ON THE QUANTUM STRUCTURE OF HORIZONS

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It was shown by 't Hooft that a black hole event horizon provides a natural cutoff length for semiclassical wave modes Here it is shown that this result is true in any spacetime with a horizon

In a recent paper [1] 't Hooft has explored the possibility of studying the physics of the back hole by erecting a "brick wall" outside the event horizon. The thermodynamics of the black hole then dictates the proper distance of the "brick wall" to the event horizon to be

$$l = (90\pi)^{-1/2} L_{\mathrm{P}} , \tag{1}$$

where L_{P} is the Planck length. We show in this note that this result is completely general and is only dependent on the behaviour of metric components near the horizon in an otherwise arbitrary spacetime.

Consider a spacetime which, in a suitable coordinate system, has the line element

$$\mathrm{d}s^2 = -A(r)\,\mathrm{d}t^2 + \mathrm{d}r^2/A(r) + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2), \tag{2}$$

or

$$\mathrm{d}s^2 = -A(x)\,\mathrm{d}t^2 + \mathrm{d}x^2/A(x) + \mathrm{d}y^2 + \mathrm{d}z^2 , \tag{3}$$

The choices

$$A_{\mathrm{S}}(r) = (1 - 2M/r) , \quad A_{\mathrm{D}}(r) = (1 - H^2 r^2) \tag{4,5}$$

in (2) represent patches of Schwarzschild and de Sitter spacetimes, while the choice

$$A_{\mathrm{NI}}(x) = (1 + 2gx) \tag{6}$$

in (3) would represent a uniformly accelerating frame. *In what follows, we shall work with metric (2)*; all the results can be extended to (3) in a simple manner. We assume that for a particular $r = r_0 > 0$ ("horizon")

$$A(r_0) = 0 , \quad (\mathrm{d}A/\mathrm{d}r)_{r_0} \equiv R = 0 , \tag{7}$$

so that, near $r = r_0$,

$$A(r) = R(r - r_0) + \mathrm{O}((r - r_0)^2) . \tag{8}$$

Consider a spinless particle of mass m in this spacetime. The action $\mathscr{A}$ for this particle satisfies the Hamilton–Jacobi equation

$$g^{ik}\partial_i \mathscr{A}\,\partial_k \mathscr{A} + m^2 = 0 , \tag{9}$$

which can be separated by

$$\mathscr{A} = -Et + J\theta + B(r) , \tag{10}$$

where

$$(\mathrm{d}B/\mathrm{d}r)^2 = [1/A^2(r)]\,[E^2 - A(r)(J^2/r^2 + m^2)]$$

$$\equiv k^2(E, J; r) . \tag{11}$$

The number of wave modes with energy less than E can be obtained by the semiclassical formula

$$N = \pm\frac{1}{\pi}\int_{r_0+h}^{L} \mathrm{d}r \int (2l+1)\,\mathrm{d}l\, k(E, l(l+1); r)$$

$$\equiv (\pi)^{-1} g(E) . \tag{12}$$

Expression (12) will reduce to eq. (3.7) of ref. [1] for the choice (4). In general, the following points must be noted about (12): (i) Following 't Hooft, we have assumed vanishing boundary conditions for the scalar field

$$\phi(x) \sim \exp(\mathrm{i}\mathscr{A}) \tag{13}$$

at $r = r_0 + h$ and at $r = L$. (ii) The sign ambiguity in (12) corresponding to $(\pm\sqrt{k^2})$ can be resolved by noting that $N > 0$. (iii) The l-integral is over the range at which k^2 is positive.

The rest of the analysis proceeds as in ref. [1]. Performing the l-integration, we get

$$g(E) = \pi N$$

$$= \frac{2}{3}\int_{r_0+h}^{L} [r^2\,\mathrm{d}r/A^2(r)]\,[E^2 - m^2 A(r)]^{3/2} . \tag{14}$$

Using (7), it is easy to see that the integral in (14) diverges in the limit of $h \to 0$ as h^{-1}. (The integral may also diverge for large L as L^3, but this is the Casimir contribution from the vacuum and can be neglected; see the comments following eq. (3.12) in ref. [11]. We get the leading contribution, as $h \to 0$, to be

$$g(E) \approx \tfrac{2}{3}E^3(r_0^2/R^2)(1/h) + \mathrm{O}(1) + \mathrm{O}(h) \;\ldots . \tag{15}$$

