikely that the anisotropy arising from generations $s=-1, -2$ can bring us into a conflict with observation. It is of course true that statistical effects would be weakened if the total count X were increased—the "signal" carried by the probability ratio would stand out more in relation to the statistical "noise" as X increased, the signal-to-noise ratio improving essentially as $X^{1/2}$. According to the recent Cambridge results, about 8000 sources would be counted on the whole sky in a survey down to $P \cong 2 \times 10^{-26}$ w.m^{-2} (c/s)$^{-1}$. Of these, about 4000 would belong to the pair $s=0$, 1 which has still to be considered. The appropriate value of X for the pair $s=-1, -2$ would thus be about 4000 in such a survey, as compared with about 1000 used in Fig. 6. Hence an improvement in "signal-to-noise ratio" by ~ 2 would be expected in such a survey, *if all sources belonged to a single luminosity class*. The mixing of sources of different luminosity classes reduces the signal-to-noise ratio, however, first by reducing the value of X appropriate for each class, and second by increasing the "noise". The "noise" is increased because the different luminosity classes give different values of $n(P)$. The distribution of patches is accordingly different for the different classes (the flux level P being specified).

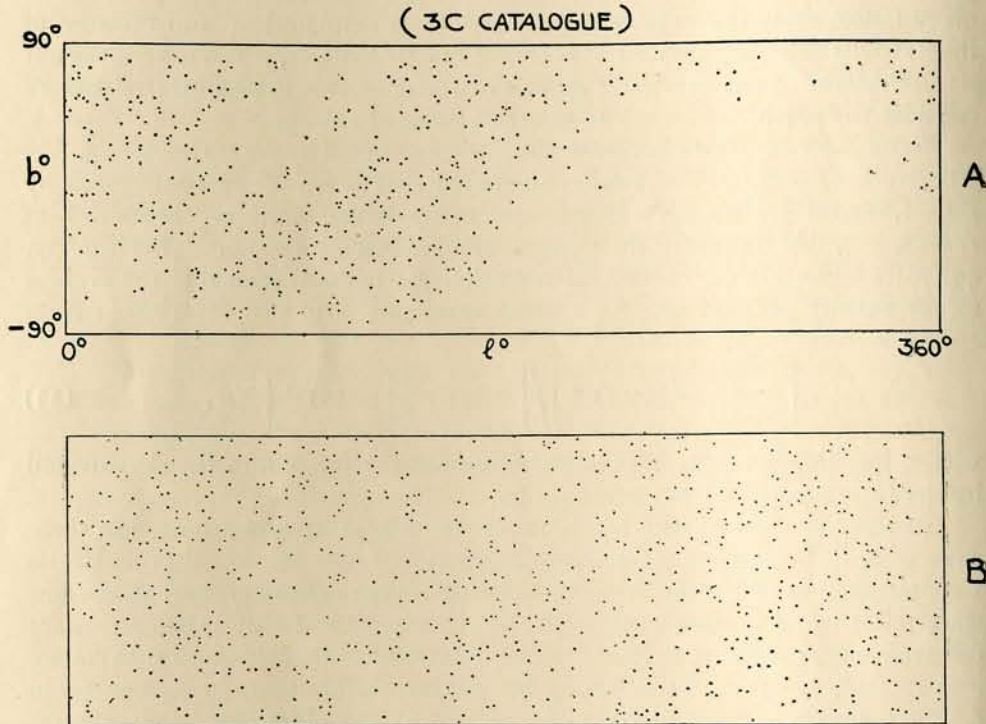

FIG. 6

We turn to the pair $s=1$, 0. Write Y for the total number of sources from the whole series $s=1, 0, -1, \ldots$. Then about 2/9 Y come from $s=1$, about 14/81 Y from $s=0$, and 49/81 Y from $s=1, -2, \ldots$ (i.e. $X \cong 0{\cdot}6\ Y$). Roughly half of the sources therefore arise in the pair $s=1$, 0.

Since the number of patches from both $s=1$ and $s=0$ together is only of order 10, it is very reasonable to regard the positions on the sky of the patches as being anticorrelated to those of the $s=0$ patches. If the two generations together are regarded *as covering the whole sky* then no crucial issue arises. The two generations cover approximately equal fractions of the sky, while they contribute numbers of sources in the ratio 9/7.

This ratio may somewhat underestimate the anisotropy, since a corresponding calculation using the discrete model gives a rather greater anisotropy. The above ratio 9/7 is derived from the value 2/9 Y for $s=1$, 14/81 Y for $s=0$, obtained in Section 9 from the continuous picture. Instead, we can argue, perhaps more simply, that each lattice unit of $s=1$ contributes $e^{4/3}$ times more sources than each lattice unit of $s=0$. With three of the former and eight of the latter, the ratio is $3e^{4/3}/8 = 1{\cdot}43/1$. But even with a ratio as great as 3/2 the fluctuation of source area density would merge with the smaller anisotropy of the generations $s=-1, -2$. A combination of the pairs $s=-1, -2$; $s=1, 0$ could hardly yield a patch ratio significantly worse than 4/3 even if the two pairs were not anticorrelated.