The contribution to the thermodynamic free energy at a temperature β^{-1} due to these wave modes is

$$F = \frac{-1}{\pi}\int_0^\infty \mathrm{d}E\, g(E)/[\exp(\beta E) - 1]$$

$$= (2/3\pi)(r_0/R)^2(1/h)\int_0^\infty E^3\,\mathrm{d}E/e$$

$$= (-2\pi^3/45)(1/h)\,\beta^{-4}(r_0/R)^2 . \tag{16}$$

The total energy U and entropy S are given by

$$U = \partial(\beta F)/\partial\beta = (2\pi^3/15)(1/h\beta^4)(r_0/R)^2 , \tag{17}$$

$$S = (U - F) = (8\pi^3/45)(1/h\beta^3)(r_0/R)^2 . \tag{18}$$

We shall next compute the invariant radial distance between $r = r_0$ and $r = r_0 + h$ for small h. We have

$$l \equiv \int_{r_0}^{r_0+h} \mathrm{d}r/\sqrt{A}$$

$$\approx (1/|R|^{1/2})\int_{r_0}^{r_0+h} \mathrm{d}r/(r - r_0)^{1/2} \approx 2(h/|R|)^{1/2} , \tag{19}$$

(Eq. (3.19) of ref. [1], which is a special case of (19) above, has a misprint; $2\sqrt{2}Mh$ should be $2\sqrt{2Mh}$. Also note that eq. (3.19) of ref. [1], as well as (19) above, are *approximate* and *valid only* in the limit of $h \to 0$.)

The area a of the surface $r = r_0$ for the metric (2) is just $4\pi r_0^2$. We equate the entropy in (18) to $\frac{1}{4}a$ getting

$$h = (8\pi^2/45)(1/\beta^3)(1/R^2) . \tag{20}$$

It can be shown, by a simple semiclassical analysis that the horizon temperature β^{-1} is related to R by

$$\beta^{-1} = |R|/4\pi . \tag{21}$$

(For a detailed discussion of this result see ref. [2]; one simple way to arrive at (21) is to note that the regularity at $r = r_0$ in the analytically extended euclidean spacetime requires periodicity with a period $4\pi/|R|$.) Using (21) in (20) we get

$$(h/|R|) = (360\pi)^{-1} , \tag{22}$$

so that the invariant distance to the "brick wall" is [see (19)]

$$l = 2(360\pi)^{-1/2} = (90\pi)^{-1/2} , \tag{23}$$

which is exactly the same as 't Hooft's result (see eq. (3.19) of ref. [1] for $Z = \lambda = 1$). In other words every metric of the form in (2) with a horizon at $r = r_0$, provides its own invariant cutoff at a proper distance l. The result depends *only* on the following assumption: *The entropy associated with the horizon is one-fourth the horizon area.* [We emphasize that (21) is not an assumption and can be derived from standard analysis.]

While the horizon in (2) is a compact surface, that of the accelerated frame in (3) is not. While the temperature in (21) can be attributed to this metric in the accelerated frame, it is doubtful whether an entropy can also be assigned. Formally equating the entropy in (18) to (πr_0^2), of course, will reproduce (23). The physical significance of this exercise, however, is not clear.

The generality of the above result, added to the fact that only the local behaviour of $A(r)$ near $r = r_0$ enters the discussion suggests that the result may be purely kinematic in origin. We shall now present such an interpretation:

Note that the free energy for the massless, spin-zero gas in flat spacetime has the form

$$F = \frac{-1}{\pi} \int_0^\infty \mathrm{d}E\, g(E)/[\exp(\beta E) - 1]\,, \qquad (24)$$

with

$$g(E) = (V/6\pi)\, E^3 = \frac{1}{6\pi} \int 4\pi r^2 E^3\, \mathrm{d}r\,. \qquad (25)$$

[See e.g. ref. [3]; p. 186, eq. (63.10) and note that spin-zero particles have half the spin degrees of freedom of photons.] In curved spacetime (25) is modified in two ways: (i) The proper volume element replaces ($\mathrm{d}r$) by ($A^{-1/2}\,\mathrm{d}r$); see (2); and (ii) The local value of the energy ($A^{-1/2}E$) should replace E. This leads to

$$g(E) = \frac{1}{6\pi} \int (4\pi r^2/A^{1/2})(E^3/A^{3/2})\, \mathrm{d}r$$

$$= \frac{2}{3} \int [r^2/A(r)^2]\, E^3\, \mathrm{d}r\,, \qquad (26)$$

which is the same as (14), for $m = 0$! . Thus all the non-trivial features arise purely from using a cutoff on the volume element and redshift factor in (26).

References

[1] G. 't Hooft, Nucl. Phys. 256 (1985) 727.
[2] T. Padmanabhan, Horizon temperature from semiclassical physics (1985), submitted to Phys. Rev. Lett.
[3] L.D. Landau and E.M. Lifshitz, Statistical physics, Part I (Pergamon, Oxford, 1980).