The serious question is whether the generations $s=1, 0$ really do cover the sky, and it appears to turn very largely on this issue as to whether a detectable anisotropy can be expected to arise or not. According to the above estimates, generations $s=1, 0$ cover 2/3 of the sky, but this estimate was very definitely a minimum value. The following argument is of interest in showing how far the fraction 0·31 for $s=0$ is a minimum.

It will be recalled that the value 0·31 was obtained for an observer at equal distances from all eight lattice units of generation $s=0$. A change of the observer to other positions increases the total solid angle subtended by these units. A calculation based upon a cubic lattice, of the effect of placing the observer in other positions, would attach too great an importance to the precise structure of the cubic lattice however. As stated in Section 4, one must take care not to use the cubic lattice model in such a way as to obtain results that are peculiar to this particular lattice, since in actuality the lattice may be expected to have no particular regularity. We therefore consider the following argument from which all lattice regularity is removed.

Write ν for the number of condensation centres per unit volume in generation $s=0$. To obtain eight such centres we require a sphere of radius a where

$$\frac{4\pi a^3 \nu}{3} \simeq 8.$$

Suppose a lattice unit at distance a subtends a solid angle Ω_a, and approximate by taking the solid angle at distance r ($<a$) as $\Omega_a(a/r)^2$—once again this can only underestimate the solid angle. Then, on the average, the total solid angle subtended by the eight lattice units inside the sphere of radius a is

$$4\pi\nu\Omega_a a^2 \int_0^a dr = 4\pi a^3 \nu \Omega_a \simeq 24\, \Omega_a$$

which is three times the value given by placing the whole eight units at equal distances a from the observer.

This argument must not be given too great weight, however, since it considers no correlation at all in the positioning of the centres of the lattice units, but it does show how far varying the observer's position, and varying the regularity of the lattice structure, could increase the total solid angle. To lift our 0·31 of Table IX to $\sim 0{\cdot}5$ we need a much smaller increase than the factor 3 given by this argument.

Turning now to $s=1$, we need to increase the 0·34 of Table IX also to $\sim$0·5. This could be achieved in two ways: by viewing more than three lattice units in a partial way—the advantages of this from the point of view of diminishing the anisotropy have already been mentioned—or by slightly decreasing the distance of the observer from the centres of each lattice unit. The estimate 0·34 was based on a maximum distance consistent with complete visibility of all the sources of the three lattice units of $s=1$. The solid angle is very sensitive to the precise distance of the observer, a slight decrease of distance producing a large increase of solid angle. To increase the fraction of the sky covered by the three lattice units from 0·34 to $\sim$0·5 it is only necessary to decrease the distance of the observer from their centres by some 20 per cent.

We may sum up the above discussion as follows: dividing the generations contributing to the counts in the manner described, the generations $s=-3, -4, \ldots$ are sensibly isotropic. It is unlikely that any anisotropy in the pair $s=-1, -2$ can readily be detected. And if the pair $s=0, 1$ fill the whole sky, it is also unlikely that anisotropy in $s=0, 1$ can be at all easily detected. If, however, there is a failure to fill the sky, a fluctuation of the source density over large areas of the sky might well be expected. The relevant areas are of order 1 steradian. In this connection it is of interest that the fluctuation reported by Mills is over an area of this order. It is possible that Mills' result arises from some systematic error, but if so the error would illustrate the experimental difficulty of giving a decisive answer to whether such large-scale fluctuations exist or not.

APPENDIX

It might appear at first sight that results of the previous section can be reproduced by introducing an age–luminosity function $L(\tau)$ instead of the age–probability function $K(\tau)$. As mentioned earlier, the two cases are not equivalent. While $K(\tau)$ affects the integrand in (19) the function $L(\tau)$ affects the limits. It is proposed to give an example where we take $K(\tau)=$ const. and introduce $L(\tau)$. We will consider the $\log N - \log P$ curve on the hypothesis that $L(\tau) \propto \exp(3H\tau)$ on analogy with the form chosen for $K(\tau)$.

The count to power level P is given by

$$N(P)=\iint \frac{z^2}{(1+z)^3} \exp(-3H\chi)\, d\chi\, dz \tag{34}$$

where the area of integration is specified in Fig. 7 by the shaded region. The upper curve α is given by

$$\chi = \ln \frac{kz}{(1+z)z_1}, \tag{35}$$

and the lower one β by

$$\chi = \tfrac{1}{3} \ln [Pz^2(1+z)^2], \tag{36}$$

where χ is measured in units of H^{-1}.

Let (z_2, χ_2), (z_3, χ_3) denote the intersections of α and β; $z_2 > z_3$. If P is large enough, $\chi_3 < 0$, and we have to integrate from z_3 to z_2. If P is small, so that $\chi_3 > 0$, we have to integrate from $z_1/k - z_1$ to z_4, and from z_4 to z_2, where z_4 is the point where the curve β intersects OZ, as shown in Fig. 7. In the absence of a power–age correlation, the curve β would simply be a straight line parallel to the χ-axis.

It is best to express N, P as functions of the parameter z_2 and to obtain their values for different values of z_2. From the figure it can be seen that P has a maximum value P_0, when the curves α, β touch one another. This happens when $z_2 = z_3 = 0{\cdot}25$. For $P > P_0$, $N = 0$. Here again we encounter the effect of discreteness on the $N-P$ curve.

The double integral can be evaluated for various values of z_2. Some values are given in the following table, with $z_1'/k = 1/15$ as before.

z_2	z_3	z_4	N	P	$NP^{3/2}$
0·25	0·25	—	0	·081	0
0·5	0·11	—	0·26	·0658	4·38
1·0	—	·086	2·618	·03125	14·46
1·5	—	·123	6·526	·0154	12·47
2·0	—	·163	15·203	·0082	11·29

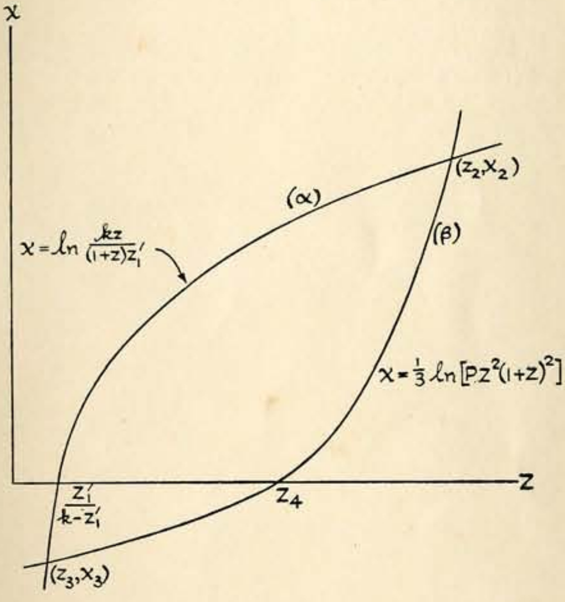

FIG. 7

Here also $NP^{3/2}$ increases initially as P decreases. But the increase is very rapid and is maintained only over a small range in P (a range ~ 2), while the decrease is gradual. It therefore does not reproduce the observed results. An exponential $L \propto \exp(4H\tau)$ would give similar behaviour, the case $L \propto \exp(3H\tau)$ was adopted because of the resulting simplicity of the integrals involved.

Thus though the amount of radiation contributed by a cluster of age τ is the same when $L(\tau) = f(\tau)$, $K(\tau) = \text{const.}$ as when $K(\tau) = f(\tau)$, $L(\tau) = \text{const.}$ the two cases are not equivalent as far as the $N-P$ curve is concerned.

This discussion, however, does not rule out the possibility of a combination of a power–age correlation with an age-dependent probability.

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References

(1) J. R. Shakeshaft, M. Ryle, J. E. Baldwin, B. Elsmore, and J. H. Thomson, *Mem. R.A.S.* **67**, 97, 1955; D. O. Edge, J. R. Shakeshaft, W. B. McAdam, J. E. Baldwin, and S. Archer, *Mem. R.A.S.*, **68**, 37, 1959; P. F. Scott, M. Ryle, and A. Hewish, *M.N.*, **122**, 95, 1961.
(2) B. Y. Mills, O. B. Slee, and E. R. Hill, *Aust. J. Phys.*, **11**, 360, 1958.
(3) K. I. Kellermann and D. E. Harris, *Observations of the California Institute of Technology Radio Observatory*, 1960.
(4) B. Y. Mills, *Aust. J. Phys.*, **13**, 550, 1960.
(5) F. Hoyle, 11th Solvay Conference, p. 53.
(6) F. Hoyle, *I.A.U. Symposium on Radio Astronomy, Paris* 1958, (Stanford), p. 529.
(7) M. Harwit, *M.N.*, **122**, 47, 1961.
(8) E. Lifshitz, *Journ. of Phys. U.S.S.R.*, **10**, 116, 1946.
(9) T. Gold and F. Hoyle, *I.A.U. Symposium, Paris* 1958, p. 583.
(10) C. D. Shane and C. A. Wirtanen, *A.J.*, **59**, 285, 1954; C. D. Shane, *A.J.*, **61**, 292, 1956.
(11) G. O. Abell, *Ap. J. Supp.*, **3**, 211, 1958.
(12) J. Crampin and F. Hoyle, *M.N.*, **122**, 27, 1961.
(13) G. Burbidge, *I.A.U. Symposium, Paris* 1958, p. 541.
(14) B. Y. Mills, O. B. Slee, and E. R. Hill, *Austr. J. Phys.*, **13**, 676, 1960.
(15) M. Ryle and R. W. Clark, *M.N.*, **122**, 349, 1961.
(16) M. Ryle, *Observatory*, **75**, 137, 1955.